

Symmetry Breaking and Algebraic Quantum Topology

Symmetry Breaking, Goldstone Bosons and Algebraic Quantum Topology

Algebraic Quantum Topology

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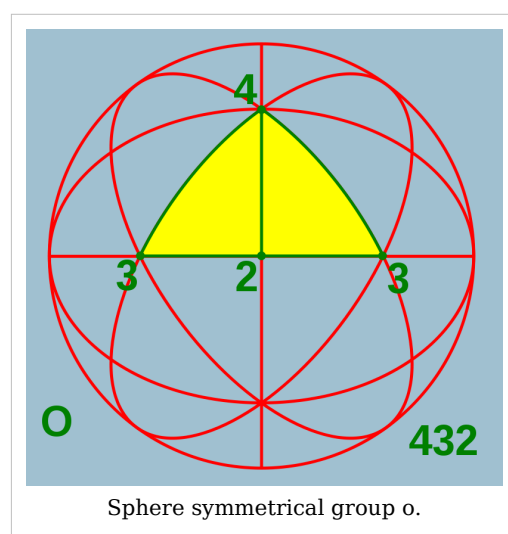
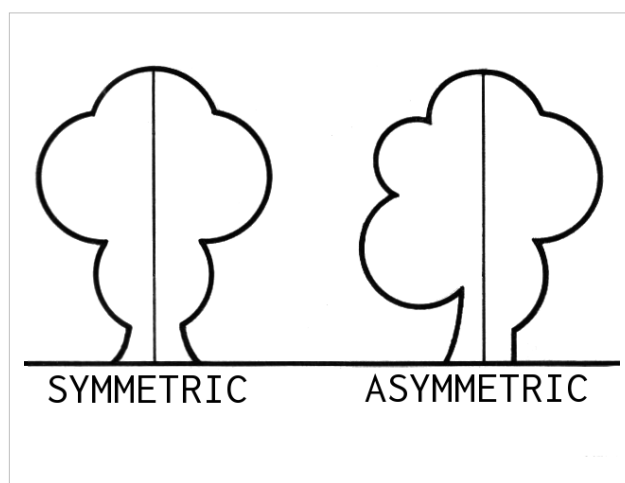
Symmetry

Symmetry generally conveys two primary meanings. The first is an imprecise sense of harmonious or aesthetically pleasing proportionality and balance; ^[1] such that it reflects beauty or perfection. The second meaning is a precise and well-defined concept of balance or "patterned self-similarity" that can be demonstrated or proved according to the rules of a formal system: by geometry, through physics or otherwise.

Although the meanings are distinguishable in some contexts, both meanings of "symmetry" are related and discussed in parallel.^{[1] [2]}

The "precise" notions of symmetry have various measures and operational definitions. For example, symmetry may be observed:

- with respect to the passage of time;
- as a spatial relationship;
- through geometric transformations such as scaling, reflection, and rotation;
- through other kinds of functional transformations^[3]; and



- as an aspect of abstract objects, theoretic models, language, music and even knowledge itself.^{[4] [5]}

This article describes these notions of symmetry from three perspectives. The first is that of mathematics, in which symmetries are defined and categorized precisely. The second perspective describes symmetry as it relates to science and technology. In this context, symmetries underlie some of the most profound results found in modern physics, including aspects of space and time. Finally, a third perspective discusses symmetry in the humanities, covering its rich and varied use in history, architecture, art, and religion.

The opposite of symmetry is asymmetry.

Symmetry in the field of mathematics

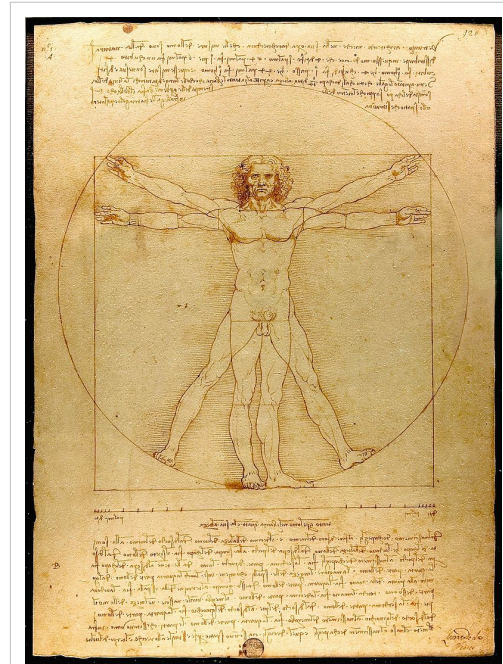
In formal terms, we say that an object is *symmetric* with respect to a given mathematical operation, if, when applied to the object, this operation does not change the object or its appearance. Two objects are symmetric to each other with respect to a given group of operations if one is obtained from the other by some of the operations (and vice versa).

Symmetries may also be found in living organisms including humans and other animals (see *symmetry in biology* below). In 2D geometry the main kinds of symmetry of interest are with respect to the basic Euclidean plane isometries: translations, rotations, reflections, and glide reflections.

Mathematical model for symmetry

The set of all symmetry operations considered, on all objects in a set X , can be modeled as a group action $g : G \times X \rightarrow X$, where the image of g in G and x in X is written as $g \cdot x$. If, for some g , $g \cdot x = x$ then x and x are said to be symmetrical to each other. For each object x , operations g for which $g \cdot x = x$ form a group, the **symmetry group** of the object, a subgroup of G . If the symmetry group of x is the trivial group then x is said to be **asymmetric**, otherwise **symmetric**. A general example is that G is a group of bijections $g : V \rightarrow V$ acting on the set of functions $x : V \rightarrow W$ by $(gx)(v) = x(g^{-1}(v))$ (or a restricted set of such functions that is closed under the group action). Thus a group of bijections of space induces a group action on "objects" in it. The symmetry group of x consists of all g for which $x(v) = x(g(v))$ for all v . G is the symmetry group of the space itself, and of any object that is uniform throughout space. Some subgroups of G may not be the symmetry group of any object. For example, if the group contains for every v and w in V a g such that $g(v) = w$, then only the symmetry groups of constant functions x contain that group. However, the symmetry group of constant functions is G itself.

In a modified version for vector fields, we have $(gx)(v) = h(g, x(g^{-1}(v)))$ where h rotates any vectors and pseudovectors in x , and inverts any vectors (but not pseudovectors) according



Leonardo da Vinci's *Vitruvian Man* (ca. 1487) is often used as a representation of symmetry in the human body and, by extension, the natural universe.

to rotation and inversion in g , see symmetry in physics. The symmetry group of x consists of all g for which $x(v)=h(g,x(g(v)))$ for all v . In this case the symmetry group of a constant function may be a proper subgroup of G : a constant vector has only rotational symmetry with respect to rotation about an axis if that axis is in the direction of the vector, and only inversion symmetry if it is zero.

For a common notion of symmetry in Euclidean space, G is the Euclidean group $E(n)$, the group of isometries, and V is the Euclidean space. The **rotation group** of an object is the symmetry group if G is restricted to $E^+(n)$, the group of direct isometries. (For generalizations, see the next subsection.) Objects can be modeled as functions x , of which a value may represent a selection of properties such as color, density, chemical composition, etc. Depending on the selection we consider just symmetries of sets of points (x is just a boolean function of position v), or, at the other extreme, e.g. symmetry of right and left hand with all their structure.

For a given symmetry group, the properties of part of the object, fully define the whole object. Considering points equivalent which, due to the symmetry, have the same properties, the equivalence classes are the orbits of the group action on the space itself. We need the value of x at one point in every orbit to define the full object. A set of such representatives forms a fundamental domain. The smallest fundamental domain does not have a symmetry; in this sense, one can say that symmetry relies upon asymmetry.

An object with a desired symmetry can be produced by choosing for every orbit a single function value. Starting from a given object x we can e.g.:

- take the values in a fundamental domain (i.e., add copies of the object)
- take for each orbit some kind of average or sum of the values of x at the points of the orbit (ditto, where the copies may overlap)

If it is desired to have no more symmetry than that in the symmetry group, then the object to be copied should be asymmetric.

As pointed out above, some groups of isometries are not the symmetry group of any object, except in the modified model for vector fields. For example, this applies in 1D for the group of all translations. The fundamental domain is only one point, so we can not make it asymmetric, so any "pattern" invariant under translation is also invariant under reflection (these are the uniform "patterns").

In the vector field version continuous translational symmetry does not imply reflectional symmetry: the function value is constant, but if it contains nonzero vectors, there is no reflectional symmetry. If there is also reflectional symmetry, the constant function value contains no nonzero vectors, but it may contain nonzero pseudovectors. A corresponding 3D example is an infinite cylinder with a current perpendicular to the axis; the magnetic field (a pseudovector) is, in the direction of the cylinder, constant, but nonzero. For vectors (in particular the current density) we have symmetry in every plane perpendicular to the cylinder, as well as cylindrical symmetry. This cylindrical symmetry without mirror planes through the axis is also only possible in the vector field version of the symmetry concept. A similar example is a cylinder rotating about its axis, where magnetic field and current density are replaced by angular momentum and velocity, respectively.

A symmetry group is said to act transitively on a repeated feature of an object if, for every pair of occurrences of the feature there is a symmetry operation mapping the first to the second. For example, in 1D, the symmetry group of $\{...,1,2,5,6,9,10,13,14,...\}$ acts

transitively on all these points, while $\{...,1,2,3,5,6,7,9,10,11,13,14,15,...\}$ does *not* act transitively on all points. Equivalently, the first set is only one conjugacy class with respect to isometries, while the second has two classes.

Non-isometric symmetry

As mentioned above, G (the symmetry group of the space itself) may differ from the Euclidean group, the group of isometries.

Examples:

- G is the group of similarity transformations, i.e. affine transformations with a matrix A that is a scalar times an orthogonal matrix. Thus dilations are added, self-similarity is considered a symmetry
- G is the group of affine transformations with a matrix A with determinant 1 or -1, i.e. the transformation which preserve area; this adds e.g. oblique reflection symmetry.
- G is the group of all bijective affine transformations
- In inversive geometry, G includes circle reflections, etc.

Directional symmetry

Reflection symmetry

Reflection symmetry, mirror symmetry, mirror-image symmetry, or bilateral symmetry is symmetry with respect to reflection.

In 1D, there is a point of symmetry. In 2D there is an axis of symmetry, in 3D a plane of symmetry. An object or figure which is indistinguishable from its transformed image is called mirror symmetric (see mirror image).

The axis of symmetry of a two-dimensional figure is a line such that, if a perpendicular is constructed, any two points lying on the perpendicular at equal distances from the axis of symmetry are identical. Another way to think about it is that if the shape were to be folded in half over the axis, the two halves would be identical: the two halves are each other's mirror image. Thus a square has four axes of symmetry, because there are four different ways to fold it and have the edges all match. A circle has infinitely many axes of symmetry, for the same reason.

If the letter T is reflected along a vertical axis, it appears the same. Note that this is sometimes called horizontal symmetry, and sometimes vertical symmetry! One can better use an unambiguous formulation, e.g. "T has a vertical symmetry axis" or "T has left-right symmetry."

The triangles with this symmetry are isosceles, the quadrilaterals with this symmetry are the kites and the isosceles trapezoids.

For each line or plane of reflection, the symmetry group is isomorphic with C_s (see point groups in three dimensions), one of the three types of order two (involutions), hence algebraically C_2 . The fundamental domain is a half-plane or half-space.

Bilateria (bilateral animals, including humans) are more or less symmetric with respect to the sagittal plane.

In certain contexts there is rotational symmetry anyway. Then mirror-image symmetry is equivalent with inversion symmetry; in such contexts in modern physics the term

P-symmetry is used for both (P stands for parity).

For more general types of reflection there are corresponding more general types of reflection symmetry. Examples:

- with respect to a non-isometric affine involution (an oblique reflection in a line, plane, etc).
- with respect to circle inversion

Rotational symmetry

Rotational symmetry is symmetry with respect to some or all rotations in m -dimensional Euclidean space. Rotations are direct isometries, i.e., isometries preserving orientation. Therefore a symmetry group of rotational symmetry is a subgroup of $E^+(m)$.

Symmetry with respect to all rotations about all points implies translational symmetry with respect to all translations, and the symmetry group is the whole $E^+(m)$. This does not apply for objects because it makes space homogeneous, but it may apply for physical laws.

For symmetry with respect to rotations about a point we can take that point as origin. These rotations form the special orthogonal group $SO(m)$, the group of $m \times m$ orthogonal matrices with determinant 1. For $m=3$ this is the rotation group.

In another meaning of the word, the rotation group of an object is the symmetry group within $E^+(n)$, the group of direct isometries; in other words, the intersection of the full symmetry group and the group of direct isometries. For chiral objects it is the same as the full symmetry group.

Laws of physics are $SO(3)$ -invariant if they do not distinguish different directions in space. Because of Noether's theorem, rotational symmetry of a physical system is equivalent to the angular momentum conservation law. See also rotational invariance.

Translational symmetry

Translational symmetry leaves an object invariant under a discrete or continuous group of translations $T_a(p) = p + a$.

Glide reflection symmetry

A glide reflection symmetry (in 3D: a glide plane symmetry) means that a reflection in a line or plane combined with a translation along the line / in the plane, results in the same object. It implies translational symmetry with twice the translation vector.

The symmetry group is isomorphic with $\mathbf{Z}$.

Rotoreflexion symmetry

In 3D, roto-reflection or improper rotation in the strict sense is rotation about an axis, combined with reflection in a plane perpendicular to that axis. As symmetry groups with regard to a roto-reflection we can distinguish:

- the angle has no common divisor with 360° , the symmetry group is not discrete
- $2n$ -fold roto-reflection (angle of $180^\circ/n$) with symmetry group S_{2n} of order $2n$ (not to be confused with symmetric groups, for which the same notation is used; abstract group C_{2n}); a special case is $n=1$, inversion, because it does not depend on the axis and the plane, it is characterized by just the point of inversion.
- C_{nh} (angle of $360^\circ/n$); for odd n this is generated by a single symmetry, and the abstract group is C_{2n} , for even n this is not a basic symmetry but a combination. See also point groups in three dimensions.

Helical symmetry

Helical symmetry is the kind of symmetry seen in such everyday objects as springs, Slinky toys, drill bits, and augers. It can be thought of as rotational symmetry along with



A drill bit with helical symmetry.

translation along the axis of rotation, the screw axis). The concept of helical symmetry can be visualized as the tracing in three-dimensional space that results from rotating an object at an even angular speed while simultaneously moving at another even speed along its axis of rotation (translation). At any one point in time, these two motions combine to give a *coiling angle* that helps define the properties of the tracing. When the tracing object rotates quickly and translates slowly, the coiling angle will be close to 0° . Conversely, if the rotation is slow and the translation speedy, the coiling angle will approach 90° .

Three main classes of helical symmetry can be distinguished based on the interplay of the angle of coiling and translation symmetries along the axis:

- **Infinite helical symmetry.** If there are no distinguishing features along the length of a helix or helix-like object, the object will have infinite symmetry much like that of a circle, but with the additional requirement of translation along the long axis of the object to return it to its original appearance. A helix-like object is one that has at every point the regular angle of coiling of a helix, but which can also have a cross section of indefinitely high complexity, provided only that precisely the same cross section exists (usually after a rotation) at every point along the length of the object. Simple examples include evenly coiled springs, slinkies, drill bits, and augers. Stated more precisely, an object has infinite helical symmetries if for any small rotation of the object around its central axis there exists a point nearby (the translation distance) on that axis at which the object will appear exactly as it did before. It is this infinite helical symmetry that gives rise to the curious illusion of movement along the length of an auger or screw bit that is being rotated. It also provides the mechanically useful ability of such devices to move materials along their length, provided that they are combined with a force such as gravity or friction that allows the materials to resist simply rotating along with the drill or auger.

- **n-fold helical symmetry.** If the requirement that every cross section of the helical object be identical is relaxed, additional lesser helical symmetries become possible. For example, the cross section of the helical object may change, but still repeats itself in a regular fashion along the axis of the helical object. Consequently, objects of this type will exhibit a symmetry after a rotation by some fixed angle θ and a translation by some fixed distance, but will not in general be invariant for any rotation angle. If the angle (rotation) at which the symmetry occurs divides evenly into a full circle (360°), the result is the helical equivalent of a regular polygon. This case is called *n-fold helical symmetry*, where $n = 360^\circ / \theta$, see e.g. double helix. This concept can be further generalized to include cases where $m\theta$ is a multiple of 360° —that is, the cycle does eventually repeat, but only after more than one full rotation of the helical object.
- **Non-repeating helical symmetry.** This is the case in which the angle of rotation θ required to observe the symmetry is an irrational number such as $\sqrt{2}$ radians that never repeats exactly no matter how many times the helix is rotated. Such symmetries are created by using a non-repeating point group in two dimensions. DNA is an example of this type of non-repeating helical symmetry.

Scale symmetry and fractals

Scale symmetry refers to the idea that if an object is expanded or reduced in size, the new object has the same properties as the original. Scale symmetry is notable for the fact that it does *not* exist for most physical systems, a point that was first discerned by Galileo. Simple examples of the lack of scale symmetry in the physical world include the difference in the strength and size of the legs of elephants versus mice, and the observation that if a candle made of soft wax was enlarged to the size of a tall tree, it would immediately collapse under its own weight.

A more subtle form of scale symmetry is demonstrated by fractals. As conceived by Mandelbrot, fractals are a mathematical concept in which the structure of a complex form looks exactly the same no matter what degree of magnification is used to examine it. A coast is an example of a naturally occurring fractal, since it retains roughly comparable and similar-appearing complexity at every level from the view of a satellite to a microscopic examination of how the water laps up against individual grains of sand. The branching of trees, which enables children to use small twigs as stand-ins for full trees in dioramas, is another example.

This similarity to naturally occurring phenomena provides fractals with an everyday familiarity not typically seen with mathematically generated functions. As a consequence, they can produce strikingly beautiful results such as the Mandelbrot set. Intriguingly, fractals have also found a place in CG, or computer-generated movie effects, where their ability to create very complex curves with fractal symmetries results in more realistic virtual worlds.

Symmetry combinations

Symmetry in science

Symmetry in physics

Symmetry in physics has been generalized to mean invariance—that is, lack of any visible change—under any kind of transformation, for example arbitrary coordinate transformations. This concept has become one of the most powerful tools of theoretical physics, as it has become evident that practically all laws of nature originate in symmetries. In fact, this role inspired the Nobel laureate PW Anderson to write in his widely read 1972 article *More is Different* that "it is only slightly overstating the case to say that physics is the study of symmetry." See Noether's theorem (which, in greatly simplified form, states that for every continuous mathematical symmetry, there is a corresponding conserved quantity; a conserved current, in Noether's original language); and also, Wigner's classification, which says that the symmetries of the laws of physics determine the properties of the particles found in nature.

Symmetry in physical objects

Classical objects

Although an everyday object may appear exactly the same after a symmetry operation such as a rotation or an exchange of two identical parts has been performed on it, it is readily apparent that such a symmetry is true only as an approximation for any ordinary physical object.

For example, if one rotates a precisely machined aluminum equilateral triangle 120 degrees around its center, a casual observer brought in before and after the rotation will be unable to decide whether or not such a rotation took place. However, the reality is that each corner of a triangle will always appear unique when examined with sufficient precision. An observer armed with sufficiently detailed measuring equipment such as optical or electron microscopes will not be fooled; he will immediately recognize that the object has been rotated by looking for details such as crystals or minor deformities.

Such simple thought experiments show that assertions of symmetry in everyday physical objects are always a matter of approximate similarity rather than of precise mathematical sameness. The most important consequence of this approximate nature of symmetries in everyday physical objects is that such symmetries have minimal or no impacts on the physics of such objects. Consequently, only the deeper symmetries of space and time play a major role in classical physics—that is, the physics of large, everyday objects.

Quantum objects

Remarkably, there exists a realm of physics for which mathematical assertions of simple symmetries in real objects cease to be approximations. That is the domain of quantum physics, which for the most part is the physics of very small, very simple objects such as electrons, protons, light, and atoms.

Unlike everyday objects, objects such as electrons have very limited numbers of configurations, called states, in which they can exist. This means that when symmetry operations such as exchanging the positions of components are applied to them, the resulting new configurations often cannot be distinguished from the originals no matter how diligent an observer is. Consequently, for sufficiently small and simple objects the generic mathematical symmetry assertion $F(x) = x$ ceases to be approximate, and instead becomes an experimentally precise and accurate description of the situation in the real world.

Consequences of quantum symmetry

While it makes sense that symmetries could become exact when applied to very simple objects, the immediate intuition is that such a detail should not affect the physics of such objects in any significant way. This is in part because it is very difficult to view the concept of exact similarity as physically meaningful. Our mental picture of such situations is invariably the same one we use for large objects: We picture objects or configurations that are very, very similar, but for which if we could "look closer" we would still be able to tell the difference.

However, the assumption that exact symmetries in very small objects should not make any difference in their physics was discovered in the early 1900s to be spectacularly incorrect. The situation was succinctly summarized by Richard Feynman in the direct transcripts of his Feynman Lectures on Physics, Volume III, Section 3.4, *Identical particles*. (Unfortunately, the quote was edited out of the printed version of the same lecture.)

"... if there is a physical situation in which it is impossible to tell which way it happened, it *always* interferes; it *never* fails."

The word "interferes" in this context is a quick way of saying that such objects fall under the rules of quantum mechanics, in which they behave more like waves that interfere than like everyday large objects.

In short, when an object becomes so simple that a symmetry assertion of the form $F(x) = x$ becomes an exact statement of experimentally verifiable sameness, x ceases to follow the rules of classical physics and must instead be modeled using the more complex—and often far less intuitive—rules of quantum physics.

This transition also provides an important insight into why the mathematics of symmetry are so deeply intertwined with those of quantum mechanics. When physical systems make the transition from symmetries that are approximate to ones that are exact, the mathematical expressions of those symmetries cease to be approximations and are transformed into precise definitions of the underlying nature of the objects. From that point on, the correlation of such objects to their mathematical descriptions becomes so close that it is difficult to separate the two.

Symmetry as a unifying principle of geometry

The German geometer Felix Klein enunciated a very influential Erlangen programme in 1872, suggesting symmetry as unifying and organising principle in geometry (at a time when that was read 'geometries'). This is a broad rather than deep principle. Initially it led to interest in the groups attached to geometries, and the slogan transformation geometry (an aspect of the New Math, but hardly controversial in modern mathematical practice). By now it has been applied in numerous forms, as kind of standard attack on problems.

Symmetry in mathematics

An example of a mathematical expression exhibiting symmetry is $a^2c + 3ab + b^2c$. If a and b are exchanged, the expression remains unchanged due to the commutativity of addition and multiplication.

Like in geometry, for the terms there are two possibilities:

- It is itself symmetric
- It has one or more other terms symmetric with it, in accordance with the symmetry kind

See also symmetric function, duality (mathematics)

Symmetry in logic

A dyadic relation R is symmetric if and only if, whenever it's true that Rab , it's true that Rba . Thus, "is the same age as" is symmetrical, for if Paul is the same age as Mary, then Mary is the same age as Paul.

Symmetric binary logical connectives are "and" ($\wedge$, $\bigwedge$, or $\&$), "or" ($\vee$), "biconditional" (if and only if) ($\iff$), NAND ("not-and"), XOR ("not-biconditional"), and NOR ("not-or").

Generalizations of symmetry

If we have a given set of objects with some structure, then it is possible for a symmetry to merely convert only one object into another, instead of acting upon all possible objects simultaneously. This requires a generalization from the concept of symmetry group to that of a groupoid. Indeed, A. Connes in his book 'Non-commutative geometry' writes that Heisenberg discovered quantum mechanics by considering the groupoid of transitions of the hydrogen spectrum.

The notion of groupoid also leads to notions of multiple groupoids, namely sets with many compatible groupoid structures, a structure which trivialises to abelian groups if one restricts to groups. This leads to prospects of 'higher order symmetry' which have been a little explored, as follows.

The automorphisms of a set, or a set with some structure, form a group, which models a homotopy 1-type. The automorphisms of a group G naturally form a crossed module $G \rightarrow \text{Aut}(G)$, and crossed modules give an algebraic model of homotopy 2-types. At the next stage, automorphisms of a crossed module fit into a structure known as a crossed square, and this structure is known to give an algebraic model of homotopy 3-types. It is not known how this procedure of generalising symmetry may be continued, although crossed n -cubes have been defined and used in algebraic topology, and these structures are only slowly being brought into theoretical physics. The web site *n-category cafe* ^[6] has much discussion of n -groups. More information is on 'Higher dimensional group theory' ^[7].

Physicists have come up with other directions of generalization, such as supersymmetry and quantum groups, yet the different options are indistinguishable during various circumstances.

Symmetry in biology

See symmetry (biology) and facial symmetry.

Symmetry in chemistry

Symmetry is important to chemistry because it explains observations in spectroscopy, quantum chemistry and crystallography. It draws heavily on group theory.

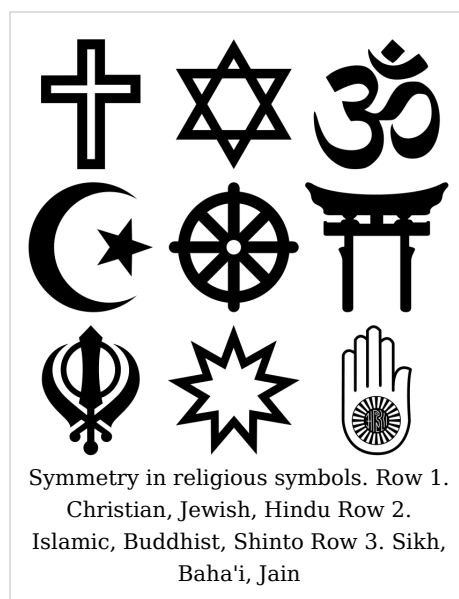
Symmetry in history, religion, and culture

In any human endeavor for which an impressive visual result is part of the desired objective, symmetries play a profound role. The innate appeal of symmetry can be found in our reactions to happening across highly symmetrical natural objects, such as precisely formed crystals or beautifully spiraled seashells. Our first reaction in finding such an object often is to wonder whether we have found an object created by a fellow human, followed quickly by surprise that the symmetries that caught our attention are derived from nature itself. In both reactions we give away our inclination to view symmetries both as beautiful and, in some fashion, informative of the world around us.

Symmetry in religious symbols

The tendency of people to see purpose in symmetry suggests at least one reason why symmetries are often an integral part of the symbols of world religions. Just a few of many examples include the sixfold rotational symmetry of Judaism's Star of David, the twofold point symmetry of Taoism's Taijitu or Yin-Yang, the bilateral symmetry of Christianity's cross and Sikhism's Khanda, or the fourfold point symmetry of Jain's ancient (and peacefully intended) version of the swastika. With its strong prohibitions against the use of representational images, Islam, and in particular the Sunni branch of Islam, has developed intricate and visually impressive use of symmetries.

The ancient Taijitu image of Taoism is a particularly fascinating use of symmetry around a central point, combined with black-and-white inversion of color at opposite distances from that central point. The image, which is often misunderstood in the Western world as representing good (white) versus evil (black), is actually intended as a graphical representative of the complementary need for two abstract concepts of "maleness" (white) and "femaleness" (black). The symmetry of the symbol in this case is used not just to create a symbol that catches the attention of the eye, but to make a significant statement about the philosophical beliefs of the people and groups that use it.



Symmetry in Social Interactions

People observe the symmetrical nature, often including asymmetrical balance, of social interactions in a variety of contexts. These include assessments of reciprocity, empathy, apology, dialog, respect, justice, and revenge. Symmetrical interactions send the message "we are all the same" while asymmetrical interactions send the message "I am special; better than you." Peer relationships are based on symmetry, power relationships are based on asymmetry. ^[8]

Symmetry in architecture

Another human endeavor in which the visual result plays a vital part in the overall result is architecture. Both in ancient times, the ability of a large structure to impress or even intimidate its viewers has often been a major part of its purpose, and the use of symmetry is an inescapable aspect of how to accomplish such goals.

Just a few examples of ancient examples of architectures that made powerful use of symmetry to impress those around them included the Egyptian Pyramids, the Greek Parthenon, the first and second Temple of Jerusalem, China's Forbidden City, Cambodia's Angkor Wat complex, and the many temples and pyramids of ancient Pre-Columbian civilizations. More recent historical examples of architectures emphasizing symmetries include Gothic architecture cathedrals, and American President Thomas Jefferson's Monticello home. India's unparalleled Taj Mahal is in a category by itself, as it may arguably be one of the most impressive and beautiful uses of symmetry in architecture that the world has ever seen.

An interesting example of a broken symmetry in architecture is the Leaning Tower of Pisa, whose notoriety stems in no small part not for the intended symmetry of its design, but for the violation of that symmetry from the lean that developed while it was still under construction. Modern examples of architectures that make impressive or complex use of various symmetries include Australia's Sydney Opera House and Houston, Texas's simpler Astrodome.

Symmetry finds its ways into architecture at every scale, from the overall external views, through the layout of the individual floor plans, and down to the design of individual building elements such as intricately caved doors, stained glass windows, tile mosaics, friezes, stairwells, stair rails, and balustradess. For sheer complexity and sophistication in the exploitation of symmetry as an architectural element, Islamic buildings such as the Taj Mahal often eclipse those of other cultures and ages, due in part to the general prohibition of Islam against using images of people or animals.

Links related to symmetry in architecture include:

- Williams: Symmetry in Architecture ^[9]
- Aslaksen: Mathematics in Art and Architecture ^[10]



Leaning Tower of Pisa

Symmetry in pottery and metal vessels

Since the earliest uses of pottery wheels to help shape clay vessels, pottery has had a strong relationship to symmetry. As a minimum, pottery created using a wheel necessarily begins with full rotational symmetry in its cross-section, while allowing substantial freedom of shape in the vertical direction. Upon this inherently symmetrical starting point cultures from ancient times have tended to add further patterns that tend to exploit or in many cases reduce the original full rotational symmetry to a point where some specific visual objective is achieved. For example, Persian pottery dating from the fourth millennium B.C. and earlier used symmetric zigzags, squares, cross-hatchings, and repetitions of figures to produce more complex and visually striking overall designs.



Persian vessel (4th millennium B.C.)

Cast metal vessels lacked the inherent rotational symmetry of wheel-made pottery, but otherwise provided a similar opportunity to decorate their surfaces with patterns pleasing to those who used them. The ancient Chinese, for example, used symmetrical patterns in their bronze castings as early as the 17th century B.C. Bronze vessels exhibited both a bilateral main motif and a repetitive translated border design.

Links:

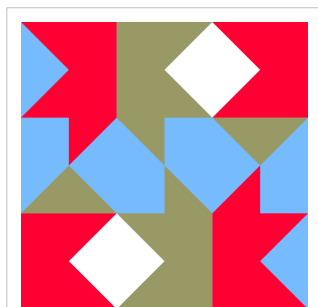
- [Chinavoc: The Art of Chinese Bronzes](#) ^[11]
- [Grant: Iranian Pottery in the Oriental Institute](#) ^[12]
- [The Metropolitan Museum of Art - Islamic Art](#) ^[13]

Symmetry in quilts

As quilts are made from square blocks (usually 9, 16, or 25 pieces to a block) with each smaller piece usually consisting of fabric triangles, the craft lends itself readily to the application of symmetry.

Links:

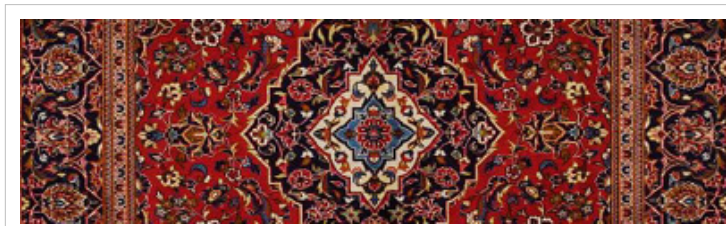
- [Quate: Exploring Geometry Through Quilts](#) ^[14]



Kitchen Kaleidoscope
Block

Symmetry in carpets and rugs

A long tradition of the use of symmetry in carpet and rug patterns spans a variety of cultures. American Navajo Indians used bold diagonals and rectangular motifs. Many Oriental rugs have intricate reflected centers and borders that translate a pattern.



Persian rug.

Not surprisingly, rectangular rugs typically use quadrilateral symmetry—that is, motifs that are reflected across both the horizontal and vertical axes.

Links:

- Mallet: Tribal Oriental Rugs ^[15]
- Dilucchio: Navajo Rugs ^[16]

Symmetry in music

Symmetry is of course not restricted to the visual arts. Its role in the history of music touches many aspects of the creation and perception of music.

Musical form

Symmetry has been used as a formal constraint by many composers, such as the arch (swell) form (ABCBA) used by Steve Reich, Béla Bartók, and James Tenney. In classical music, Bach used the symmetry concepts of permutation and invariance; see (external link "Fugue No. 21," pdf ^[17] or Shockwave ^[18]).

Pitch structures

Symmetry is also an important consideration in the formation of scales and chords, traditional or tonal music being made up of non-symmetrical groups of pitches, such as the diatonic scale or the major chord. Symmetrical scales or chords, such as the whole tone scale, augmented chord, or diminished seventh chord (diminished-diminished seventh), are said to lack direction or a sense of forward motion, are ambiguous as to the key or tonal center, and have a less specific diatonic functionality. However, composers such as Alban Berg, Béla Bartók, and George Perle have used axes of symmetry and/or interval cycles in an analogous way to keys or non-tonal tonal centers.

Perle (1992) explains "C-E, D-F#, [and] Eb-G, are different instances of the same interval...the other kind of identity. ..has to do with axes of symmetry. C-E belongs to a family of symmetrically related dyads as follows:"

D	D#	E	F	F#	G	G#
D	C#	C	B	A#	A	G#

Thus in addition to being part of the interval-4 family, C-E is also a part of the sum-4 family (with C equal to 0).

+	2	3	4	5	6	7	8
	2	1	0	11	10	9	8
	4	4	4	4	4	4	4

Interval cycles are symmetrical and thus non-diatonic. However, a seven pitch segment of C5 (the cycle of fifths, which are enharmonic with the cycle of fourths) will produce the diatonic major scale. Cyclic tonal progressions in the works of Romantic composers such as Gustav Mahler and Richard Wagner form a link with the cyclic pitch successions in the atonal music of Modernists such as Bartók, Alexander Scriabin, Edgard Varèse, and the Vienna school. At the same time, these progressions signal the end of tonality.

The first extended composition consistently based on symmetrical pitch relations was probably Alban Berg's *Quartet*, Op. 3 (1910). (Perle, 1990)

Equivalency

Tone rows or pitch class sets which are invariant under retrograde are horizontally symmetrical, under inversion vertically. See also Asymmetric rhythm.

Symmetry in other arts and crafts

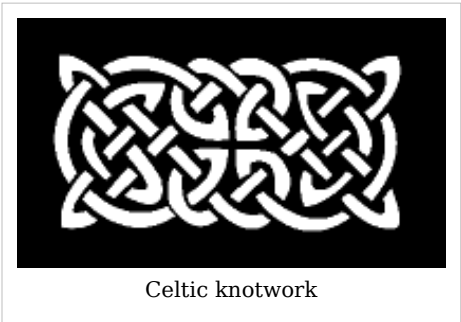
The concept of symmetry is applied to the design of objects of all shapes and sizes. Other examples include beadwork, furniture, sand paintings, knotwork, masks, musical instruments, and many other endeavors.

Symmetry in aesthetics

The relationship of symmetry to aesthetics is complex. Certain simple symmetries, and in particular bilateral symmetry, seem to be deeply ingrained in the inherent perception by humans of the likely health or fitness of other living creatures, as can be seen by the simple experiment of distorting one side of the image of an attractive face and asking viewers to rate the attractiveness of the resulting image. Consequently, such symmetries that mimic biology tend to have an innate appeal that in turn drives a powerful tendency to create artifacts with similar symmetry. One only needs to imagine the difficulty in trying to market a highly asymmetrical car or truck to general automotive buyers to understand the power of biologically inspired symmetries such as bilateral symmetry.

Another more subtle appeal of symmetry is that of simplicity, which in turn has an implication of safety, security, and familiarity. A highly symmetrical room, for example, is unavoidably also a room in which anything out of place or potentially threatening can be identified easily and quickly. For example, people who have grown up in houses full of exact right angles and precisely identical artifacts can find their first experience in staying in a room with *no* exact right angles and *no* exactly identical artifacts to be highly disquieting. Symmetry thus can be a source of comfort not only as an indicator of biological health, but also of a safe and well-understood living environment.

Opposed to this is the tendency for excessive symmetry to be perceived as boring or uninteresting. Humans in particular have a powerful desire to exploit new opportunities or explore new possibilities, and an excessive degree of symmetry can convey a lack of such



opportunities.

Yet another possibility is that when symmetries become too complex or too challenging, the human mind has a tendency to "tune them out" and perceive them in yet another fashion: as noise that conveys no useful information.

Finally, perceptions and appreciation of symmetries are also dependent on cultural background. The far greater use of complex geometric symmetries in many Islamic cultures, for example, makes it more likely that people from such cultures will appreciate such art forms (or, conversely, to rebel against them).

As in many human endeavors, the result of the confluence of many such factors is that effective use of symmetry in art and architecture is complex, intuitive, and highly dependent on the skills of the individuals who must weave and combine such factors within their own creative work. Along with texture, color, proportion, and other factors, symmetry is a powerful ingredient in any such synthesis; one only need to examine the Taj Mahal to powerful role that symmetry plays in determining the aesthetic appeal of an object.

Modernist architecture rejects symmetry, stating *only a bad architect relies on symmetry*; instead of symmetrical layout of blocks, masses and structures, Modernist architecture relies on wings and balance of masses. This notion of getting rid of symmetry was first encountered in International style. Some people find asymmetrical layouts of buildings and structures revolutionizing; other find them restless, boring and unnatural.

A few examples of the more explicit use of symmetries in art can be found in the remarkable art of M. C. Escher, the creative design of the mathematical concept of a wallpaper group, and the many applications (both mathematical and real world) of tiling.

Symmetry in games and puzzles

- See also symmetric games.
- See sudoku.

Symmetry in literature

See palindrome.

Moral symmetry

- Tit for tat
 - Reciprocity
 - Golden Rule
 - Empathy & Sympathy
 - Reflective equilibrium
-

See also

- Symmetry group
- Chirality
- Fixed points of isometry groups in Euclidean space - center of symmetry
- Spontaneous symmetry breaking
- Gödel, Escher, Bach
- M. C. Escher
- Wallpaper group
- Asymmetry
- Asymmetric rhythm
- Even and odd functions
- Symmetries of polyominoes
- Symmetries of polyiamonds
- Burnside's lemma
- Symmetry (biology)
- Spacetime symmetries
- Time symmetry
- Semimetric, which is sometimes translated as symmetric in Russian texts.
- Symmetric relation
- Phase transition

Notes

- [1] For example, Aristotle ascribed spherical shape to the heavenly bodies, attributing this formally defined geometric measure of symmetry to the natural order and perfection of the cosmos.
- [2] Weyl 1982
- [3] For example, operations such as moving across a regularly patterned tile floor or rotating an eight-sided vase, or complex transformations of equations or in the way music is played.
- [4] See e.g., Mainzer, Klaus (2005). *Symmetry And Complexity: The Spirit and Beauty of Nonlinear Science*. World Scientific. ISBN 9812561927.
- [5] Symmetric objects can be material, such as a person, crystal, quilt, floor tiles, or molecule, or it can be an abstract structure such as a mathematical equation or a series of tones (music).
- [6] <http://golem.ph.utexas.edu/category/>
- [7] <http://www.bangor.ac.uk/r.brown/hdaweb2.htm>
- [8] Emotional Competency (<http://www.emotionalcompetency.com/symmetry.htm>) Entry describing Symmetry
- [9] <http://members.tripod.com/vismath/kim/>
- [10] <http://www.math.nus.edu.sg/aslaksen/teaching/math-art-arch.shtml>
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- [15] <http://www.marlamallett.com/default.htm>
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External links

- An Analysis of the first movement of the Fourth String Quartet (1928) (<http://home.earthlink.net/~akuster/music/bartok/quartet4.htm>) by Andrew Kuster
- Skaalid: Design Theory (<http://www.usask.ca/education/coursework/skaalid/theory/theory.htm>)
- Mathforum: Symmetry/Tessellations (http://mathforum.org/library/topics/sym_tess/)
- Calotta: A World of Symmetry (<http://www.teachersnetwork.org/teachnet/westchester/symmetry.htm>)
- Dutch: Symmetry Around a Point in the Plane (<http://www.uwgb.edu/dutchs/SYMMETRY/2DPTGRP.HTM>)
- Saw: Design Notes (<http://daphne.palomar.edu/design/conclude.html>)
- Chapman: Aesthetics of Symmetry (<http://home.earthlink.net/~jdc24/symmetry.htm>)
- Abas: The Wonder Of Symmetry (<http://www.bangor.ac.uk/~mas009/psym.htm>)
- ISIS Symmetry (<http://www.mi.sanu.ac.rs/~jablans/isis0.htm>)
- Symmetry and Asymmetry at The Dictionary of the History of Ideas (<http://etext.lib.virginia.edu/cgi-local/DHI/dhi.cgi?id=dv4-46>)
- Examples of asymmetry in musical waveforms (<http://www.st-and.demon.co.uk/AudioMisc/asymmetry/asym.html>)
- *International Symmetry Association - ISA* (http://www.symmetry.hu/isa_articles.html)
- Institute *Symmetrion* (http://www.symmetry.hu/aus_symmetrion.html)
- Professor Ian Stewart on the history of symmetry (<http://www2.warwick.ac.uk/newsandevents/audio/more/symmetry/>)

- Photographs of Symmetrical Cloisters (<http://www.hayquesufrir.com>)
- Symmetry and Symmetry Measures: (<http://petitjeanmichel.free.fr/itoweb.petitjean.symmetry.html>) General Definitions

Time symmetry

1. redirect T-symmetry

Spacetime symmetries

Spacetime symmetries refers to aspects of spacetime that can be described as exhibiting some form of symmetry. The role of symmetry in physics is important, for example, in simplifying solutions to many problems. Spacetime symmetries are used to simplify problems and find ample application in the study of exact solutions of Einstein's field equations of general relativity.

Physical motivation

Quite often, physical problems may be investigated and solved by noticing features of the problem which have some form of symmetry. For example, in the Schwarzschild solution, the role of spherical symmetry is important in deriving the Schwarzschild solution and deducing the physical consequences of this symmetry (for example, the non-existence of gravitational radiation in a spherically pulsating star). In cosmological problems, symmetry finds a role to play in the cosmological principle which restricts the type of universes that are consistent with large-scale observations (e.g. the Friedmann-Robertson-Walker (FRW) metric). Symmetries usually require some form of preserving property, the most important of which in general relativity include the following:

- preserving geodesics of the spacetime
- preserving the metric tensor
- preserving the curvature tensor

These and other symmetries will be discussed in more detail later. This preservation feature can be used to motivate a useful definition of symmetries.

Mathematical definition

A rigorous definition of symmetries in general relativity has been given by Hall (2004). In this approach, the idea is to use (smooth) vector fields whose local flow diffeomorphisms preserve some property of the spacetime. This preserving property of the diffeomorphisms is made precise as follows. A smooth vector field X on a spacetime M is said to **preserve** a smooth tensor T on M (or T is **invariant** under X) if, for each smooth local flow diffeomorphism ϕ_t associated with X , the tensors T and $\phi_t^*(T)$ are equal on the domain of ϕ_t . This statement is equivalent to the more usable condition that the Lie derivative of the tensor under the vector field vanishes:

$$\mathcal{L}_X T = 0$$

on M . This has the consequence that, given any two points p and q on M , the coordinates of T in a coordinate system around p are equal to the coordinates of T in a coordinate system around q . A **symmetry on the spacetime** is a smooth vector field whose local flow diffeomorphisms preserve some (usually geometrical) feature of the spacetime. The (geometrical) feature may refer to specific tensors (such as the metric, or the energy-momentum tensor) or to other aspects of the spacetime such as its geodesic structure. The vector fields are sometimes referred to as **collineations**, **symmetry vector fields** or just **symmetries**. The set of all symmetry vector fields on M forms a Lie algebra under the Lie bracket operation as can be seen from the identity:

$$\mathcal{L}_{[X,Y]}T = \mathcal{L}_X(\mathcal{L}_Y T) - \mathcal{L}_Y(\mathcal{L}_X T)$$

the term on the right usually being written, with an abuse of notation, as $[\mathcal{L}_X, \mathcal{L}_Y]T$. Various examples of symmetries are briefly described below.

Killing symmetry

Killing vector fields are one of the most important types of symmetries and are defined to be those smooth vector fields that preserve the metric tensor:

$$\mathcal{L}_X g_{ab} = 0$$

This is usually written in the expanded form as:

$$X_{a;b} + X_{b;a} = 0$$

Killing vector fields find extensive applications (including in classical mechanics) and are related to conservation laws.

Homothetic symmetry

A homothetic vector field is one which satisfies:

$$\mathcal{L}_X g_{ab} = 2c g_{ab}$$

where c is a real constant. Homothetic vector fields find application in the study of singularities in general relativity.

Affine symmetry

An affine vector field is one that satisfies:

$$(\mathcal{L}_X g_{ab})_{;c} = 0$$

An affine vector field preserves geodesics and preserves the affine parameter.

The above three vector field types are special cases of projective vector fields which preserve geodesics without necessarily preserving the affine parameter.

Conformal symmetry

A **conformal vector field** is one which satisfies:

$$\mathcal{L}_X g_{ab} = \phi g_{ab}$$

where ϕ is a smooth real-valued function on M .

Curvature symmetry

A **curvature collineation** is a vector field which preserves the Riemann tensor:

$$\mathcal{L}_X R^a{}_{bcd} = 0$$

where $R^a{}_{bcd}$ are the components of the Riemann tensor. The set of all smooth curvature collineations forms a Lie algebra under the Lie bracket operation (if the smoothness condition is dropped, the set of all curvature collineations need not form a Lie algebra). The Lie algebra is denoted by $CC(M)$ and may be infinite-dimensional. Every affine vector field is a curvature collineation.

Matter symmetry

A less well-known form of symmetry concerns vector fields that preserve the energy-momentum tensor. These are variously referred to as **matter collineations** or **matter symmetries** and are defined by:

$$\mathcal{L}_X T_{ab} = 0$$

where T_{ab} are the energy-momentum tensor components. The intimate relation between geometry and physics may be highlighted here, as the vector field X is regarded as preserving certain physical quantities along the flow lines of X , this being true for any two observers. In connection with this, it may be shown that **every Killing vector field is a matter collineation** (by the Einstein field equations, with or without cosmological constant). Thus, given a solution of the EFE, *a vector field that preserves the metric necessarily preserves the corresponding energy-momentum tensor*. When the energy-momentum tensor represents a perfect fluid, every Killing vector field preserves the energy density, pressure and the fluid flow vector field. When the energy-momentum tensor represents an electromagnetic field, a Killing vector field does *not necessarily* preserve the electric and magnetic fields.

Applications of symmetry vector fields

As mentioned at the start of this article, the main application of these symmetries occur in general relativity, where solutions of Einstein's equations may be classified by imposing some certain symmetries on the spacetime.

Classifications of spacetimes

Classifying solutions of the EFE constitutes a large part of general relativity research. Various approaches to classifying spacetimes, including using the Segre classification of the energy-momentum tensor or the Petrov classification of the Weyl tensor have been studied extensively by many researchers, most notably Stephani et al. (2003). They also classify spacetimes using symmetry vector fields (especially Killing and homothetic symmetries). For example, Killing vector fields may be used to classify spacetimes, as there is a limit to the number of global, smooth Killing vector fields that a spacetime may possess (the maximum being 10 for 4-dimensional spacetimes). Generally speaking, the higher the

dimension of the algebra of symmetry vector fields on a spacetime, the more symmetry the spacetime admits. For example, the Schwarzschild solution has a Killing algebra of dimension 4 (3 spatial rotational vector fields and a time translation), whereas the Friedmann-Robertson-Walker (FRW) metric (excluding the Einstein static subcase) has a Killing algebra of dimension 6 (3 translations and 3 rotations). The Einstein static metric has a Killing algebra of dimension 7 (the previous 6 plus a time translation).

The assumption of a spacetime admitting a certain symmetry vector field can place severe restrictions on the spacetime.

See also

- Field (physics)
- Killing tensor
- Lie groups
- Noether's theorem
- Ricci decomposition
- Symmetry in physics

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-

Symmetry (biology)

1. REDIRECT Symmetry in biology

Symmetric relation

In mathematics, a binary relation R over a set X is **symmetric** if it holds for all a and b in X that if a is related to b then b is related to a .

In mathematical notation, this is:

$$\forall a, b \in X, aRb \Rightarrow bRa.$$

Note: symmetry is **not** the exact opposite of *antisymmetry* (aRb and bRa implies $b = a$). There are relations which are both symmetric and antisymmetric (equality and its subrelations, including, vacuously, the empty relation), there are relations which are neither symmetric nor antisymmetric ('preys on' in biological sciences), there are relations which are symmetric and not antisymmetric (congruence modulo n), and there are relations which are not symmetric but are antisymmetric ("is less than or equal to").

Properties containing the symmetric relation

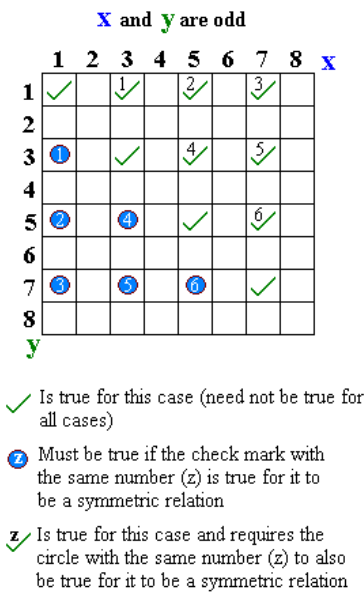
equivalence relation - A symmetric relation that is also transitive and reflexive.

Graph-theoretic interpretation

In an undirected graph, the relation over the set of vertices of the graph under which v and w are related if and only if they are adjacent forms a symmetric relation. Conversely, if R is a symmetric relation over a set X , one can interpret it as describing an undirected graph with the elements of X as the vertices and the pairs in R as the edges. Thus, symmetric relations and undirected graphs are combinatorially equivalent objects.

Examples

- "is married to" is a symmetric relation, while "is less than" is not.
- "is equal to" (equality)
- "... is odd and ... is odd too":

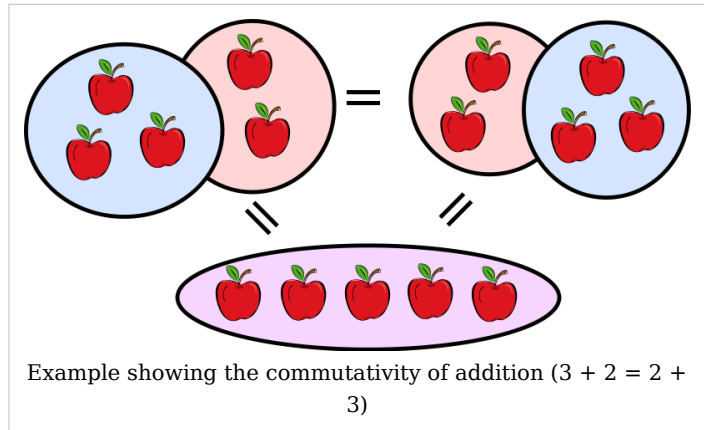


See also

- Symmetry in mathematics
- Asymmetric relation
- Antisymmetric relation

Commutativity

In mathematics, **commutativity** is the property that changing the order of something does not change the end result. It is a fundamental property of many binary operations, and many mathematical proofs depend on it. The commutativity of simple operations, such as multiplication and addition of numbers, was for many years implicitly assumed and the property was not named until the 19th century when mathematicians began to formalize the theory of mathematics.



Common uses

The *commutative property* (or *commutative law*) is a property associated with binary operations and functions. Similarly, if the commutative property holds for a pair of elements under a certain binary operation then it is said that the two elements *commute* under that operation.

In group and set theory, many algebraic structures are called commutative when certain operands satisfy the commutative property. In higher branches of math, such as analysis and linear algebra the commutativity of well known operations (such as addition and multiplication on real and complex numbers) is often used (or implicitly assumed) in proofs.^{[1] [2] [3]}

Mathematical definitions

The term "commutative" is used in several related senses.^{[4] [5]}

1. A binary operation $\square$ on a set S is said to be *commutative* if:

$$\forall x, y \in S : x * y = y * x$$

- An operation that does not satisfy the above property is called *noncommutative*.

2. One says that x *commutes* with y under $\square$ if:

$$x * y = y * x$$

3. A binary function $f: A \times A \rightarrow B$ is said to be *commutative* if:

$$\forall x, y \in A : f(x, y) = f(y, x)$$

History and etymology

(14)
 f et f' , sont telles qu'elles donnent
 el que soit l'ordre dans lequel on les
 ppelees *commutatives* entre elles.

; $aEz = Eaz$;

The first known use of the term was in a French
 Journal published in 1814

Records of the implicit use of the commutative property go back to ancient times. The Egyptians used the commutative property of multiplication to simplify computing products.^{[6] [7]} Euclid is known to have assumed the commutative property of multiplication in his book *Elements*.^[8] Formal uses of the commutative property arose in the late 18th and early 19th century when mathematicians began to work on a theory of functions. Today the

commutative property is a well known and basic property used in most branches of mathematics. Simple versions of the commutative property are usually taught in beginning mathematics courses.

The first use of the actual term *commutative* was in a memoir by François Servois in 1814,^{[9] [10]} which used the word *commutatives* when describing functions that have what is now called the commutative property. The word is a combination of the French word *commuter* meaning "to substitute or switch" and the suffix *-ative* meaning "tending to" so the word literally means "tending to substitute or switch." The term then appeared in English in *Philosophical Transactions of the Royal Society* in 1844.^[11]

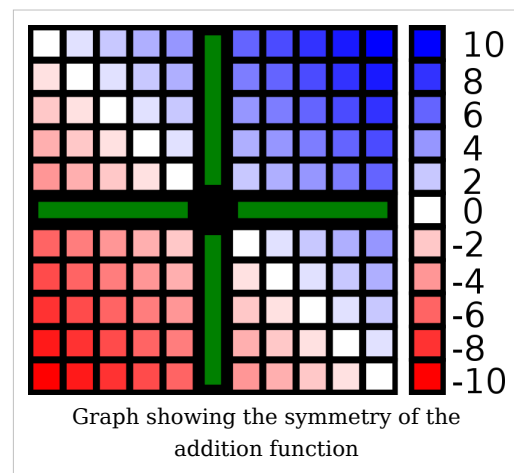
Related properties

Associativity

The associative property is closely related to the commutative property. The associative property states that the order in which operations are performed does not affect the final result as long as the order of terms is not changed. In contrast, the commutative property states that the order of the terms does not affect the final result.

Symmetry

Symmetry can be directly linked to commutativity. When a commutative operator is written as a binary function then the resulting function is symmetric across the line $y = x$. As an example, if we let a function f represent addition (a commutative operation) so that $f(x,y) = x + y$ then f is a symmetric function which can be seen in the image on the right.



Examples

Commutative operations in everyday life

- Putting your shoes on resembles a commutative operation since it doesn't matter if you put the left or right shoe on first, the end result (having both shoes on), is the same.
- When making change we take advantage of the commutativity of addition. It doesn't matter what order we put the change in, it always adds to the same total.

Commutative operations in math

Two well-known examples of commutative binary operations are:^[4]

- The addition of real numbers, which is commutative since

$$y + z = z + y \quad \forall y, z \in \mathbb{R}$$

For example $4 + 5 = 5 + 4$, since both expressions equal 9.

- The multiplication of real numbers, which is commutative since

$$yz = zy \quad \forall y, z \in \mathbb{R}$$

For example, $3 \times 5 = 5 \times 3$, since both expressions equal 15.

- Further examples of commutative binary operations include addition and multiplication of complex numbers, addition of vectors, and intersection and union of sets.

Noncommutative operations in everyday life

EA + ≠ + TEA

() ≠ () EATEA

Concatenation, the act of joining character strings together, is a noncommutative operation.

- Washing and drying your clothes resembles a noncommutative operation, if you dry first and then wash, you get a significantly different result than if you wash first and then dry.
- The Rubik's Cube is noncommutative. For example, twisting the front face clockwise, the top face clockwise and the front face

counterclockwise (FUF') does not yield the same result as twisting the front face clockwise, then counterclockwise and finally twisting the top clockwise (FF'U). The twists do not commute. This is studied in group theory.

Noncommutative operations in mathematics

Some noncommutative binary operations are:^[12]

- subtraction is noncommutative since $0 - 1 \neq 1 - 0$
- division is noncommutative since $1/2 \neq 2/1$
- matrix multiplication is noncommutative since

$$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Mathematical structures and commutativity

- An abelian group is a group whose group operation is commutative.^[2]
- A commutative ring is a ring whose multiplication is commutative. (Addition in a ring is by definition always commutative.)^[13]
- In a field both addition and multiplication are commutative.^[14]

See also

- Anticommutativity
- Binary operation
- Commutant
- Commutative diagram
- Commutative (neurophysiology)
- Commutator
- Distributivity
- Particle statistics (for commutativity in physics)

Notes

- [1] Axler, p.2
- [2] Gallian, p.34
- [3] p. 26,87
- [4] Krowne, p.1
- [5] Weisstein, *Commute*, p.1
- [6] Lumpkin, p.11
- [7] Gay and Shute, p.?
- [8] O'Conner and Robertson, *Real Numbers*
- [9] Cabillón and Miller, *Commutative and Distributive*
- [10] O'Conner and Robertson, *Servois*
- [11] Cabillón and Miller, *Commutative and Distributive*
- [12] Yark, p.1
- [13] Gallian p.236
- [14] Gallian p.250

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Biography of Francois Servois, who first used the term

Abelian group

An **abelian group**, also called a **commutative group**, is a group in which the result of applying the group operation to two group elements does not depend on their order (the axiom of commutativity). Abelian groups generalize the arithmetic of addition of integers. They are named after Niels Henrik Abel.

The concept of an abelian group is one of the first concepts encountered in undergraduate abstract algebra, with many other basic objects, such as a module and a vector space, being its refinements. The theory of abelian groups is generally simpler than that of their non-abelian counterparts, and finite abelian groups are very well understood. On the other hand, the theory of infinite abelian groups is an area of current research.

Definition

An abelian group is a set, A , together with an operation " $\bullet$ " that combines any two elements a and b to form another element denoted $a \bullet b$. The symbol " $\bullet$ " is a general placeholder for a concretely given operation. To qualify as an abelian group, the set and operation, $(A, \bullet)$, must satisfy five requirements known as the *abelian group axioms*:

Closure

For all a, b in A , the result of the operation $a \bullet b$ is also in A .

Associativity

For all a, b and c in A , the equation $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ holds.

Identity element

There exists an element e in A , such that for all elements a in A , the equation $e \bullet a = a \bullet e = a$ holds.

Inverse element

For each a in A , there exists an element b in A such that $a \bullet b = b \bullet a = e$, where e is the identity element.

Commutativity

For all a, b in A , $a \bullet b = b \bullet a$.

More compactly, an abelian group is a commutative group. A group in which the group operation is not commutative is called a "non-abelian group" or "non-commutative group".

Facts

Notation

There are two main notational conventions for abelian groups — additive and multiplicative.

Convention	Operation	Identity	Powers	Inverse
Addition	$x + y$	0	nx	$-x$
Multiplication	$x * y$ or xy	e or 1	x^n	x^{-1}

Generally, the multiplicative notation is the usual notation for groups, while the additive notation is the usual notation for modules. The additive notation may also be used to

emphasize that a particular group is abelian, whenever both abelian and non-abelian groups are considered.

Multiplication table

To verify that a finite group is abelian, a table (matrix) - known as a Cayley table - can be constructed in a similar fashion to a multiplication table. If the group is $G = \{g_1 = e, g_2, \dots, g_n\}$ under the operation $\square$, the (i, j) 'th entry of this table contains the product $g_i \square g_j$. The group is abelian if and only if this table is symmetric about the main diagonal (i.e. if the matrix is a symmetric matrix).

This is true since if the group is abelian, then $g_i \square g_j = g_j \square g_i$. This implies that the (i, j) 'th entry of the table equals the (j, i) 'th entry - i.e. the table is symmetric about the main diagonal.

Examples

- For the integers and the operation addition "+", denoted $(\mathbf{Z}, +)$, the operation + combines any two integers to form a third integer, addition is associative, zero is the additive identity, every integer n has an additive inverse, $-n$, and the addition operation is commutative since $m + n = n + m$ for any two integers m and n .
- Every cyclic group G is abelian, because if x, y are in G , then $xy = a^m a^n = a^{m+n} = a^{n+m} = a^n a^m = yx$. Thus the integers, $\mathbf{Z}$, form an abelian group under addition, as do the integers modulo n , $\mathbf{Z}/n\mathbf{Z}$.
- Every ring is an abelian group with respect to its addition operation. In a commutative ring the invertible elements, or units, form an abelian multiplicative group. In particular, the real numbers are an abelian group under addition, and the nonzero real numbers are an abelian group under multiplication.
- Every subgroup of an abelian group is normal, so each subgroup gives rise to a quotient group. Subgroups, quotients, and direct sums of abelian groups are again abelian.

In general, matrices, even invertible matrices, do not form an abelian group under multiplication because matrix multiplication is generally not commutative. However, some groups of matrices are abelian groups under matrix multiplication - one example is the group of 2x2 rotation matrices.

Historical remarks

Abelian groups were named for Norwegian mathematician Niels Henrik Abel by Camille Jordan who was first to observe their importance in connection with the problem of solvability by radicals, first posed by Abel.

Properties

If n is a natural number and x is an element of an abelian group G written additively, then nx can be defined as $x + x + \dots + x$ (n summands) and $(-n)x = -(nx)$. In this way, G becomes a module over the ring $\mathbf{Z}$ of integers. In fact, the modules over $\mathbf{Z}$ can be identified with the abelian groups.

Theorems about abelian groups (i.e. modules over the principal ideal domain $\mathbf{Z}$) can often be generalized to theorems about modules over an arbitrary principal ideal domain. A

typical example is the classification of finitely generated abelian groups which is a specialization of the structure theorem for finitely generated modules over a principal ideal domain. In the case of finitely generated abelian groups, this theorem guarantees that an abelian group splits as a direct sum of a torsion group and a free abelian group. The former may be written as a direct sum of finitely many groups of the form $\mathbf{Z}/p^k\mathbf{Z}$ for p prime, and the latter is a direct sum of finitely many copies of $\mathbf{Z}$.

If $f, g : G \rightarrow H$ are two group homomorphisms between abelian groups, then their sum $f + g$, defined by $(f + g)(x) = f(x) + g(x)$, is again a homomorphism. (This is not true if H is a non-abelian group.) The set $\text{Hom}(G, H)$ of all group homomorphisms from G to H thus turns into an abelian group in its own right.

Somewhat akin to the dimension of vector spaces, every abelian group has a *rank*. It is defined as the cardinality of the largest set of linearly independent elements of the group. The integers and the rational numbers have rank one, as well as every subgroup of the rationals.

Finite abelian groups

Cyclic groups of integers modulo n , $\mathbf{Z}/n\mathbf{Z}$, were among the first examples of groups. It turns out that an arbitrary finite abelian group is isomorphic to a direct sum of finite cyclic groups of prime power order, and these orders are uniquely determined, forming a complete system of invariants. The automorphism group of a finite abelian group can be described directly in terms of these invariants. The theory had been first developed in the 1879 paper of Georg Frobenius and Ludwig Stickelberger and later was both simplified and generalized to finitely generated modules over a principal ideal domain, forming an important chapter of linear algebra.

Classification

The **fundamental theorem of finite abelian groups** states that every finite abelian group G can be expressed as the direct sum of cyclic subgroups of prime-power order. This is a special case of the fundamental theorem of finitely generated abelian groups when G has zero rank.

The cyclic group $\mathbb{Z}_{mn}$ of order mn is isomorphic to the direct sum of $\mathbb{Z}_m$ and $\mathbb{Z}_n$ if and only if m and n are coprime. It follows that any finite abelian group G is isomorphic to a direct sum of the form

$$\mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_u}$$

in either of the following canonical ways:

- the numbers $k_1, \dots, k_u$ are powers of primes
- k_1 divides k_2 , which divides k_3 , and so on up to k_u .

For example, $\mathbb{Z}/15\mathbb{Z} \cong \mathbb{Z}_{15}$ can be expressed as the direct sum of two cyclic subgroups of order 3 and 5: $\mathbb{Z}_{15} \cong \{0, 5, 10\} \oplus \{0, 3, 6, 9, 12\}$. The same can be said for any abelian group of order 15, leading to the remarkable conclusion that all abelian groups of order 15 are isomorphic.

For another example, every abelian group of order 8 is isomorphic to either $\mathbb{Z}_8$ (the integers 0 to 7 under addition modulo 8), $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ (the odd integers 1 to 15 under multiplication modulo 16), or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

See also list of small groups for finite abelian groups of order 16 or less.

Automorphisms

One can apply the fundamental theorem to count (and sometimes determine) the automorphisms of a given finite abelian group G . To do this, one uses the fact (which will not be proved here) that if G splits as a direct sum $H \oplus K$ of subgroups of coprime order, then $\text{Aut}(H \oplus K) \cong \text{Aut}(H) \oplus \text{Aut}(K)$.

Given this, the fundamental theorem shows that to compute the automorphism group of G it suffices to compute the automorphism groups of the Sylow p -subgroups separately (that is, all direct sums of cyclic subgroups, each with order a power of p). Fix a prime p and suppose the exponents e_i of the cyclic factors of the Sylow p -subgroup are arranged in increasing order:

$$e_1 \leq e_2 \leq \cdots \leq e_n$$

for some $n > 0$. One needs to find the automorphisms of

$$\mathbb{Z}_{p^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{e_n}}.$$

One special case is when $n = 1$, so that there is only one cyclic prime-power factor in the Sylow p -subgroup P . In this case the theory of automorphisms of a finite cyclic group can be used. Another special case is when n is arbitrary but $e_i = 1$ for $1 \leq i \leq n$. Here, one is considering P to be of the form

$$\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p,$$

so elements of this subgroup can be viewed as comprising a vector space of dimension n over the finite field of p elements $\mathbb{F}_p$. The automorphisms of this subgroup are therefore given by the invertible linear transformations, so

$$\text{Aut}(P) \cong \text{GL}(n, \mathbb{F}_p),$$

where GL is the appropriate general linear group. This is easily shown to have order

$$|\text{Aut}(P)| = (p^n - 1) \cdots (p^n - p^{n-1}).$$

In the most general case, where the e_i and n are arbitrary, the automorphism group is more difficult to determine. It is known, however, that if one defines

$$d_k = \max\{r \mid e_r = e_k\}$$

and

$$c_k = \min\{r \mid e_r = e_k\}$$

then one has in particular $d_k \geq k$, $c_k \leq k$, and

$$|\text{Aut}(P)| = \left(\prod_{k=1}^n p^{d_k} - p^{k-1} \right) \left(\prod_{j=1}^n (p^{e_j})^{n-d_j} \right) \left(\prod_{i=1}^n (p^{e_i-1})^{n-c_i+1} \right).$$

One can check that this yields the orders in the previous examples as special cases (see [Hillar,Rhea]).

Infinite abelian groups

The theory of infinite abelian groups is far from complete. Two important special classes with diametrically opposite properties are *torsion groups* and *torsion-free groups*.

Torsion groups

An abelian group is called **periodic** or torsion **if every element has finite order**.

Important areas of the theory of torsion groups are:

- Direct sums of cyclic groups, also called pure projective modules
- Bounded groups, an example of pure injective modules
- Ulm invariants

Torsion-free groups

An abelian group is called **torsion-free** if every non-zero element has infinite order. Important areas of torsion-free groups are:

- Rank of an abelian group
- Basic subgroups
- Cotorsion and algebraically compact torsion-free groups such as the p -adic integers
- Slender groups

Mixed groups

An abelian group is called **mixed** if it is neither torsion nor torsion-free. Important topics in the theory of mixed groups are:

- Ext functor
- Pure subgroups
- Divisible groups

In each case, the new ideas help to approximate a mixed group as a direct sum of a torsion and a torsion-free group.

Additive groups of rings

The additive group of a ring is an abelian group, but not all abelian groups are additive groups of rings. Some important topics in this area of study are:

- Tensor product
 - Corner's results on countable torsion-free groups
 - Shelah's work to remove cardinality restrictions
-

Relation to other mathematical topics

Many large abelian groups possess a natural topology, which turns them into topological groups.

The collection of all abelian groups, together with the homomorphisms between them, forms the category **Ab**, the prototype of an abelian category.

Nearly all well-known algebraic structures other than Boolean algebra, are undecidable. Hence it is surprising that Tarski's student Szmielew (1955) proved that the first order theory of abelian groups, unlike its nonabelian counterpart, is decidable. This decidability, plus the fundamental theorem of finite abelian groups described above, highlight some of the successes in abelian group theory, but there are still many areas of current research:

- Amongst torsion-free abelian groups of finite rank, only the finitely generated case and the rank 1 case are well understood;
- There are many unsolved problems in the theory of infinite-rank torsion-free abelian groups;
- While countable torsion abelian groups are well understood through simple presentations and Ulm invariants, the case of countable mixed groups is much less mature.
- Many mild extensions of the first order theory of abelian groups are known to be undecidable.
- Finite abelian groups remain a topic of research in computational group theory.

Moreover, abelian groups of infinite order lead, quite surprisingly, to deep questions about the set theory commonly assumed to underlie all of mathematics. Take the Whitehead problem: are all Whitehead groups of infinite order also free abelian groups? In the 1970s, Saharon Shelah proved that the Whitehead problem is:

- Undecidable in ZFC, the conventional axiomatic set theory from which nearly all of present day mathematics can be derived. The Whitehead problem is also the first question in ordinary mathematics proved undecidable in ZFC;
- Undecidable even if ZFC is augmented by taking the generalized continuum hypothesis as an axiom;
- Decidable if ZFC is augmented with the axiom of constructibility (see statements true in L).

A note on the typography

Among mathematical adjectives derived from the proper name of a mathematician, the word "abelian" is rare in that it is spelled with a lowercase **a**, rather than an uppercase **A**, indicating how ubiquitous the concept is in modern mathematics.^[1]

See also

- Abelianization
- Class field theory
- Commutator subgroup
- Elementary abelian group
- Finitely generated abelian group
- Free abelian group
- Pontryagin duality
- Torsion-free abelian groups of rank 1

Notes

[1] Abel Prize Awarded: The Mathematicians' Nobel (http://www.maa.org/devlin/devlin_04_04.html)

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Nonabelian group

In mathematics, a **nonabelian group**, also sometimes called a **non-commutative group**, is a group $(G, *)$ such that there are at least two elements a and b of G such that $a * b \neq b * a$. The term *non-abelian* is used to distinguish from the idea of an abelian group, where all of the elements of the group commute.

Nonabelian groups are pervasive in mathematics and physics. One of the simplest examples of a non-abelian group is the dihedral group of order 6. A common example from physics is the rotation group in three dimensions.

Both discrete groups and continuous groups may be non-abelian; most of the interesting Lie groups are nonabelian. The term *nonabelian* is used primarily by physicists, rather than mathematicians, and is frequently taken to be a synonym for the collection of Lie groups. This usage is particularly common in gauge theory.

See also

- Associative algebra
 - Non-commutative geometry
-

Symmetry group

The **symmetry group** of an object (image, signal, etc.) is the group of all isometries under which it is invariant with composition as the operation. It is a subgroup of the isometry group of the space concerned.

If not stated otherwise, this article considers symmetry groups in Euclidean geometry, but the concept may also be studied in wider contexts; see below.

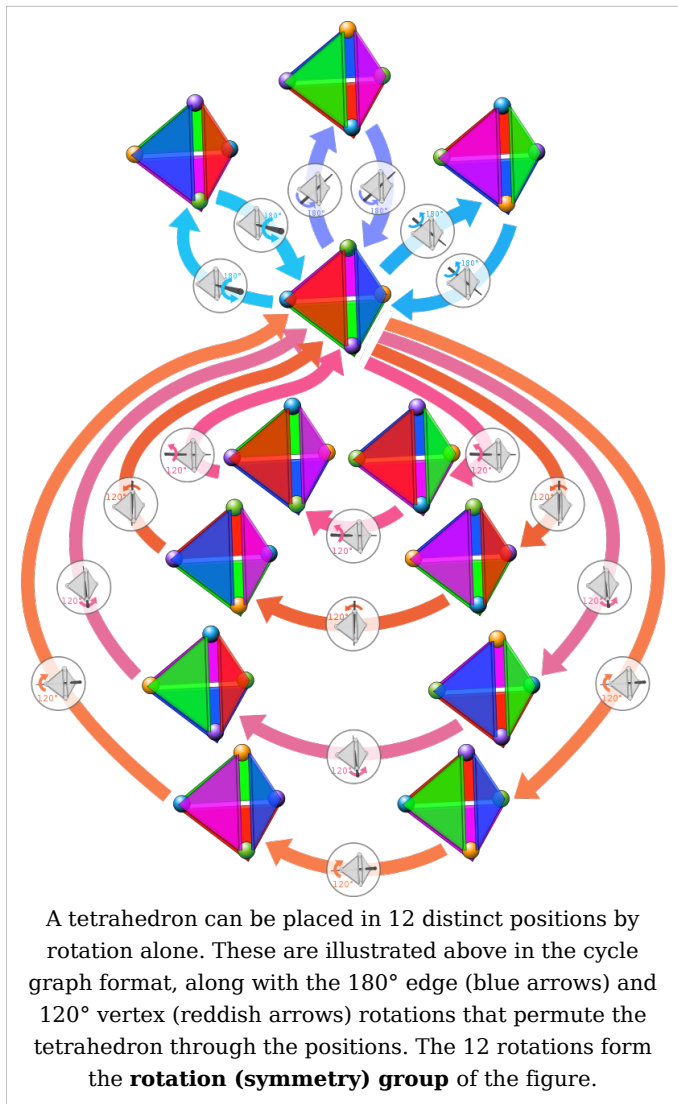
Introduction

The "objects" may be geometric figures, images and patterns, such as a wallpaper pattern. The definition can be made more precise by specifying what is meant by image or pattern, e.g., a function of position with values in a set of colors. For symmetry of physical objects, one may also want to take physical composition into account. The group of isometries of space induces a group action on objects in it.

The symmetry group is sometimes also called **full symmetry group** in order to emphasize that it includes the orientation-reversing isometries (like reflections, glide reflections and improper rotations) under which the figure is invariant. The subgroup of orientation-preserving isometries (i.e. translations, rotations, and compositions of these) which leave the figure invariant is called its **proper symmetry group**. The proper symmetry group of an object is equal to its full symmetry group if and only if the object is chiral (and thus there are no orientation-reversing isometries under which it is invariant).

Any symmetry group whose elements have a common fixed point, which is true for all finite symmetry groups and also for the symmetry groups of bounded figures, can be represented as a subgroup of orthogonal group $O(n)$ by choosing the origin to be a fixed point. The proper symmetry group is a subgroup of the special orthogonal group $SO(n)$ then, and therefore also called **rotation group** of the figure.

Discrete symmetry groups come in three types: (1) finite **point groups**, which include only rotations, reflections, inversion and rotoinversion - they are in fact just the finite subgroups of $O(n)$, (2) infinite **lattice groups**, which include only translations, and (3) infinite **space groups** which combines elements of both previous types, and may also include extra



transformations like screw axis and glide reflection. There are also *continuous* symmetry groups, which contain rotations of arbitrarily small angles or translations of arbitrarily small distances. The group of all symmetries of a sphere $O(3)$ is an example of this, and in general such continuous symmetry groups are studied as Lie groups. With a categorization of subgroups of the Euclidean group corresponds a categorization of symmetry groups.

Two geometric figures are considered to be of the same symmetry type if their symmetry groups are conjugate subgroups of the Euclidean group $E(n)$ (the isometry group of $\mathbb{R}^n$), where two subgroups H_1, H_2 of a group G are *conjugate*, if there exists $g \in G$ such that $H_1 = g^{-1}H_2g$. For example:

- two 3D figures have mirror symmetry, but with respect to different mirror planes.
- two 3D figures have 3-fold rotational symmetry, but with respect to different axes.
- two 2D patterns have translational symmetry, each in one direction; the two translation vectors have the same length but a different direction.

When considering isometry groups, one may restrict oneself to those where for all points the set of images under the isometries is topologically closed. This excludes for example in 1D the group of translations by a rational number. A "figure" with this symmetry group is non-drawable and up to arbitrarily fine detail homogeneous, without being really homogeneous.

One dimension

The isometry groups in 1D where for all points the set of images under the isometries is topologically closed are:

- the trivial group C_1
- the groups of two elements generated by a reflection in a point; they are isomorphic with C_2
- the infinite discrete groups generated by a translation; they are isomorphic with $\mathbf{Z}$
- the infinite discrete groups generated by a translation and a reflection in a point; they are isomorphic with the generalized dihedral group of $\mathbf{Z}$, $\text{Dih}(\mathbf{Z})$, also denoted by D_∞ (which is a semidirect product of $\mathbf{Z}$ and C_2).
- the group generated by all translations (isomorphic with $\mathbf{R}$); this group cannot be the symmetry group of a "pattern": it would be homogeneous, hence could also be reflected. However, a uniform 1D vector field has this symmetry group.
- the group generated by all translations and reflections in points; they are isomorphic with the generalized dihedral group of $\mathbf{R}$, $\text{Dih}(\mathbf{R})$.

See also symmetry groups in one dimension.

Two dimensions

Up to conjugacy the discrete point groups in 2 dimensional space are the following classes:

- cyclic groups $C_1, C_2, C_3, C_4, \dots$ where C_n consists of all rotations about a fixed point by multiples of the angle $360^\circ/n$
- dihedral groups $D_1, D_2, D_3, D_4, \dots$ where D_n (of order $2n$) consists of the rotations in C_n together with reflections in n axes that pass through the fixed point.

C_1 is the trivial group containing only the identity operation, which occurs when the figure has no symmetry at all, for example the letter **F**. C_2 is the symmetry group of the letter **Z**, C_3 that of a triskelion, C_4 of a swastika, and C_5, C_6 etc. are the symmetry groups of similar

swastika-like figures with five, six etc. arms instead of four.

D_1 is the 2-element group containing the identity operation and a single reflection, which occurs when the figure has only a single axis of bilateral symmetry, for example the letter **A**. D_2 , which is isomorphic to the Klein four-group, is the symmetry group of a non-equilateral rectangle, and D_3, D_4 etc. are the symmetry groups of the regular polygons. The actual symmetry groups in each of these cases have two degrees of freedom for the center of rotation, and in the case of the dihedral groups, one more for the positions of the mirrors.

The remaining isometry groups in 2D with a fixed point, where for all points the set of images under the isometries is topologically closed are:

- the special orthogonal group $SO(2)$ consisting of all rotations about a fixed point; it is also called the circle group S^1 , the multiplicative group of complex numbers of absolute value 1. It is the *proper* symmetry group of a circle and the continuous equivalent of C_n . There is no figure which has as *full* symmetry group the circle group, but for a vector field it may apply (see the 3D case below).
- the orthogonal group $O(2)$ consisting of all rotations about a fixed point and reflections in any axis through that fixed point. This is the symmetry group of a circle. It is also called $Dih(S^1)$ as it is the generalized dihedral group of S^1 .

For non-bounded figures, the additional isometry groups can include translations; the closed ones are:

- the 7 frieze groups
- the 17 wallpaper groups
- for each of the symmetry groups in 1D, the combination of all symmetries in that group in one direction, and the group of all translations in the perpendicular direction
- ditto with also reflections in a line in the first direction

Three dimensions

Up to conjugacy the set of 3D point groups consists of 7 infinite series, and 7 separate ones. In crystallography they are restricted to be compatible with the discrete translation symmetries of a crystal lattice. This crystallographic restriction of the infinite families of general point groups results in 32 crystallographic point groups (27 from the 7 infinite series, and 5 of the 7 others).

See **point groups in three dimensions**.

The continuous symmetry groups with a fixed point include those of:

- cylindrical symmetry without a symmetry plane perpendicular to the axis, this applies for example often for a bottle
- cylindrical symmetry with a symmetry plane perpendicular to the axis
- spherical symmetry

For objects and scalar fields the cylindrical symmetry implies vertical planes of reflection. However, for vector fields it does not: in cylindrical coordinates with respect to some axis, $\mathbf{A} = A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{z}$ has cylindrical symmetry with respect to the axis if and only if A_ρ, A_ϕ , and A_z have this symmetry, i.e., they do not depend on ϕ . Additionally there is reflectional symmetry if and only if $A_\phi = 0$.

For spherical symmetry there is no such distinction, it implies planes of reflection.

The continuous symmetry groups without a fixed point include those with a screw axis, such as an infinite helix. See also subgroups of the Euclidean group.

Symmetry groups in general

In wider contexts, a **symmetry group** may be any kind of **transformation group**, or automorphism group. Once we know what kind of mathematical structure we are concerned with, we should be able to pinpoint what mappings preserve the structure. Conversely, specifying the symmetry can define the structure, or at least clarify what we mean by an invariant, geometric language in which to discuss it; this is one way of looking at the Erlangen programme.

For example, automorphism groups of certain models of finite geometries are not "symmetry groups" in the usual sense, although they preserve symmetry. They do this by preserving *families* of point-sets rather than point-sets (or "objects") themselves.

Like above, the group of automorphisms of space induces a group action on objects in it.

For a given geometric figure in a given geometric space, consider the following equivalence relation: two automorphisms of space are equivalent if and only if the two images of the figure are the same (here "the same" does not mean something like e.g. "the same up to translation and rotation", but it means "exactly the same"). Then the equivalence class of the identity is the symmetry group of the figure, and every equivalence class corresponds to one isomorphic version of the figure.

There is a bijection between every pair of equivalence classes: the inverse of a representative of the first equivalence class, composed with a representative of the second.

In the case of a finite automorphism group of the whole space, its order is the order of the symmetry group of the figure multiplied by the number of isomorphic versions of the figure.

Examples:

- Isometries of the Euclidean plane, the figure is a rectangle: there are infinitely many equivalence classes; each contains 4 isometries.
- The space is a cube with Euclidean metric; the figures include cubes of the same size as the space, with colors or patterns on the faces; the automorphisms of the space are the 48 isometries; the figure is a cube of which one face has a different color; the figure has a symmetry group of 8 isometries, there are 6 equivalence classes of 8 isometries, for 6 isomorphic versions of the figure.

Compare Lagrange's theorem (group theory) and its proof.

See also

- crystallography
 - crystal system
 - Euclidean plane isometry
 - fixed points of isometry groups in Euclidean space
 - group action
 - permutation group
 - point group
 - space group
 - symmetric group
 - symmetry
-

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External links

- Eric W. Weisstein, *Symmetry Group*^[1] at MathWorld.
- Eric W. Weisstein, *Tetrahedral Group*^[2] at MathWorld.
- Overview of the 32 crystallographic point groups^[3] - form the first parts (apart from skipping $n=5$) of the 7 infinite series and 5 of the 7 separate 3D point groups

References

- [1] <http://mathworld.wolfram.com/SymmetryGroup.html>
 [2] <http://mathworld.wolfram.com/TetrahedralGroup.html>
 [3] <http://newton.ex.ac.uk/research/qsystems/people/goss/symmetry/Solids.html>

Isometry

For the mechanical engineering and architecture usage, see isometric projection. For isometry in differential geometry, see isometry (Riemannian geometry).

In mathematics, an **isometry**, **isometric isomorphism** or **congruence mapping** is a distance-preserving isomorphism between metric spaces. Geometric figures which can be related by an isometry are called congruent.

Isometries are often used in constructions where one space is embedded in another space. For instance, the completion of a metric space M involves an isometry from M into M' , a quotient set of the space of Cauchy sequences on M . The original space M is thus isometrically isomorphic to a subspace of a complete metric space, and it is usually identified with this subspace. Other embedding constructions show that every metric space is isometrically isomorphic to a closed subset of some normed vector space and that every complete metric space is isometrically isomorphic to a closed subset of some Banach space.

Definitions

The notion of isometry comes in two main flavors: *global isometry* and a weaker notion *path isometry* or *arcwise isometry*. Both are often called just *isometry* and one should determine from context which one is intended.

Let X and Y be metric spaces with metrics d_X and d_Y . A map $f : X \rightarrow Y$ is called **distance preserving** if for any $a, b \in X$ one has

$$d_Y(f(a), f(b)) = d_X(a, b).$$

A distance preserving map is automatically injective. Clearly, every isometry between metric spaces is necessarily a topological embedding.

A **global isometry** is a bijective distance preserving map. A **path isometry** or **arcwise isometry** is a map which preserves the lengths of curves (not necessarily bijective).

Two metric spaces X and Y are called **isometric** if there is an isometry from X to Y . The set of isometries from a metric space to itself forms a group with respect to function composition, called the **isometry group**.

Examples

- Any reflection, translation and rotation is a global isometry on Euclidean spaces. See also Euclidean group.
- The map $\mathbf{R} \rightarrow \mathbf{R}$ defined by $x \mapsto |x|$ is a path isometry but not a global isometry.
- The isometric linear maps from $\mathbf{C}^n$ to itself are the unitary matrices.

Linear isometries

Given two normed vector spaces V and W , a **linear isometry** is a linear map $f: V \rightarrow W$ that preserves the norms:

$$\|f(v)\| = \|v\|$$

for all v in V . Linear isometries are distance-preserving maps in the above sense. They are global isometries if and only if they are surjective.

By the Mazur-Ulam theorem, any isometry of normed vector spaces over $\mathbf{R}$ is affine.

Generalizations

- Given a positive real number ε , an **ε -isometry** or **almost isometry** (also called a **Hausdorff approximation**) is a map $f: X \rightarrow Y$ between metric spaces such that
 - for $x, x' \in X$ one has $|d_Y(f(x), f(x')) - d_X(x, x')| < \varepsilon$, and
 - for any point $y \in Y$ there exists a point $x \in X$ with $d_Y(y, f(x)) < \varepsilon$

That is, an ε -isometry preserves distances to within ε and leaves no element of the codomain further than ε away from the image of an element of the domain. Note that ε -isometries are not assumed to be continuous.

- Quasi-isometry** is yet another useful generalization.

Beckman-Quarles theorem

The **Beckman-Quarles theorem** states that for a Euclidean space E of dimension d at least 2, any mapping f from E to itself that preserves the property of being at a unit distance apart must be an isometry.

See also

- Isometric projection
- Congruence (geometry)
- Euclidean plane isometry
- 3D isometries which leave the origin fixed
- space group
- involution
- Isometries in physics

- Isometry group
- Homeomorphism group

References

- F. S. Beckman and D. A. Quarles, Jr., *On isometries of Euclidean space*, Proc. Amer. Math. Soc., 4 (1953) 810-815.

Space group

The **space group** of a crystal or **crystallographic group** is a mathematical description of the symmetry inherent in the structure. The word 'group' in the name comes from the mathematical notion of a group, which is used to build the set of space groups.

Crystallography

The space groups in three dimensions are made from combinations of the 32 crystallographic point groups with the 14 Bravais lattices which belong to one of 7 crystal systems. This results in a space group being some combination of the translational symmetry of a unit cell including lattice centering, and the point group symmetry operations of reflection, rotation and improper rotation (also called rotoinversion). Furthermore one must consider the screw axis and glide plane symmetry operations. These are called compound symmetry operations and are combinations of a rotation or reflection with a translation less than the unit cell size. The combination of all these symmetry operations results in a total of 230 unique space groups describing all possible crystal symmetries.

Glide planes and screw axes

Two of the symmetry operations involved in the space groups are not contained in the corresponding point group or Bravais lattice. These are the compound symmetry operations called the glide plane and the screw axis.

A glide plane is a reflection in a plane, followed by a translation parallel with that plane. This is noted by a , b or c , depending on which axis the glide is along. There is also the n glide, which is a glide along the half of a diagonal of a face, and the d glide, which is a fourth of the way along either a face or space diagonal of the unit cell. The latter is often called the diamond glide plane as it features in the diamond structure.

A screw axis is a rotation about an axis, followed by a translation along the direction of the axis. These are noted by a number, n , to describe the degree of rotation, where the number is how many operations must be applied to complete a full rotation (e.g., 3 would mean a rotation one third of the way around the axis each time). The degree of translation is then added as a subscript showing how far along the axis the translation is, as a portion of the parallel lattice vector. So, 2_1 is a twofold rotation followed by a translation of $1/2$ of the lattice vector.

Notation

There are a number of methods of identifying space groups. The International Union of Crystallography publishes a table (more correctly, a hefty tome of tables) of all space groups, and assigns each a unique number. Other than this numbering scheme there are two main forms of notation, the Hermann-Mauguin notation and Schönflies notation.

The Hermann-Mauguin (or international) notation is the one most commonly used in crystallography, and consists of a set of four symbols. The first describes the centering of the Bravais lattice (P , A , B , C , I , R or F). The next three describe the most prominent symmetry operation visible when projected along one of the high symmetry directions of the crystal. These symbols are the same as used in point groups, with the addition of glide planes and screw axis, described above. By way of example, the space group of quartz is $P3_121$, showing that it exhibits primitive centering of the motif (i.e., once per unit cell), with a threefold screw axis and a twofold rotation axis. Note that it does not explicitly contain the crystal system, although this is unique to each space group (in the case of $P3_121$, it is trigonal).

In HM notation the first symbol (3_1 in this example) denotes the symmetry along the major axis (c-axis in trigonal cases), the second (2 in this case) along axes of secondary importance (a and b) and the third symbol the symmetry in another direction. In the trigonal case there also exists a space group $P3_112$. In this space group the twofold axes are not along the a and b-axes but in a direction rotated by 30° .

Group theory

Mathematically, a space group is a symmetry group or symmetry group *type* of n -dimensional structures with translational symmetry in n independent directions, such as, for $n = 3$, a crystal. Only discrete symmetry groups are included in the categorization; i.e., infinitely fine structure or homogeneity in one or more directions is excluded. This comes from the necessity to describe *discrete* sets of 'points' (i.e. atoms or ions in a crystal), as opposed to continuous media (see Symmetry in physics for the latter case). See the articles Bravais lattices, Crystals, and Translation (geometry) for a fuller discussion.

Two symmetry groups are of the same **crystallographic space group type** if they are the same up to an affine transformation of space that preserves orientation. Thus e.g. a change of angle between translation vectors does not affect the space group type if it does not add or remove any symmetry. A more formal definition involves conjugacy, see Symmetry group.

Two symmetry groups are of the same **affine space group type** if they are the same up to an affine transformation, even if that inverts orientation.

This can be expressed by saying that two symmetry groups which are chiral and each other's mirror image, are of different crystallographic space group type, but of the same affine space group type.

In 1D and 2D space groups of the same affine space group type are also of the same crystallographic space group type, but in 3D this need not be the case: in 2D, the mirror image of a rotation is a reversed rotation, which is in the group anyway, and the mirror image of a mirror is still a mirror, but the mirror image of a righthand screw operation is a lefthand one, not the inverse of the righthand screw operation.

The Bieberbach theorem states that in each dimension all affine space group types are different even as abstract groups (as opposed to e.g. Frieze groups, of which two are isomorphic with $\mathbf{Z}$).

The term "space group" is often used for space group *type*. It is often clear from the context what is meant. However, when considering subgroup relationships a specific symmetry group should not be confused with the space group type.

Various dimensions

In 1D there are two space group types: those with and without mirror image symmetry, see symmetry groups in one dimension.

In 2D there are 17; these **2D space groups** are also called **wallpaper groups** or **plane groups**.

In 3D there are 230 crystallographic space group types, which reduces to 219 affine space group types because of some types being different from their mirror image; these are said to differ by "enantiomorphous character" (e.g. $P3_112$ and $P3_212$). Usually "space group" refers to 3D. They are by themselves purely mathematical, but play a large role in crystallography.

In 4 dimensions there are 4895 crystallographic space group types, or 4783 affine space group types.^[1]

The number of affine space group types in n dimensions is given by sequence A004029 in OEIS; the number of crystallographic space group types in n dimensions is given by A006227.

Double groups and time reversal

In addition to crystallographic space groups there are also magnetic space groups or double groups. These symmetries contain an element known as time reversal. They are of importance in magnetic structures that contain ordered unpaired spins, i.e. ferro-, ferri- or antiferromagnetic structures as studied by neutron diffraction. The time reversal element flips a magnetic spin while leaving all other structure the same and it can be combined with a number of other symmetry elements. Including time reversal there are 1651 magnetic space groups in 3D.^[2]

Grouping by point group

A symmetry group consists of isometric affine transformations; each is given by an orthogonal matrix and a translation vector (which may be the zero vector). Space groups can be grouped by the matrices involved, i.e. ignoring the translation vectors (see also Euclidean group). This corresponds to discrete symmetry groups with a fixed point: the point groups. However, not all point groups are compatible with translational symmetry; those that are compatible are called the crystallographic point groups. This is expressed in the crystallographic restriction theorem. (In spite of these names, this is a geometric limitation, not just a physical one.)

In 1D both space group types correspond to their own "crystallographic point group".

In 2D the 17 wallpaper groups are grouped according to 10 associated crystallographic point groups: 1-, 2-, 3-, 4-, and 6-fold rotational symmetry, each with or without reflections. Thus a wallpaper group with glide reflection axes is associated with the same point group

as the wallpaper group with reflection axes parallel to these glide reflection axes.

In 3D this gives a grouping of the 230 space group types into 32 crystal classes, one for each associated crystallographic point group. A space group with a screw axis is in the same crystal class as one with a corresponding pure axis of rotation. Similarly a space group with a glide plane is in the same crystal class as one with a corresponding pure reflection.

In addition to translations, and the point operations of reflection, rotation and improper rotation, there are combinations of reflections and rotations with translation: the screw axis and the glide plane.

The number of crystallographic point groups in n dimensions is given by OEIS:A004028.

Further categorizing

Space groups are categorized by Bravais lattice and crystal class. However, for some combinations there are multiple space groups, while other combinations are not possible.

The 230 space group types can be subdivided in two categories:

- 73 symmorphic space group types: a space group is symmorphic if all symmetries can be described in terms of rotation axes and reflection planes all through the same point (including roto reflections), without screw axes and glide planes). Equivalently, a space group is symmorphic if it is equivalent to a semidirect product of its point group with its translation subgroup.
- 157 nonsymmorphic space group types.

Conway and Thurston gave another classification of the space groups, where they divided the 230 groups into reducible and irreducible groups. The reducible groups fall into 17 classes corresponding to the 17 wallpaper groups, and the remaining 35 irreducible groups are classified separately.

See also

- Isometries in up to three dimensions
- Table of all space groups
- Yevgraf Fyodorov - the creator of space group theory

Bibliography

- International Tables for Crystallography, Volume A, edited by Th. Hahn. Reidel Publishing Company, Dordrecht, Boston, 1996.
- Conway, John H.; Delgado Friedrichs, Olaf; Huson, Daniel H.; Thurston, William P. *On three-dimensional space groups*.^[3] Beiträge Algebra Geom. 42 (2001), no. 2, 475--507. From the summary: "An entirely new and independent enumeration of the crystallographic space groups is given, based on obtaining the groups as fibrations over the plane crystallographic groups, when this is possible."

[1] H. Brown, R. Bülow, J. Neubüser, H. Wondratschek and H. Zassenhaus, Crystallographic Groups of Four-Dimensional Space. Wiley, NY, 1978, p. 52.

[2] p.428 Group Theoretical Methods and Applications to Molecules and Crystals. By Shoon Kyung Kim.1999. Cambridge University. Press.ISBN 0521640628

[3] <http://arxiv.org/abs/math.MG/9911185>

External links

- International Union of Crystallography (<http://www.iucr.org>)
- Point Groups and Bravais Lattices (<http://neon.mems.cmu.edu/degraeef/pointgroups/>)
- Bilbao Crystallographic Server (<http://www.cryst.ehu.es/>)
- Space Group Info (old) (<http://cci.lbl.gov/sginfo/>)
- Space Group Info (new) (http://cci.lbl.gov/cctbx/explore_symmetry.html)
- Crystal Lattice Structures: Index by Space Group (<http://cst-www.nrl.navy.mil/lattice/spcgrp/>)
- Full list of 230 crystallographic space groups (<http://img.chem.ucl.ac.uk/sgp/mainmenu.htm>)

Crystal system

A **crystal system** is a category of space groups, which characterize symmetry of structures in three dimensions with translational symmetry in three directions, having a discrete class of point groups. A major application is in crystallography, to categorize crystals, but by itself the topic is one of 3D Euclidean geometry.

Overview

There are 7 crystal systems:

- Triclinic, all cases not satisfying the requirements of any other system. There is no necessary symmetry other than translational symmetry, although inversion is possible.
- Monoclinic, requires either 1 twofold axis of rotation or 1 mirror plane.
- Orthorhombic, requires either 3 twofold axes of rotation or 1 twofold axis of rotation and two mirror planes.
- Tetragonal, requires 1 fourfold axis of rotation.
- Rhombohedral, also called trigonal, requires 1 threefold axis of rotation.
- Hexagonal, requires 1 sixfold axis of rotation.
- Isometric or cubic, requires 4 threefold axes of rotation.

There are 2, 13, 59, 68, 25, 27, and 36 space groups per crystal system, respectively, for a total of 230. The following table gives a brief characterization of the various crystal systems:

Crystal system	No. of point groups	No. of bravais lattices	No. of space groups
Triclinic	2	1	2
Monoclinic	3	2	13
Orthorhombic	3	4	59
Tetragonal	7	2	68
Rhombohedral	5	1	25
Hexagonal	7	1	27
Cubic	5	3	36
Total	32	14	230

Within a crystal system there are two ways of categorizing space groups:

- by the linear parts of symmetries, i.e. by crystal class, also called crystallographic point group; each of the 32 crystal classes applies for one of the 7 crystal systems
- by the symmetries in the translation lattice, i.e. by Bravais lattice; each of the 14 Bravais lattices applies for one of the 7 crystal systems.

The 73 symmorphic space groups (see space group) are largely combinations, within each crystal system, of each applicable point group with each applicable Bravais lattice: there are 2, 6, 12, 14, 5, 7, and 15 combinations, respectively, together 61.

Crystallographic point groups

A symmetry group consists of isometric affine transformations; each is given by an orthogonal matrix and a translation vector (which may be the zero vector). Space groups can be grouped by the matrices involved, i.e. ignoring the translation vectors (see also Euclidean group). This corresponds to discrete symmetry groups with a fixed point. There are infinitely many of these point groups in three dimensions. However, only part of these are compatible with translational symmetry: the crystallographic point groups. This is expressed in the crystallographic restriction theorem. (In spite of these names, this is a geometric limitation, not just a physical one.)

The point group of a crystal, among other things, determines the symmetry of the crystal's optical properties. For instance, one knows whether it is birefringent, or whether it shows the Pockels effect, by simply knowing its point group.

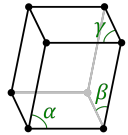
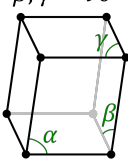
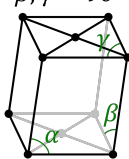
Overview of point groups by crystal system

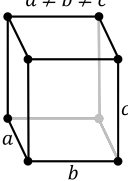
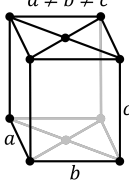
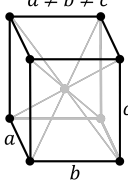
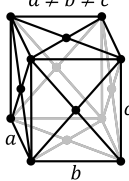
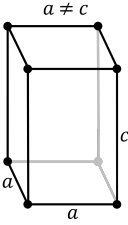
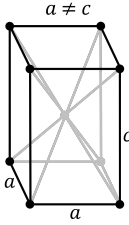
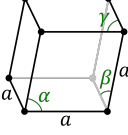
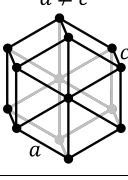
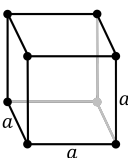
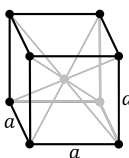
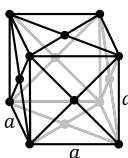
crystal system	point group / crystal class	Schönflies	Hermann-Mauguin	orbifold	Type
triclinic	triclinic-pedial	C_1		11	enantiomorphic polar
	triclinic-pinacoidal	C_i	$\bar{1}$	1x	centrosymmetric
monoclinic	monoclinic-sphenoidal	C_2		22	enantiomorphic polar
	monoclinic-domatic	C_s		1*	polar
	monoclinic-prismatic	C_{2h}		2*	centrosymmetric
orthorhombic	orthorhombic-sphenoidal	D_2		222	enantiomorphic
	orthorhombic-pyramidal	C_{2v}		*22	polar
	orthorhombic-bipyramidal	D_{2h}		*222	centrosymmetric
tetragonal	tetragonal-pyramidal	C_4		44	enantiomorphic polar
	tetragonal-disphenoidal	S_4	$\bar{4}$	2x	
	tetragonal-dipyramidal	C_{4h}		4*	centrosymmetric
	tetragonal-trapezoidal	D_4		422	enantiomorphic
	ditetragonal-pyramidal	C_{4v}		*44	polar
	tetragonal-scalenoidal	D_{2d}	or $\bar{4}m2$	2*2	
	ditetragonal-dipyramidal	D_{4h}		*422	centrosymmetric

rhombohedral (trigonal)	trigonal-pyramidal	C_3	3	33	enantiomorphic polar
	rhombohedral	$S_6 (C_{3i})$	$\bar{3}$	3x	centrosymmetric
	trigonal-trapezoidal	D_3	or or	322	enantiomorphic
	ditrigonal-pyramidal	C_{3v}	or or	*33	polar
	ditrigonal-scalahedral	D_{3d}	or $\bar{3}m1$ or $\bar{3}1m$	2*3	centrosymmetric
hexagonal	hexagonal-pyramidal	C_6		66	enantiomorphic polar
	trigonal-dipyramidal	C_{3h}	$\bar{6}$	3*	
	hexagonal-dipyramidal	C_{6h}		6*	centrosymmetric
	hexagonal-trapezoidal	D_6		622	enantiomorphic
	dihexagonal-pyramidal	C_{6v}		*66	polar
	ditrigonal-dipyramidal	D_{3h}	$\bar{6}m2$ or $\bar{6}2m$	*322	
	dihexagonal-dipyramidal	D_{6h}		*622	centrosymmetric
cubic	tetartohedral	T		332	enantiomorphic
	diploidal	T_h		3*2	centrosymmetric
	gyroidal	O		432	enantiomorphic
	tetrahedral	T_d	$\bar{4}3m$	*332	
	hexoctahedral	O_h	$m\bar{3}m$	*432	centrosymmetric

The crystal structures of biological molecules (such as protein structures) can only occur in the 11 enantiomorphic point groups, as biological molecules are invariably chiral. The protein assemblies themselves may have symmetries other than those given above, because they are not intrinsically restricted by the Crystallographic restriction theorem. For example the Rad52 DNA binding protein has an 11-fold rotational symmetry (in human), however, it must form crystals in one of the 11 enantiomorphic point groups given above.

Classification of lattices

The 7 Crystal systems	The 14 Bravais Lattices			
triclinic (parallelepiped)	$\alpha, \beta, \gamma \neq 90^\circ$ 			
monoclinic (right prism with parallelogram base; here seen from above)	simple	centered		
	$\alpha \neq 90^\circ$ $\beta, \gamma = 90^\circ$ 	$\alpha \neq 90^\circ$ $\beta, \gamma = 90^\circ$ 		

orthorhombic (cuboid)	simple	base-centered	body-centered	face-centered
	$a \neq b \neq c$ 	$a \neq b \neq c$ 	$a \neq b \neq c$ 	$a \neq b \neq c$ 
tetragonal (square cuboid)	simple	body-centered		
	$a \neq c$ 	$a \neq c$ 		
rhombohedral or trigonal (trigonal trapezohedron)	$\alpha, \beta, \gamma \neq 90^\circ$ 			
hexagonal (centered regular hexagon)	$a \neq c$ 			
cubic (isometric; cube)	simple	body-centered	face-centered	
				

In geometry and crystallography, a **Bravais lattice** is a category of symmetry groups for translational symmetry in three directions, or correspondingly, a category of translation lattices.

Such symmetry groups consist of translations by vectors of the form

$$\mathbf{R} = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2 + n_3 \mathbf{a}_3,$$

where n_1 , n_2 , and n_3 are integers and a_1 , a_2 , and a_3 are three non-coplanar vectors, called *primitive vectors*.

These lattices are classified by space group of the translation lattice itself; there are 14 Bravais lattices in three dimensions; each can apply in one crystal system only. They represent the maximum symmetry a structure with the translational symmetry concerned can have.

All crystalline materials must, by definition fit in one of these arrangements (not including quasicrystals).

For convenience a Bravais lattice is depicted by a unit cell which is a factor 1, 2, 3 or 4 larger than the primitive cell. Depending on the symmetry of a crystal or other pattern, the fundamental domain is again smaller, up to a factor 48.

The Bravais lattices were studied by Moritz Ludwig Frankenheim (1801-1869), in 1842, who found that there were 15 Bravais lattices. This was corrected to 14 by A. Bravais in 1848.

See also

- Crystal structure
- Point group
- List of the 230 crystallographic 3D space groups

External links

- Overview of the 32 groups ^[3]
- Mineral galleries - Symmetry ^[1]
- all cubic crystal classes, forms and stereographic projections (interactive java applet) ^[2]

References

[1] <http://mineral.galleries.com/minerals/symmetry/symmetry.htm>

[2] http://www.ifg.uni-kiel.de/kubische_Formen

Mirror image

A **mirror image** is a reflected duplication that appears identical but in reverse. As an optical effect it results from reflection off of substances such as a mirror or water. It is also a concept in geometry and can also be used in a conceptualization process for 3-D structures.

In geometry

In two dimensions

In geometry, the mirror image of an object or two-dimensional figure is the virtual image formed by reflection in a plane mirror; it is of the same size as the original object, yet different, unless the object or figure has reflection symmetry (also known as a P-symmetry).

If a point of an object has coordinates $(-x, -y, z)$ then the image of this point (as reflected from the mirror in y, z plane) has coordinates $(-x, y, z)$ - so mirror reflection is a reversal of the coordinate axis perpendicular to the mirror's surface. Thus, a mirror image does not have reversed right and left (or up and down), but rather reversed front and back.

Two-dimensional mirror images can be seen in the reflections of mirrors or other reflecting surfaces, or on a printed surface seen inside out.



Mirror image

In three dimensions

The concept of mirror image can be extended to three-dimensional objects, including the inside parts, even if they are not transparent. The term then relates to structural as well as visual aspects. This is also called enantiomer or enantiomorph.

A mirror image appears three-dimensional if the observer moves. This is because the relative position of objects changes as the observer's perspective changes.^[1]

Looking through a mirror from different positions (but necessarily with the point of observation restricted to the halfspace on one side of the mirror) is like looking at the 3D mirror image of space; without further mirrors only the mirror image of the halfspace before the mirror is relevant; if there is another mirror, the mirror image of the other halfspace is too.

Uses

A text is sometimes deliberately displayed in mirror image, in order to be read through a mirror. Emergency vehicles such as ambulances or fire engines use mirror images in order to be read from a driver's rear-view mirror. Some movie theaters also use a Rear Window Captioning System to assist individuals with hearing impairments watching the film.



The word fire and its mirror image are displayed on the front of this fire engine

Systems of mirrors

In the case of two mirrors, in planes at an angle α , looking through both from the sector which is the intersection of the two halfspaces, is like looking at a version of the world rotated by an angle of 2α ; the points of observations and directions of looking for which this applies correspond to those for looking through a frame like that of the first mirror, and a frame at the mirror image with respect to the first plane, of the second mirror. If the mirrors have vertical edges then the left edge of the field of view is the plane through the right edge of the first mirror and the edge of the second mirror which is on the right when looked at directly, but on the left in the mirror image.

In the case of two parallel mirrors, looking through both once is like looking at a version of the world which is translated by twice the distance between the mirrors, in the direction perpendicular to them, away from the observer. Since the plane of the mirror in which one looks directly is beyond that of the other mirror, one always looks at an oblique angle, and the translation just mentioned has not only a component away from the observer, but also one in a perpendicular direction. The translated view can also be described by a translation of the observer in opposite direction. For example, with a vertical periscope, the shift of the world is away from the observer and down, both by the length of the periscope, but it is more practical to consider the equivalent shift of the observer: up, and backward.

See also

- relative direction
- handedness
- mirror writing
- flopped image
- flipped image

References

- [1] Adams, Cecil (1985-09-27). http://www.straightdope.com/classics/a2_071b.html "Are dogs unable to see 2-D images (mirrors, photos, TV)?" The Straight Dope. http://www.straightdope.com/classics/a2_071b.html. Retrieved on 2008-01-31.

External links

- Why do mirrors reverse images left to right? Why not up and down? (<http://www.d7s.com/mirror.htm>)
- The same question explained a little differently, with examples (<http://amoebacrunch.blogspot.com/2008/04/fun-with-mirror-images.html>)

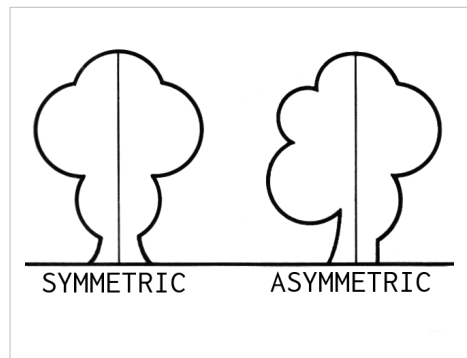
Asymmetry

Asymmetry is the absence of, or a violation of, a symmetry.

In organisms

Due to how cells divide in organisms, asymmetry in organisms is fairly usual in at least one dimension, with biological symmetry also being common in at least one dimension.

Louis Pasteur proposed that biological molecules are asymmetric because the cosmic [i.e. physical] forces that preside over their formation are themselves asymmetric. While at his time, and even now, the symmetry of physical processes are highlighted, it is known that there are fundamental physical asymmetries, starting with time. Further, truly fundamental left-right symmetry violation is now known in particle physics (see *Parity violation* below).



Usefulness to organisms

Asymmetry and important evolutionary traits, such as the left dolphin lung being smaller than the right to make room for the asymmetrical heart.

- Handedness is an asymmetry in skill development in people and animals. Training the neural pathways in a skill with one hand (or paw) takes less effort than doing the same with both hands.

Nature also provides several examples of handedness in traits that are usually symmetric. The following are examples of animals with obvious left-right asymmetries:

- Fiddler crabs have one big claw and one small claw.
- The narwhal's tusk is a left incisor which can grow up to 10 feet in length and forms a left-handed helix.
- Flatfish have evolved to swim with one side upward, and as a result have both eyes on one side of their heads.
- Several species of owls exhibit asymmetries in the size and positioning of their ears, which is thought to help locate prey.



Fiddler crab, *Uca pugnax*

As an indicator of unfitness

- Certain disturbances during the development of the organism, resulting in birth defects.
- Injuries after cell division that cannot be biologically repaired, such as a lost limb from an accident.

Since birth defects and injuries are likely to indicate poor health of the organism, defects resulting in asymmetry often put an animal at a disadvantage when it comes to finding a mate. In particular, a degree of facial symmetry is associated with physical attractiveness, but complete symmetry is both impossible and probably unattractive.

In chemistry

Certain molecules are chiral; that is, they cannot be superposed upon their mirror image.

Some sugars are chiral: glucose (also called *dextrose*) and fructose (sometimes called *levulose* or *invert sugar*) are chiral isomers of the same molecule, $C_6H_{12}O_6$. The word *invert* comes from the way that sugar syrups rotate plane-polarized light. A sucrose or glucose solution rotates the plane of polarization of the light to the right, while a fructose syrup rotates it strongly to the left.

In physics

Asymmetry arises in physics in a number of different realms.

Thermodynamics

Thermodynamics is asymmetrical in time: the entropy in a closed system can only increase with time. A consequence of this is Clausius' Second Law, which states that there is no thermodynamic process whose sole effect is to extract a quantity of heat from a colder reservoir and deliver it to a hotter reservoir.

Particle physics

Symmetry is one of the most powerful tools in particle physics, because it has become evident that practically all laws of nature originate in symmetries. Violations of symmetry therefore present theoretical and experimental puzzles that lead to a deeper understanding of nature. Asymmetries in experimental measurements also provide powerful handles that are often relatively free from background or systematic uncertainties.

Parity violation

Until the 1950s, it was believed that fundamental physics was left-right symmetric; i.e., that interactions were invariant under parity. Although parity is conserved in electromagnetism, strong interactions and gravity, it turns out to be violated in weak interactions. The Standard Model incorporates parity violation by expressing the weak interaction as a chiral gauge interaction. Only the left-handed components of particles and right-handed components of antiparticles participate in weak interactions in the Standard Model. A consequence of parity violation in particle physics is that neutrinos have only been observed as left-handed particles (and antineutrinos as right-handed particles).

In 1956-1957 Chien-Shiung Wu, E. Ambler, R. W. Hayward, D. D. Hoppes, and R. P. Hudson found a clear violation of parity conservation in the beta decay of cobalt-60. Simultaneously, R. L. Garwin, Leon Lederman, and R. Weinrich modified an existing cyclotron experiment and immediately verified parity violation.

CP violation

After the discovery of the violation of parity in 1956-57, it was believed that the combined symmetry of parity (P) and simultaneous charge conjugation (C), called *CP*, was preserved. For example, *CP* transforms a left-handed neutrino into a right-handed antineutrino. In 1964, however, James Cronin and Val Fitch provided clear evidence that *CP* symmetry was also violated in an experiment with neutral kaons.

CP violation is one of the necessary conditions for the generation of a baryon asymmetry in the universe.

Combining the *CP* symmetry with simultaneous time reversal (T) produces a combined symmetry called *CPT* symmetry. *CPT* symmetry must be preserved in any Lorentz invariant local quantum field theory with a Hermitian Hamiltonian. As of 2006, no violations of *CPT* symmetry have been observed.

Baryon asymmetry of the universe

The baryons (i.e., the protons and neutrons and the atoms that they comprise) observed in the universe are overwhelmingly matter as opposed to anti-matter. This asymmetry is called the baryon asymmetry of the universe.

Isospin violation

Isospin is the symmetry transformation of the weak interactions. The concept was first introduced by Werner Heisenberg in nuclear physics based on the observations that the masses of the neutron and the proton are almost identical and that the strength of the strong interaction between any pair of nucleons is the same, independent of whether they are protons or neutrons. This symmetry arises at a more fundamental level as a symmetry between up-type and down-type quarks. Isospin symmetry in the strong interactions can be considered as a subset of a larger flavor symmetry group, in which the strong interactions are invariant under interchange of different types of quarks. Including the strange quark in this scheme gives rise to the Eight-fold Way scheme for classifying mesons and baryons.

Isospin is violated by the fact that the masses of the up and down quarks are different, as well as by their different electric charges. Because this violation is only a small effect in most processes that involve the strong interactions, isospin symmetry remains a useful calculational tool, and its violation introduces corrections to the isospin-symmetric results.

In collider experiments

Because the weak interactions violate parity, collider processes that can involve the weak interactions typically exhibit asymmetries in the distributions of the final-state particles. These asymmetries are typically sensitive to the *difference* in the interaction between particles and antiparticles, or between left-handed and right-handed particles. They can thus be used as a sensitive measurement of differences in interaction strength and/or to distinguish a small asymmetric signal from a large but symmetric background.

- A **forward-backward asymmetry** is defined as $A_{FB} = (N_F - N_B) / (N_F + N_B)$, where N_F is the number of events in which some particular final-state particle is moving "forward" with respect to some chosen direction (e.g., a final-state electron moving in the same direction as the initial-state electron beam in electron-positron collisions), while N_B is the number of events with the final-state particle moving "backward". Forward-backward asymmetries were used by the LEP experiments to measure the difference in the interaction strength of the Z boson between left-handed and right-handed fermions, which provides a precision measurement of the weak mixing angle.
- A **left-right asymmetry** is defined as $A_{LR} = (N_L - N_R) / (N_L + N_R)$, where N_L is the number of events in which some initial- or final-state particle is left-polarized, while N_R is the corresponding number of right-polarized events. Left-right asymmetries in Z boson production and decay were measured at the Stanford Linear Collider using the event rates obtained with left-polarized versus right-polarized initial electron beams. Left-right asymmetries can also be defined as asymmetries in the polarization of final-state particles whose polarizations can be measured; e.g., tau leptons.
- A **charge asymmetry** or particle-antiparticle asymmetry is defined in a similar way. This type of asymmetry has been used to constrain the parton distribution functions of protons at the Tevatron from events in which a produced W boson decays to a charged lepton. The asymmetry between positively and negatively charged leptons as a function of the direction of the W boson relative to the proton beam provides information on the relative distributions of up and down quarks in the proton. Particle-antiparticle asymmetries are also used to extract measurements of CP violation from B meson and anti-B meson production at the BaBar and Belle experiments.

Lexical

Asymmetry is also relevant to grammar and linguistics, especially in the contexts of lexical analysis and transformational grammar.

Enumeration example: In English, there are grammatical rules for specifying coordinate items in an enumeration or series. Similar rules exist for programming languages and mathematical notation. These rules vary, and some require lexical asymmetry to be considered grammatically correct.

For example in standard written English:

We sell domesticated cats, dogs, and goldfish.	### in-line
asymmetric and grammatical	
We sell domesticated animals (cats, dogs, goldfish).	### in-line
symmetric and grammatical	
We sell domesticated animals (cats, dogs, goldfish,).	### in-line
symmetric and ungrammatical	
We sell domesticated animals:	### outline

symmetric and grammatical

- cats
- dogs
- goldfish

See also

- Information asymmetry
- Asymmetric multiprocessing

References

- Yuh-Nung Jan and Lily Yeh Jan, 1999. *Asymmetry across species*. Nature Cell Biology 1, E42 - E44 PMID 10559895

Non-abelian

Non-abelian may describe:

- Non-abelian group, in mathematics, a group that is not abelian (commutative)
- Non-abelian gauge theory, in physics, a gauge group that is non-abelian

See also:

- Abelian

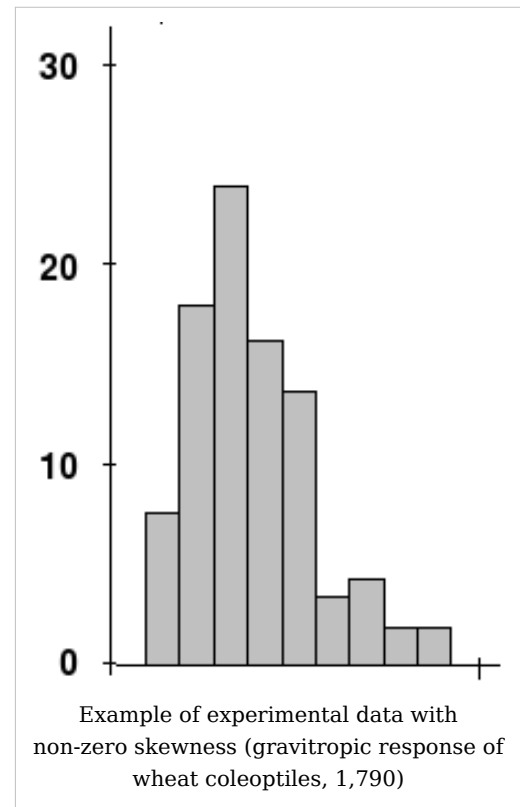
Skewness

In probability theory and statistics, **skewness** is a measure of the asymmetry of the probability distribution of a real-valued random variable.

Introduction

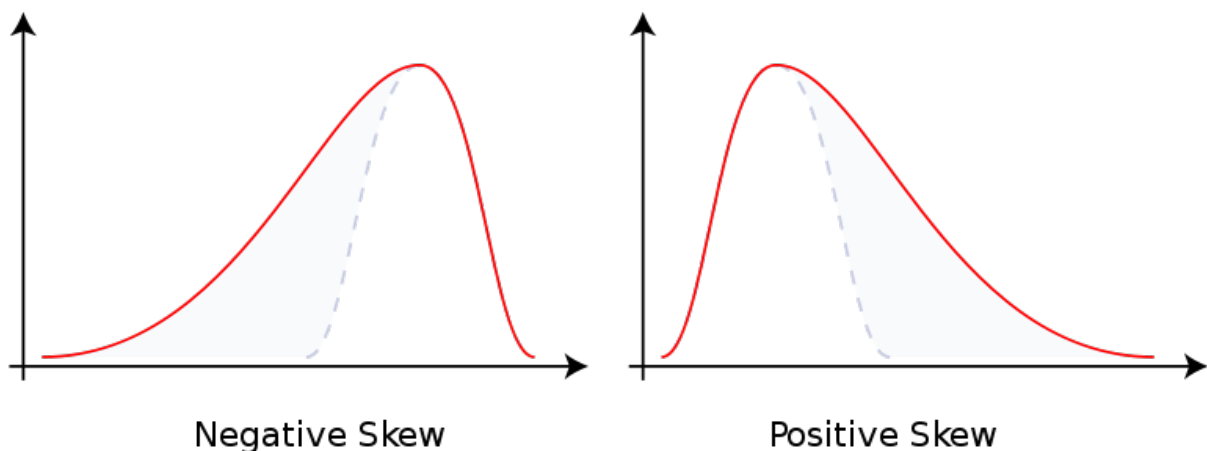
Consider the distribution in the figure. The bars on the right side of the distribution taper differently than the bars on the left side. These tapering sides are called *tails*, and they provide a visual means for determining which of the two kinds of skewness a distribution has:

1. **negative skew:** The left tail is longer; the mass of the distribution is concentrated on the right of the figure. It has relatively few low values. The distribution is said to be *left-skewed*. Example (observations): 1,1000,1001,1002,1003
2. **positive skew:** The right tail is longer; the mass of the distribution is concentrated on the left of the figure. It has relatively few high values. The distribution is said to be *right-skewed*. Example (observations): 1,2,3,4,100



In a skewed (unbalanced, lopsided) distribution, the mean is farther out in the long tail than is the median. If there is no skewness or the distribution is symmetric like the bell-shaped normal curve then the mean = median = mode.

Many textbooks teach a rule of thumb stating that the mean is right of the median under right skew, and left of the median under left skew. This rule fails with surprising frequency. It can fail in multimodal distributions, or in distributions where one tail is long but the other is heavy. Most commonly, though, the rule fails in discrete distributions where the areas to the left and right of the median are not equal. Such distributions not only contradict the textbook relationship between mean, median, and skew, they also contradict the textbook interpretation of the median.^[1]



Definition

Skewness, the third standardized moment, is written as γ_1 and defined as

$$\gamma_1 = \frac{\mu_3}{\sigma^3},$$

where μ_3 is the third moment about the mean and σ is the standard deviation. Equivalently, skewness can be defined as the ratio of the third cumulant κ_3 and the third power of the square root of the second cumulant κ_2 :

$$\gamma_1 = \frac{\kappa_3}{\kappa_2^{3/2}}.$$

This is analogous to the definition of kurtosis, which is expressed as the fourth cumulant divided by the fourth power of the square root of the second cumulant.

The skewness of a random variable X is sometimes denoted $\text{Skew}[X]$.

Sample skewness

For a sample of n values the *sample skewness* is

$$g_1 = \frac{m_3}{m_2^{3/2}} = \frac{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^3}{\left(\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2\right)^{3/2}},$$

where x_i is the i^{th} value, $\bar{x}$ is the sample mean, m_3 is the sample third central moment, and m_2 is the sample variance.

Given samples from a population, the equation for the sample skewness g_1 above is a biased estimator of the population skewness. The usual estimator of skewness is

$$G_1 = \frac{k_3}{k_2^{3/2}} = \frac{\sqrt{n(n-1)}}{n-2} g_1,$$

where k_3 is the unique symmetric unbiased estimator of the third cumulant and k_2 is the symmetric unbiased estimator of the second cumulant. Unfortunately G_1 is, nevertheless, generally biased. Its expected value can even have the opposite sign from the true skewness; compare unbiased estimation of standard deviation.

Properties

If Y is the sum of n independent random variables, all with the same distribution as X , then it can be shown that $\text{Skew}[Y] = \text{Skew}[X] / \sqrt{n}$.

Applications

Skewness has benefits in many areas. Many simplistic models assume normal distribution i.e. data is symmetric about the mean. The normal distribution has a skewness of zero. But in reality, data points are not perfectly symmetric. So, an understanding of the skewness of the dataset indicates whether deviations from the mean are going to be positive or negative.

D'Agostino's K-squared test is a goodness-of-fit normality test based on sample skewness and sample kurtosis.

Pearson's skewness coefficients

Karl Pearson suggested simpler calculations as a measure of skewness: The Pearson mode skewness^[2], defined by

- $(\text{mean} - \text{mode}) / \text{standard deviation}$,

Pearson's first skewness coefficient^[3], defined by

- $3 (\text{mean} - \text{mode}) / \text{standard deviation}$,

as well as Pearson's second skewness coefficient, defined by

- $3 (\text{mean} - \text{median}) / \text{standard deviation}$.

There is no guarantee that these will be the same sign as each other or as the ordinary definition of skewness.

See also

- Skewness risk
- Kurtosis risk
- Shape parameters
- Skew normal distribution
- Lake Wobegon effect#Asymmetric distributions
- D'Agostino's K-squared test

References

[1] Textbook rule often fails (<http://www.amstat.org/publications/jse/v13n2/vonhippel.html>)

[2] Eric W. Weisstein, *Pearson Mode Skewness* (<http://mathworld.wolfram.com/PearsonModeSkewness.html>) at MathWorld.

[3] Eric W. Weisstein, *Pearson's skewness coefficients* (<http://mathworld.wolfram.com/PearsonsSkewnessCoefficients.html>) at MathWorld.

External links

- An Asymmetry Coefficient for Multivariate Distributions (<http://petitjeanmichel.free.fr/itoweb.petitjean.skewness.html>) by Michel Petitjean

Chirality

Chirality, or "**handedness**", (Greek, *χειρ*, *kheir*: "hand") is a property of asymmetry important in several branches of science.

An object or a system is **chiral** if it cannot be superposed on its mirror image. A chiral object and its mirror image are called **enantiomorphs** (Greek *opposite forms*) or, when referring to molecules, **enantiomers**. A non-chiral object is called **achiral** (sometimes also **amphichiral**) and can be superposed on its mirror image.

Chirality may also refer to:

- Chirality (chemistry) of some molecules
- Chirality (mathematics) of mathematical objects
- Chirality (physics) of some subatomic particles
- The chirality of certain crystalline solids. Of the 230 existing space groups 65 are chiral. Sodium chlorate is an achiral ionic compound but crystallizes in a chiral $P2_13$ space group. An example of an achiral organic compound forming chiral crystals is benzil. Racemic acid is the racemic form of tartaric acid forming a mixture of two enantiomorphic crystals each form consisting of one of the two enantiomers.
- The chirality of surfaces. Materials with bulk chirality can be cleaved exposing a chiral surface.
- Chirality (electromagnetics) is an indication of the direction of the rotation of the electric and magnetic fields of a circularly-polarized wave.
- Chirality (manga)
- Chirality is important in forensic science, as it can indicate whether a knot was tied by a left- or right-handed person.

See also

- Handedness
 - Rigid body
 - Symmetry
-

Chirality (mathematics)

In geometry, a figure is **chiral** (and said to have **chirality**) if it is not identical to its mirror image, or more particularly if it cannot be mapped to its mirror image by rotations and translations alone. A chiral object and its mirror image are said to be **enantiomorphs**. The word *chirality* is derived from the Greek χείρ (cheir), the hand, the most familiar chiral object; the word *enantiomorph* stems from the Greek εναντιος (enantios) 'opposite' and μορφη (morphē) 'form'. A non-chiral figure is called **achiral** or **amphichiral**.

The helix (and by extension a spun string, a screw, a propeller, etc.) and Möbius strip are chiral three-dimensional objects. The J, L, S and Z-shaped *tetrominoes* of the popular video game Tetris also exhibit chirality, but only in a two-dimensional space.

Many other familiar objects exhibit the same chiral symmetry of the human body: gloves, glasses, shoes, legs on a pair of pants, etc. A similar notion of chirality is considered in knot theory, as explained below.

Some chiral three-dimensional objects, such as the helix, can be assigned a right or left handedness, according to the right-hand rule.

Chirality and symmetry group

A figure is achiral if and only if its symmetry group contains at least one *orientation-reversing* isometry. (In Euclidean geometry any isometry can be written as $v \mapsto Av + b$ with an orthogonal matrix A and a vector b . The determinant of A is either 1 or -1 then. If it is -1 the isometry is *orientation-reversing*, otherwise it is *orientation-preserving*.)

Chirality in three dimensions

In three dimensions, every figure which possesses a plane of symmetry or a center of symmetry is achiral. (A *plane of symmetry* of a figure F is a plane P , such that F is invariant under the mapping $(x, y, z) \mapsto (x, y, -z)$, when P is chosen to be the x - y -plane of the coordinate system. A *center of symmetry* of a figure F is a point C , such that F is invariant under the mapping $(x, y, z) \mapsto (-x, -y, -z)$, when C is chosen to be the origin of the coordinate system.) Note, however, that there are achiral figures lacking both plane and center of symmetry. An example is the figure

$$F_0 = \{(1, 0, 0), (0, 1, 0), (-1, 0, 0), (0, -1, 0), (2, 1, 1), (-1, 2, -1), (-2, -1, 1), (1, -2, -1)\}$$

which is invariant under the orientation reversing isometry $(x, y, z) \mapsto (-y, x, -z)$ and thus achiral, but it has neither plane nor center of symmetry. The figure

$$F_1 = \{(1, 0, 0), (-1, 0, 0), (0, 2, 0), (0, -2, 0), (1, 1, 1), (-1, -1, -1)\}$$

also is achiral as the origin is a center of symmetry, but it lacks a plane of symmetry.

Note also that achiral figures can have a center axis.

Chirality in two dimensions

In two dimensions, every figure which possesses an axis of symmetry is achiral, and it can be shown that every *bounded* achiral figure must have an axis of symmetry. (An *axis of symmetry* of a figure F is a line L , such that F is invariant under the mapping $(x, y) \mapsto (x, -y)$, when L is chosen to be the x -axis of the coordinate system.) Consider the following pattern:

```
> > > > > > > > > >
> > > > > > > > > >
```

This figure is chiral, as it is not identical to its mirror image from either axis:

```
> > > > > > > > > >      or      <
< < < < < < < < < <
> > > > > > > > > >      <
< < < < < < < < < <
```

But if one prolongs the pattern in both directions to infinity, one receives an (unbounded) achiral figure which has no axis of symmetry. Its symmetry group is a frieze group generated by a single glide reflection.

Knot theory

A knot is called achiral if it can be continuously deformed into its mirror image, otherwise it is called chiral. For example the unknot and the figure-eight knot are achiral, whereas the trefoil knot is chiral.

See also

- Chirality (physics)
- Chirality (chemistry)
- Orientation (mathematics)
- Handedness
- Asymmetry
- Skewness
- Vertex algebra

External links

- The Mathematical Theory of Chirality ^[1] by Michel Petitjean
- Chiral Polyhedra ^[2] by Eric W. Weisstein, The Wolfram Demonstrations Project.

References

- [1] <http://petitjeanmichel.free.fr/itoweb.petitjean.html>
 [2] <http://demonstrations.wolfram.com/ChiralPolyhedra/>

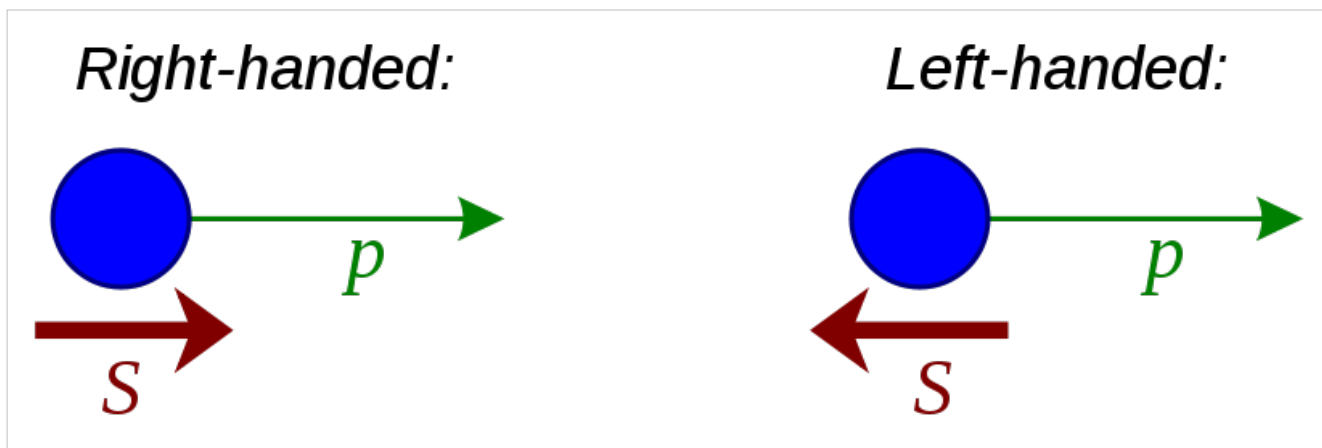
Chirality (physics)

A phenomenon is said to be **chiral** if it is not identical to its mirror image (see Chirality). The spin of a particle may be used to define a **handedness** (aka **chirality**) for that particle. A symmetry transformation between the two is called **parity**. The action of parity acting on a Dirac fermion is called **chiral symmetry**.

An experiment on the weak decay of cobalt-60 nuclei carried out by Chien-Shiung Wu and collaborators in 1957 demonstrated that parity is not a symmetry of the universe.

Chirality and helicity

The helicity of a particle is **Right-handed** if the direction of its spin is the same as the direction of its motion. It is **Left-handed** if the directions of spin and motion are opposite. By convention for rotation, a standard clock, tossed with its face directed forwards, has **Left-handed** helicity. Mathematically, helicity is the sign of the projection of the spin vector onto the momentum vector: **Left** is negative, **Right** is positive.



The chirality of a particle is more abstract. It is determined by whether the particle transforms in a right or left-handed representation of the Poincaré group. (However, some representations, such as Dirac spinors, have both right and left-handed components. In cases like this, we can define projection operators that project out either the right or left hand components and discuss the right and left-handed portions of the representation.)

For massless particles — such as the photon, the gluon, and the (hypothetical) graviton — chirality is the same as helicity; a given massless particle appears to spin in the same direction along its axis of motion regardless of point of view of the observer.

For particles that do have mass — such as electrons, quarks, and neutrinos — chirality and helicity must be distinguished. In the case of these particles, it is possible for an observer to change to a reference frame that overtakes the spinning particle, in which case the particle will then appear to move backwards, and its helicity (which may be thought of as 'apparent chirality') will be reversed.

A *massless* particle moves with the speed of light, so a real observer (who must always travel at less than the speed of light) cannot be in any reference frame where the particle appears to reverse its relative direction, meaning that all real observers see the same chirality. Because of this, the direction of spin of massless particles is not affected by a Lorentz boost (change of viewpoint) in the direction of motion of the particle, and the sign

of the projection (helicity) is fixed for all reference frames: the helicity is a *relativistic invariant*.

With the discovery of neutrino oscillation, which implies that neutrinos have mass, the only observed massless particle is the photon. The gluon also is expected to be massless, although the assumption that it is massless has not been tested. Hence, these are the only two particles now known for which helicity could be identical to chirality, and only one that has been confirmed by measurement. All other observed particles have mass and thus may have different helicities in different reference frames. It is still possible that as-yet unobserved particles, like the graviton, might be massless, and hence have invariant helicity like the photon. It is also not known for certain that the gluon is actually massless, it is only supposed; all that is certain from measurement is that if it is not zero then its mass must be very small. Because of confinement, observation of gluons is complicated and difficult; it may be that they cannot exist as a free particle and only come in bound states called glueballs.

Chiral theories

It has been observed that only left-handed fermions interact with the weak interaction. In most circumstances, two left-handed fermions interact more strongly than right-handed or opposite-handed fermions. Experiments sensitive to this effect imply that the universe has a preference for left-handed chirality, which violates a symmetry of the other forces of nature.

Chirality for a Dirac fermion ψ is defined by the operator γ^5 , which has eigenvalues ± 1 . Any Dirac field can therefore be projected into its left- or right-handed component by the operation of the projection operator $(1-\gamma^5)/2$ or $(1+\gamma^5)/2$ acting on ψ . The coupling of the weak interaction to fermions is proportional to such a projection operator, which is responsible for its parity symmetry violation.

A common source of confusion is due to conflating this operator with the helicity operator. Since the helicity of massive particles is frame-dependent, it might seem that the same particle would interact with the weak force according to one frame of reference, but not another. The resolution to this paradox is that the chirality operator is equivalent to helicity for massless fields only, for which helicity is not frame-dependent. For massive particles, chirality is not the same as helicity so there is no frame dependence of the weak interaction: a particle that interacts with the weak force does so in every frame.

A theory that is asymmetric between chiralities is called a *chiral theory*, while a parity symmetric theory is sometimes called a *vector theory*. Most pieces of the Standard Model of physics are non-chiral, which may be due to problems of anomaly cancellation in chiral theories. Quantum chromodynamics is an example of a vector theory since both chiralities of all quarks appear in the theory, and couple the same way.

The electroweak theory developed in the mid twentieth century is an example of a *chiral theory*. Originally, it assumed that neutrinos were massless, and only assumed the existence of left-handed neutrinos (along with their complementary right-handed antineutrinos). After the observation of neutrino oscillations, which imply that neutrinos are massive like all other fermions, the revised theories of the electroweak interaction now include both right- and left-handed neutrinos. However, it is still a chiral theory, as it does not respect parity symmetry.

The exact nature of the neutrino is still unsettled and so the electroweak theories that have been proposed are different, but most accommodate the chirality of neutrinos in the same way as was already done for all other fermions.

Chiral symmetry

Vector gauge theories with massless Dirac fermion fields ψ exhibit chiral symmetry, i.e., rotating the left-handed and the right-handed components independently makes no difference to the theory. We can write this as the action of rotation on the fields:

$$\psi_L \rightarrow e^{i\theta_L} \psi_L \quad \text{and} \quad \psi_R \rightarrow \psi_R$$

or

$$\psi_L \rightarrow \psi_L \quad \text{and} \quad \psi_R \rightarrow e^{i\theta_R} \psi_R.$$

With N flavors, we have unitary rotations instead: $SU(N)_L \times SU(N)_R$.

Massive fermions do not exhibit chiral symmetry. One also says that the mass term in the Lagrangian, $m\bar{\psi}\psi$ breaks chiral symmetry explicitly. Spontaneous chiral symmetry breaking may also occur in some theories, most notably in quantum chromodynamics.

References and external links

- History of science: parity violation ^[1]
- D.A. Bromley (2000). *Gauge Theory of Weak Interactions*. Springer. ISBN 3-540-67672-4.
- Gordon L. Kane (1987). *Modern Elementary Particle Physics*. Perseus Books. ISBN 0-201-11749-5.
- To see a summary of the differences and similarities between chirality and helicity (those covered here and more) in chart form, go to Pedagogic Aids to Quantum Field Theory ^[2] and click on the link near the bottom of the page entitled "Chirality and Helicity Summary". To see an in depth discussion of the two with examples, which also shows how chirality and helicity approach the same thing as speed approaches that of light, click the link entitled "Chirality and Helicity in Depth" on the same page.

See also

- Electroweak theory
- Chirality (chemistry)
- Chirality (mathematics)
- Chiral symmetry
- Handedness
- Spinors and Dirac fields

References

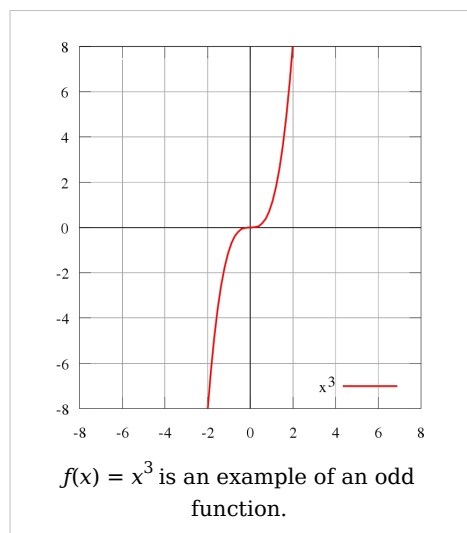
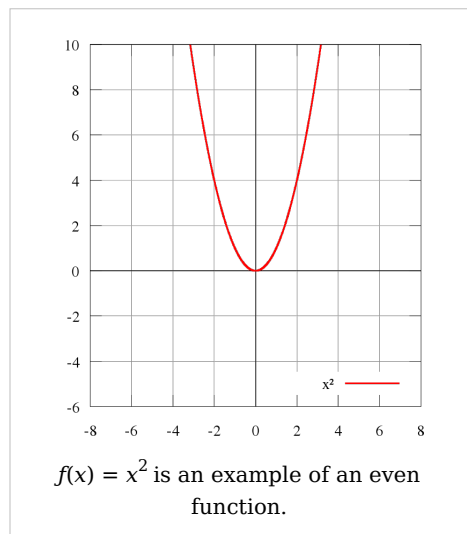
- [1] <http://ccreweb.org/documents/parity/parity.html>
 [2] <http://www.quantumfieldtheory.info>

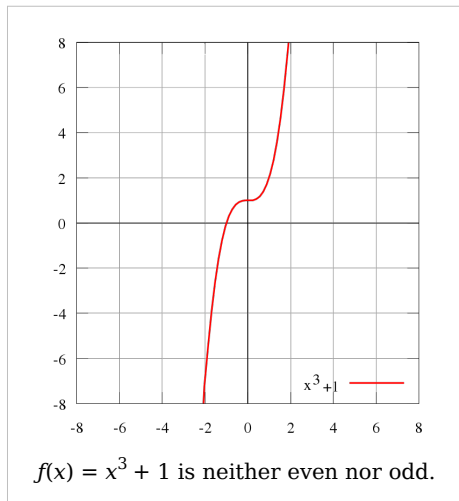
Even and odd functions

In mathematics, **even functions** and **odd functions** are functions which satisfy particular symmetry relations, with respect to taking additive inverses. They are important in many areas of mathematical analysis, especially the theory of power series and Fourier series. They are named for the parity of the powers of the power functions which satisfy each condition: the function $f(x) = x^n$ is an even function if n is an even integer, and it is an odd function if n is an odd integer.

Even functions

Let $f(x)$ be a real-valued function of a real variable. Then f is **even** if the following equation holds for all x in the domain of f :





$$f(x) = f(-x).$$

Geometrically, the graph of an even function is symmetric with respect to the y -axis, meaning that its graph remains unchanged after reflection about the y -axis.

Examples of even functions are $|x|$, x^2 , x^4 , $\cos(x)$, and $\cosh(x)$.

Odd functions

Again, let $f(x)$ be a real-valued function of a real variable. Then f is **odd** if the following equation holds for all x in the domain of f :

$$-f(x) = f(-x),$$

or

$$f(x) + f(-x) = 0.$$

Geometrically, the graph of an odd function has rotational symmetry with respect to the origin, meaning that its graph remains unchanged after rotation of 180 degrees about the origin.

Examples of odd functions are x , x^3 , $\sin(x)$, $\sinh(x)$, and $\operatorname{erf}(x)$.

Some facts

Note: A function's being odd or even does not imply differentiability, or even continuity. For example, the Dirichlet function is even, but is nowhere continuous. Properties involving Fourier series, Taylor series, derivatives and so on may only be used when they can be assumed to exist.

Basic properties

- The only function which is *both* even and odd is the constant function which is identically zero (i.e., $f(x) = 0$ for all x).
- The sum of an even and odd function is neither even nor odd, unless one of the functions is identically zero.
- The sum of two even functions is even, and any constant multiple of an even function is even.
- The sum of two odd functions is odd, and any constant multiple of an odd function is odd.
- The product of two even functions is an even function.

- The product of two odd functions is an even function.
- The product of an even function and an odd function is an odd function.
- The quotient of two even functions is an even function.
- The quotient of two odd functions is an even function.
- The quotient of an even function and an odd function is an odd function.
- The derivative of an even function is odd.
- The derivative of an odd function is even.
- The composition of two even functions is even, and the composition of two odd functions is odd.
- The composition of an even function and an odd function is even.
- The composition of any function with an even function is even (but not vice versa).
- The integral of an odd function from $-A$ to $+A$ is zero (where A is finite, and the function has no vertical asymptotes between $-A$ and A).
- The integral of an even function from $-A$ to $+A$ is twice the integral from 0 to $+A$ (where A is finite, and the function has no vertical asymptotes between $-A$ and A).

Series

- The Maclaurin series of an even function includes only even powers.
- The Maclaurin series of an odd function includes only odd powers.
- The Fourier series of a periodic even function includes only cosine terms.
- The Fourier series of a periodic odd function includes only sine terms.

Algebraic structure

- Any linear combination of even functions is even, and the even functions form a vector space over the reals. Similarly, any linear combination of odd functions is odd, and the odd functions also form a vector space over the reals. In fact, the vector space of *all* real-valued functions is the direct sum of the subspaces of even and odd functions. In other words, every function $f(x)$ can be written uniquely as the sum of an even function and an odd function:

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x),$$

where

$$f_{\text{even}}(x) = \frac{f(x) + f(-x)}{2}$$

is even and

$$f_{\text{odd}}(x) = \frac{f(x) - f(-x)}{2}$$

is odd.

- The even functions form a commutative algebra over the reals. However, the odd functions do *not* form an algebra over the reals.

Harmonics

In signal processing, harmonic distortion occurs when a sine wave signal is multiplied by a non-linear transfer function. The type of harmonics produced depend on the transfer function:^[1]

- When the transfer function is even, the resulting signal will consist of only even harmonics of the input sine wave;
 - The fundamental is also an odd harmonic, so will not be present.
 - A simple example is a full-wave rectifier.
- When it is odd, the resulting signal will consist of only odd harmonics of the input sine wave;
 - The output signal will be half-wave symmetric.
 - A simple example is clipping in a symmetric push-pull amplifier.
- When it is asymmetric, the resulting signal may contain either even or odd harmonics;
 - A simple example is clipping in an asymmetrical class A amplifier.

References

[1] Ask the Doctors: Tube vs. Solid-State Harmonics (<http://www.uaudio.com/webzine/2005/october/content/content2.html>)

See also

- Hermitian function for a generalization in complex numbers.
 - Taylor series
 - Fourier series
-

Spin group

In mathematics the **spin group** $\text{Spin}(n)$ is the double cover of the special orthogonal group $\text{SO}(n)$, such that there exists a short exact sequence of Lie groups

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}(n) \rightarrow \text{SO}(n) \rightarrow 1.$$

As a Lie group $\text{Spin}(n)$ therefore shares its dimension, $n(n-1)/2$, and its Lie algebra with the special orthogonal group. For $n > 2$, $\text{Spin}(n)$ is simply connected and so coincides with the universal cover of $\text{SO}(n)$.

The non-trivial element of the kernel is denoted -1 , which should not be confused with the orthogonal transform of reflection through the origin, generally denoted $-I$.

$\text{Spin}(n)$ can be constructed as a subgroup of the invertible elements in the Clifford algebra $\text{Cl}(n)$.

Accidental isomorphisms

In low dimensions, there are isomorphisms among the classical Lie groups called *accidental isomorphisms*. For instance, there are isomorphisms between low dimensional spin groups and certain classical Lie groups, due to low dimensional isomorphisms between the root systems of the different families of simple Lie algebras. Specifically, we have

$$\text{Spin}(1) = \text{O}(1)$$

$$\text{Spin}(2) = \text{U}(1) = \text{SO}(2)$$

$$\text{Spin}(3) = \text{Sp}(1) = \text{SU}(2)$$

$$\text{Spin}(4) = \text{Sp}(1) \times \text{Sp}(1)$$

$$\text{Spin}(5) = \text{Sp}(2)$$

$$\text{Spin}(6) = \text{SU}(4)$$

There are certain vestiges of these isomorphisms left over for $n = 7, 8$ (see $\text{Spin}(8)$ for more details). For higher n , these isomorphisms disappear entirely.

Indefinite signature

In indefinite signature, the spin group $\text{Spin}(p, q)$ is constructed through Clifford algebras in a similar way to standard spin groups. It is a connected double cover of $\text{SO}_0(p, q)$, the connected component of the identity of the indefinite orthogonal group $\text{SO}(p, q)$ (there are a variety of conventions on the connectedness of $\text{Spin}(p, q)$; in this article, it is taken to be connected for $p+q > 2$). As in definite signature, there are some accidental isomorphisms in low dimensions:

$$\text{Spin}(1, 1) = \text{GL}(1, \mathbf{R})$$

$$\text{Spin}(2, 1) = \text{SL}(2, \mathbf{R})$$

$$\text{Spin}(3, 1) = \text{SL}(2, \mathbf{C})$$

$$\text{Spin}(2, 2) = \text{SL}(2, \mathbf{R}) \times \text{SL}(2, \mathbf{R})$$

$$\text{Spin}(4, 1) = \text{Sp}(1, 1)$$

$$\text{Spin}(3, 2) = \text{Sp}(4, \mathbf{R})$$

$$\text{Spin}(5, 1) = \text{SL}(2, \mathbf{H})$$

$$\text{Spin}(4, 2) = \text{SU}(2, 2)$$

$$\text{Spin}(3,3) = \text{SL}(4, \mathbf{R})$$

Note that $\text{Spin}(p,q) = \text{Spin}(q,p)$.

Topological considerations

Connected and simply connected Lie groups are classified by their Lie algebra. So if G is a connected Lie group with a simple Lie algebra, with $G^\square$ the universal cover of G , there is an inclusion

$$\pi_1(G) \subset Z(G'),$$

with $Z(G^\square)$ the centre of $G^\square$. This inclusion and the Lie algebra $\mathfrak{g}$ of G determine G entirely (note that it is not the fact that $\mathfrak{g}$ and $\pi_1(G)$ determine G entirely; for instance $\text{SL}(2, \mathbf{R})$ and $\text{PSL}(2, \mathbf{R})$ have the same Lie algebra and same fundamental group $\mathbb{Z}$, but are not isomorphic).

The definite signature $\text{Spin}(n)$ are all simply connected for $(n > 2)$, so they are the universal coverings for $\text{SO}(n)$. In indefinite signature, the maximal compact connected subgroup of $\text{Spin}(p,q)$ is

$$(\text{Spin}(p) \times \text{Spin}(q)) / \{(1, 1), (-1, -1)\}.$$

This allows us to calculate the fundamental groups of $\text{Spin}(p,q)$:

$$\pi_1(\text{Spin}(p, q)) = \begin{cases} \{0\} & (p, q) = (1, 1) \text{ or } (1, 0) \\ \{0\} & p > 2, q = 0, 1 \\ \mathbb{Z} & (p, q) = (2, 0) \text{ or } (2, 1) \\ \mathbb{Z} \times \mathbb{Z} & (p, q) = (2, 2) \\ \mathbb{Z} & p > 2, q = 2 \\ \mathbb{Z}_2 & p > 2, q > 2 \end{cases}$$

For $p, q > 2$, this implies that the map $\pi_1(\text{Spin}(p, q)) \rightarrow \pi_1(\text{SO}(p, q))$ is given by $1 \in \mathbb{Z}_2$ going to $(1, 1) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. For $p=2, q>2$, this map is given by $1 \in \mathbb{Z} \rightarrow (1, 1) \in \mathbb{Z} \times \mathbb{Z}_2$. And finally, for $p=q=2$, $(1, 0) \in \mathbb{Z} \times \mathbb{Z}$ is sent to $(1, 1) \in \mathbb{Z} \times \mathbb{Z}$ and $(0, 1)$ is sent to $(1, -1)$.

See also

- Pin group
- Spinor
- Spinor bundle
- Anyon
- Spin structure
- Clifford algebra
- Orientation entanglement
- Complex Spin Group
- Clifford analysis

Lagrangian mechanics

Lagrangian mechanics is a re-formulation of classical mechanics that combines conservation of momentum with conservation of energy. It was introduced by Italian mathematician Lagrange in 1788. In Lagrangian mechanics, the trajectory of a system of particles is derived by solving Lagrange's equation, given herein, for each of the system's generalized coordinates. The fundamental lemma of the calculus of variations shows that solving Lagrange's equation is equivalent to finding the path that minimizes the action functional, a quantity that is the integral of the Lagrangian over time.

The use of generalized coordinates may considerably simplify a system's analysis. For example, consider a small frictionless bead traveling in a groove. If one is tracking the bead as a particle, calculation of the motion of the bead using Newtonian mechanics would require solving for the time-varying constraint force required to keep the bead in the groove. For the same problem using Lagrangian mechanics, one looks at the path of the groove and chooses a set of *independent* generalized coordinates that completely characterize the possible motion of the bead. This choice eliminates the need for the constraint force to enter into the resultant system of equations. There are fewer equations since one is not directly calculating the influence of the groove on the bead at a given moment.

Lagrange's equations

The equations of motion in Lagrangian mechanics are *Lagrange's equations*, also known as *Euler-Lagrange equations*. Below, we sketch out the derivation of Lagrange's equation. Please note that in this context, V is used rather than U for potential energy and T replaces K for kinetic energy. See the references for more detailed and more general derivations.

Start with D'Alembert's principle for the virtual work of applied forces, $\mathbf{F}_i$, and inertial forces on a three dimensional accelerating system of n particles, i , whose motion is consistent with its constraints:^{[1] :269}

$$\delta W = \sum_{i=1}^n (\mathbf{F}_i - m_i \mathbf{a}_i) \cdot \delta \mathbf{r}_i = 0.$$

δW is the virtual work

$\delta \mathbf{r}_i$ is the virtual displacement of the system, consistent with the constraints

m_i are the masses of the particles in the system

$\mathbf{a}_i$ are the accelerations of the particles in the system

$m_i \mathbf{a}_i$ together as products represent the time derivatives of the system momenta, aka. inertial forces

i is an integer used to indicate (via subscript) a variable corresponding to a particular particle

n is the number of particles under consideration

Break out the two terms:

$$\delta W = \sum_{i=1}^n \mathbf{F}_i \cdot \delta \mathbf{r}_i - \sum_{i=1}^n m_i \mathbf{a}_i \cdot \delta \mathbf{r}_i = 0.$$

Assume that the following transformation equations from m independent generalized coordinates, q_j , hold:^{[1] :260}

$$\mathbf{r}_1 = \mathbf{r}_1(q_1, q_2, \dots, q_m, t),$$

$$\mathbf{r}_2 = \mathbf{r}_2(q_1, q_2, \dots, q_m, t), \dots$$

$$\mathbf{r}_n = \mathbf{r}_n(q_1, q_2, \dots, q_m, t).$$

m (without a subscript) indicates the total number generalized coordinates

An expression for the virtual displacement (differential), $\delta \mathbf{r}_i$ of the system for *time-independent constraints* is^{[1] :264}

$$\delta \mathbf{r}_i = \sum_{j=1}^m \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j.$$

j is an integer used to indicate (via subscript) a variable corresponding to a generalized coordinate

The applied forces may be expressed in the generalized coordinates as generalized forces, Q_j ,^{[1] :265}

$$Q_j = \sum_{i=1}^n \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}.$$

Combining the equations for δW , $\delta \mathbf{r}_i$, and Q_j yields the following result after pulling the sum out of the dot product in the second term:^{[1] :269}

$$\delta W = \sum_{j=1}^m Q_j \delta q_j - \sum_{j=1}^m \sum_{i=1}^n m_i \mathbf{a}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = 0.$$

Substituting in the result from the kinetic energy relations to change the inertial forces into a function of the kinetic energy leaves^{[1] :270}

$$\delta W = \sum_{j=1}^m Q_j \delta q_j - \sum_{j=1}^m \left(\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right) \delta q_j = 0.$$

In the above equation, δq_j is arbitrary, though it is—by definition—consistent with the constraints. So the relation must hold term-wise:^{[1] :270}

$$Q_j = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j}.$$

If the $\mathbf{F}_i$ are conservative, they may be represented by a scalar potential field, V .^{[1] :266 & 270}

$$\mathbf{F}_i = -\nabla V \Rightarrow Q_j = - \sum_{i=1}^n \nabla V \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = - \frac{\partial V}{\partial q_j}.$$

The previous result may be easier to see by recognizing that V is a function of the $\mathbf{r}_i$, which are in turn functions of q_j , and then applying the chain rule to the derivative of V with respect to q_j .

The definition of the Lagrangian is^{[1] :270}

$$\mathcal{L} = T - V.$$

Since the potential field is only a function of position, not velocity, **Lagrange's equations** are as follows:^{[1] :270}

$$0 = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) - \frac{\partial \mathcal{L}}{\partial q_j}.$$

This is consistent with the results derived above and may be seen by differentiating the right side of the Lagrangian with respect to $\dot{q}_j$ and time, and solely with respect to q_j , adding the results and associating terms with the equations for F_i and Q_j .

In a more general formulation, the forces could be both potential and viscous. If an appropriate transformation can be found from the F_i , Rayleigh suggests using a dissipation function, D , of the following form:^{[1] :271}

$$D = \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m C_{jk} \dot{q}_j \dot{q}_k.$$

C_{jk} are constants that are related to the damping coefficients in the physical system, though not necessarily equal to them

If D is defined this way, then^{[1] :271}

$$Q_j = -\frac{\partial V}{\partial q_j} - \frac{\partial D}{\partial \dot{q}_j} \text{ and}$$

$$0 = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) - \frac{\partial \mathcal{L}}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j}.$$

Kinetic energy relations

The kinetic energy, T , for the system of particles is defined by^{[1] :269}

$$T = \frac{1}{2} \sum_{i=1}^n m_i \mathbf{v}_i \cdot \mathbf{v}_i.$$

The partial derivative of T with respect to the time derivatives of the generalized coordinates, $\dot{q}_j$, is^{[1] :269}

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^n m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}.$$

The previous result may be difficult to visualize. As a result of the product rule, the derivative of a general dot product $\frac{d}{dx}(\mathbf{f}(x) \cdot \mathbf{g}(x))$ is $\mathbf{f}(x) \cdot \frac{d}{dx} \mathbf{g}(x) + \mathbf{g}(x) \cdot \frac{d}{dx} \mathbf{f}(x)$. This general result may be seen by briefly stepping into a Cartesian coordinate system, recognizing that the dot product is (there) a term-by-term product sum, and also recognizing that the derivative of a sum is the sum of its derivatives. In our case, $\mathbf{f}$ and $\mathbf{g}$ are equal to $\mathbf{v}$, which is why the factor of one half disappears.

According to the chain rule and the coordinate transformation equations given above for $\mathbf{r}$, its time derivative, $\mathbf{v}$, is:^{[1] :264}

$$\mathbf{v}_i = \sum_{k=1}^m \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t}.$$

Together, the definition of $\mathbf{v}_i$ and the total differential, $d\mathbf{r}_i$, suggest that^{[1] :269}

$$\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}.$$

[Remember that : $\frac{\partial}{\partial \dot{q}_k} A \dot{q}_k = A$. Also remember that in the sum, there is only one $\dot{q}_j$.]

Substituting this relation back into the expression for the partial derivative of T gives^{[1] :269}

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^n m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{r}_i}{\partial \dot{q}_j}.$$

Taking the time derivative gives^{[1] :270}

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) = \sum_{i=1}^n \left[m_i \mathbf{a}_i \cdot \frac{\partial \mathbf{r}_i}{\partial \dot{q}_j} + m_i \mathbf{v}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial \dot{q}_j} \right) \right].$$

Using the chain rule on the last term gives^{[1] :270}

$$\frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial \dot{q}_j} \right) = \sum_{k=1}^m \frac{\partial^2 \mathbf{r}_i}{\partial \dot{q}_j \partial \dot{q}_k} \dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial \dot{q}_j \partial t}.$$

From the expression for $\mathbf{v}_i$, one sees that^{[1] :270}

$$\frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial \dot{q}_j} \right) = \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}.$$

This allows simplification of the last term,^{[1] :270}

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) = \sum_{i=1}^n \left[m_i \mathbf{a}_i \cdot \frac{\partial \mathbf{r}_i}{\partial \dot{q}_j} + m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \right].$$

The partial derivative of T with respect to the generalized coordinates, q_j , is^{[1] :270}

$$\frac{\partial T}{\partial q_j} = \sum_{i=1}^n \frac{\partial [\frac{1}{2} m_i v_i^2]}{\partial q_j} = \sum_{i=1}^n \frac{\partial [\frac{1}{2} m_i (\mathbf{v}_i \cdot \mathbf{v}_i)]}{\partial q_j} = \frac{1}{2} \sum_{i=1}^n \left[m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j} + m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j} \right] = \sum_{i=1}^n m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j}.$$

[This last result may be obtained by doing a partial differentiation directly on the kinetic energy definition represented by the first equation.] The last two equations may be combined to give an expression for the inertial forces in terms of the kinetic energy:^{[1] :270}

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = \sum_{i=1}^n m_i \mathbf{a}_i \cdot \frac{\partial \mathbf{r}_i}{\partial \dot{q}_j}$$

Old Lagrange's equations

Consider a single particle with mass m and position vector $\mathbf{r}$, moving under an applied force, $\mathbf{F}$, which can be expressed as the gradient of a scalar potential energy function $V(\mathbf{r}, t)$:

$$\mathbf{F} = -\nabla V.$$

Such a force is independent of third- or higher-order derivatives of $\mathbf{r}$, so Newton's second law forms a set of 3 second-order ordinary differential equations. Therefore, the motion of the particle can be completely described by 6 independent variables, or *degrees of freedom*. An obvious set of variables is $\{\mathbf{r}_j, \dot{\mathbf{r}}_j | j = 1, 2, 3\}$, the Cartesian components of $\mathbf{r}$ and their time derivatives, at a given instant of time (i.e. position (x,y,z) and velocity (v_x, v_y, v_z)).

More generally, we can work with a set of generalized coordinates, q_j , and their time derivatives, the generalized velocities, $\dot{q}_j$. The position vector, $\mathbf{r}$, is related to the generalized coordinates by some *transformation equation*:

$$\mathbf{r} = \mathbf{r}(q_i, q_j, q_k, t).$$

For example, for a simple pendulum of length l , a logical choice for a generalized coordinate is the angle of the pendulum from vertical, θ , for which the transformation equation would be

$$\mathbf{r}(\theta, \dot{\theta}, t) = (l \sin \theta, l \cos \theta).$$

The term "generalized coordinates" is really a holdover from the period when Cartesian coordinates were the default coordinate system.

Consider an arbitrary displacement $\delta \mathbf{r}$ of the particle. The work done by the applied force $\mathbf{F}$ is $W = \mathbf{F} \cdot \delta \mathbf{r}$. Using Newton's second law, we write:

$$\mathbf{F} \cdot \delta \mathbf{r} = m \ddot{\mathbf{r}} \cdot \delta \mathbf{r}.$$

Since work is a physical scalar quantity, we should be able to rewrite this equation in terms of the generalized coordinates and velocities. On the left hand side,

$$\begin{aligned} \mathbf{F} \cdot \delta \mathbf{r} &= -\nabla V \cdot \sum_i \frac{\partial \mathbf{r}}{\partial q_i} \delta q_i \\ &= -\sum_{i,j} \frac{\partial V}{\partial r_j} \frac{\partial r_j}{\partial q_i} \delta q_i \\ &= -\sum_i \frac{\partial V}{\partial q_i} \delta q_i. \end{aligned}$$

On the right hand side, carrying out a change of coordinates, we obtain:

$$m \ddot{\mathbf{r}} \cdot \delta \mathbf{r} = m \sum_{i,j} \ddot{r}_i \frac{\partial r_i}{\partial q_j} \delta q_j$$

Rearranging Slightly:

$$m \ddot{\mathbf{r}} \cdot \delta \mathbf{r} = m \sum_j \left[\sum_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} \right] \delta q_j$$

Now, by performing an "integration by parts" transformation, with respect to t:

$$m \ddot{\mathbf{r}} \cdot \delta \mathbf{r} = m \sum_j \left[\sum_i \left[\frac{d}{dt} \left(\dot{r}_i \frac{\partial r_i}{\partial q_j} \right) - \dot{r}_i \frac{d}{dt} \left(\frac{\partial r_i}{\partial q_j} \right) \right] \right] \delta q_j$$

Recognizing that $\frac{d}{dt} \frac{\partial r_j}{\partial q_i} = \frac{\partial \dot{r}_j}{\partial q_i}$ and $\frac{\partial r_j}{\partial q_i} = \frac{\partial \dot{r}_j}{\partial \dot{q}_i}$, we obtain:

$$m \ddot{\mathbf{r}} \cdot \delta \mathbf{r} = m \sum_j \left[\sum_i \left[\frac{d}{dt} \left(\dot{r}_i \frac{\partial r_i}{\partial \dot{q}_j} \right) - \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_j} \right] \right] \delta q_j$$

Now, by changing the order of differentiation, we obtain:

$$m \ddot{\mathbf{r}} \cdot \delta \mathbf{r} = m \sum_j \left[\sum_i \left[\frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} \dot{r}_i^2 \right) - \frac{\partial}{\partial q_j} \left(\frac{1}{2} \dot{r}_i^2 \right) \right] \right] \delta q_j$$

Finally, we change the order of summation:

$$m \ddot{\mathbf{r}} \cdot \delta \mathbf{r} = \sum_j \left[\frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m \dot{r}_i^2 \right) - \frac{\partial}{\partial q_j} \left(\sum_i \frac{1}{2} m \dot{r}_i^2 \right) \right] \delta q_j$$

Which is equivalent to:

$$m \ddot{\mathbf{r}} \cdot \delta \mathbf{r} = \sum_i \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} \right] \delta q_i$$

where $T = \frac{1}{2}m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$ is the kinetic energy of the particle. Our equation for the work done becomes

$$\sum_i \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial (T - V)}{\partial q_i} \right] \delta q_i = 0.$$

However, this must be true for *any* set of generalized displacements δq_i , so we must have

$$\left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial (T - V)}{\partial q_i} \right] = 0$$

for *each* generalized coordinate q_i . We can further simplify this by noting that V is a function solely of $\mathbf{r}$ and t , and $\mathbf{r}$ is a function of the generalized coordinates and t . Therefore, V is independent of the generalized velocities:

$$\frac{d}{dt} \frac{\partial V}{\partial \dot{q}_i} = 0.$$

Inserting this into the preceding equation and substituting $L = T - V$, called the Lagrangian, we obtain Lagrange's equations:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}.$$

There is one Lagrange equation for each generalized coordinate q_i . When $q_i = r_i$ (i.e. the generalized coordinates are simply the Cartesian coordinates), it is straightforward to check that Lagrange's equations reduce to Newton's second law.

The above derivation can be generalized to a system of N particles. There will be $6N$ generalized coordinates, related to the position coordinates by $3N$ transformation equations. In each of the $3N$ Lagrange equations, T is the total kinetic energy of the system, and V the total potential energy.

In practice, it is often easier to solve a problem using the Euler-Lagrange equations than Newton's laws. This is because not only may more appropriate generalized coordinates q_i be chosen to exploit symmetries in the system, but constraint forces are replaced with simpler relations.

Examples

In this section two examples are provided in which the above concepts are applied. The first example establishes that in a simple case, the Newtonian approach and the Lagrangian formalism agree. The second case illustrates the power of the above formalism, in a case which is hard to solve with Newton's laws.

Falling mass

Consider a point mass m falling freely from rest. By gravity a force $F = m g$ is exerted on the mass (assuming g constant during the motion). Filling in the force in Newton's law, we find $\ddot{x} = g$ from which the solution

$$x(t) = \frac{1}{2}gt^2$$

follows (choosing the origin at the starting point). This result can also be derived through the Lagrange formalism. Take x to be the coordinate, which is 0 at the starting point. The kinetic energy is $T = \frac{1}{2}mv^2$ and the potential energy is $V = -mgx$, hence

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{x}^2 + mgx.$$

Now we find

$$0 = \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = mg - m \frac{d\dot{x}}{dt}$$

which can be rewritten as $\ddot{x} = g$, yielding the same result as earlier.

Pendulum on a movable support

Consider a pendulum of mass m and length l , which is attached to a support with mass M which can move along a line in the x -direction. Let x be the coordinate along the line of the support, and let us denote the position of the pendulum by the angle θ from the vertical. The kinetic energy can then be shown to be

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}_{\text{pend}}^2 + \dot{y}_{\text{pend}}^2) = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m \left[(\dot{x} + l\dot{\theta} \cos \theta)^2 + (l\dot{\theta} \sin \theta)^2 \right],$$

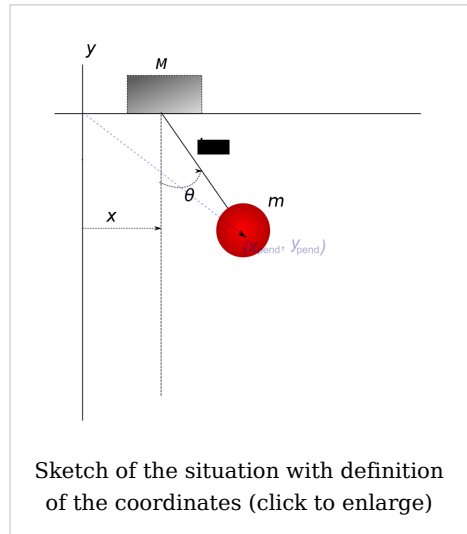
and the potential energy of the system is

$$V = mgy_{\text{pend}} = -mgl \cos \theta.$$

The Lagrangian is therefore

$$\mathcal{L} = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m \left[(\dot{x} + l\dot{\theta} \cos \theta)^2 + (l\dot{\theta} \sin \theta)^2 \right] + mgl \cos \theta = \frac{1}{2}(M + m)\dot{x}^2 + m\dot{x}l\dot{\theta} \cos \theta + \frac{1}{2}ml^2\dot{\theta}^2$$

Now carrying out the differentiations gives for the support coordinate x



$$\frac{d}{dt} \left[(M + m)\dot{x} + ml\dot{\theta} \cos \theta \right] = 0,$$

therefore:

$$(M + m)\ddot{x} + ml\ddot{\theta} \cos \theta - ml\dot{\theta}^2 \sin \theta = 0$$

indicating the presence of a constant of motion. Performing the same procedure for the variable θ yields:

$$\frac{d}{dt} \left[m(\dot{x}l \cos \theta + l^2\dot{\theta}) \right] + m(\dot{x}l\dot{\theta} + gl) \sin \theta = 0;$$

therefore

$$\ddot{\theta} + \frac{\ddot{x}}{l} \cos \theta + \frac{g}{l} \sin \theta = 0.$$

These equations may look quite complicated, but finding them with Newton's laws would have required carefully identifying all forces, which would have been much harder and prone to errors. By considering limit cases ($\ddot{x} \rightarrow 0$ should give the equations of motion for a pendulum, $\ddot{\theta} \rightarrow 0$ should give the equations for a pendulum in a constantly accelerating system, etc.) the correctness of this system can be verified.

Hamilton's principle

The action, denoted by S , is the time integral of the Lagrangian:

$$S = \int \mathcal{L} dt.$$

Let q_0 and q_1 be the coordinates at respective initial and final times t_0 and t_1 . Using the calculus of variations, it can be shown the Lagrange's equations are equivalent to *Hamilton's principle*:

The system undergoes the trajectory between t_0 and t_1 whose action has a stationary value.

By *stationary*, we mean that the action does not vary to first-order for infinitesimal deformations of the trajectory, with the end-points (q_0, t_0) and (q_1, t_1) fixed. Hamilton's principle can be written as:

$$\delta S = 0.$$

Thus, instead of thinking about particles accelerating in response to applied forces, one might think of them picking out the path with a stationary action.

Hamilton's principle is sometimes referred to as the *principle of least action*. However, this is a misnomer: the action only needs to be stationary, and the correct trajectory could be produced by a maximum, saddle point, or minimum in the action.

We can use this principle instead of Newton's Laws as the fundamental principle of mechanics, this allows us to use an integral principle (Newton's Laws are based on differential equations so they are a differential principle) as the basis for mechanics. However it is not widely stated that Hamilton's principle is a variational principle only with holonomic constraints, if we are dealing with nonholonomic systems then the variational principle should be replaced with one involving d'Alembert principle of virtual work. Working only with holonomic constraints is the price we have to pay for using an elegant variational formulation of mechanics.

Extensions of Lagrangian mechanics

The Hamiltonian, denoted by H , is obtained by performing a Legendre transformation on the Lagrangian, which introduces new variables, canonically conjugate to the original variables. This doubles the number of variables, but linearizes the differential equations. The Hamiltonian is the basis for an alternative formulation of classical mechanics known as Hamiltonian mechanics. It is a particularly ubiquitous quantity in quantum mechanics (see Hamiltonian (quantum mechanics)).

In 1948, Feynman invented the path integral formulation extending the principle of least action to quantum mechanics for electrons and photons. In this formulation, particles travel every possible path between the initial and final states; the probability of a specific final state is obtained by summing over all possible trajectories leading to it. In the classical regime, the path integral formulation cleanly reproduces Hamilton's principle, and Fermat's

principle in optics.

See also

- Canonical coordinates
- Functional derivative
- Generalized coordinates
- Hamiltonian mechanics
- Lagrangian analysis (applications of Lagrangian mechanics)
- Nielsen form
- Restricted three-body problem

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- Principle of least action interactive (<http://www.eftaylor.com/software/ActionApplets/LeastAction.html>) Excellent interactive explanation/webpage
- Aerospace dynamics lecture notes on Lagrangian mechanics (<http://ocw.mit.edu/NR/rdonlyres/Aeronautics-and-Astronautics/16-61Aerospace-DynamicsSpring2003/D453E02B-5218-4154-8531-DB35ECD76A6C/0/lecture9.pdf>)
- Aerospace dynamics lecture notes on Rayleigh dissipation function (<http://ocw.mit.edu/NR/rdonlyres/Aeronautics-and-Astronautics/16-61Aerospace-DynamicsSpring2003/53F21B11-4F88-4870-967A-0C05AD85B104/0/lecture10.pdf>)
- Introduction to Lagrangian Mechanics (http://www.yaronhadad.com/Site/Philosophy%20Naturalis/Entries/2008/9/3_Introduction_to_Lagrangian_Mechanics.html)
- (<http://www.sydgram.nsw.edu.au/CollegeSt/extension/lagrangian.html>) Sydney Grammar School Academic Extension notes

Hamiltonian mechanics

Hamiltonian mechanics is a reformulation of classical mechanics that was introduced in 1833 by Irish mathematician William Rowan Hamilton. It arose from Lagrangian mechanics, a previous reformulation of classical mechanics introduced by Joseph Louis Lagrange in 1788, but can be formulated *without* recourse to Lagrangian mechanics using symplectic spaces (see *Mathematical formalism*, below). The Hamiltonian method differs from the Lagrangian method in that instead of expressing second-order differential constraints on an n -dimensional coordinate space (where n is the number of degrees of freedom of the system), it expresses first-order constraints on a $2n$ -dimensional phase space.^[1]

As with Lagrangian mechanics, Hamilton's equations provide a new and equivalent way of looking at classical mechanics. Generally, these equations do not provide a more convenient way of solving a particular problem. Rather, they provide deeper insights into both the general structure of classical mechanics and its connection to quantum mechanics as understood through Hamiltonian mechanics, as well as its connection to other areas of science.

Simplified overview of uses

For a closed system the sum of the kinetic and potential energy in the system is represented by a set of differential equations known as the *Hamilton equations* for that system. Hamiltonians can be used to describe such simple systems as a bouncing ball, a pendulum or an oscillating spring in which energy changes from kinetic to potential and back again over time. Hamiltonians can also be employed to model the energy of other more complex dynamic systems such as planetary orbits in celestial mechanics and also in quantum mechanics.^[2]

The Hamilton equations are generally written as follows:

$$\begin{aligned}\dot{p} &= -\frac{\partial \mathcal{H}}{\partial q} \\ \dot{q} &= \frac{\partial \mathcal{H}}{\partial p}\end{aligned}$$

In the above equations, the dot denotes the ordinary derivative with respect to time of the functions $p = p(t)$ (called generalized momenta) and $q = q(t)$ (called generalized coordinates), taking values in some vector space, and $\mathcal{H} = \mathcal{H}(p, q, t)$ is the so-called Hamiltonian, or (scalar valued) Hamiltonian function. Thus, a little more explicitly, one can equivalently write

$$\begin{aligned}\frac{d}{dt}p(t) &= -\frac{\partial}{\partial q}\mathcal{H}(p(t), q(t), t) \\ \frac{d}{dt}q(t) &= \frac{\partial}{\partial p}\mathcal{H}(p(t), q(t), t)\end{aligned}$$

and specify the domain of values in which the parameter t ("time") varies.

For a detailed derivation of these equations from Lagrangian mechanics, see below.

Basic physical interpretation

The simplest interpretation of the Hamilton Equations is as follows, applying them to a one-dimensional system consisting of one particle of mass m under time independent boundary conditions and exhibiting conservation of energy: The Hamiltonian $\mathcal{H}$ represents the energy of the system, which is the sum of kinetic and potential energy, traditionally denoted T & V , respectively. Here q is the x -coordinate and p is the momentum, mv . Then

$$\mathcal{H} = T + V, \quad T = \frac{p^2}{2m}, \quad V = V(q) = V(x).$$

Note that T is a function of p alone, while V is a function of x (or q) alone.

Now the time-derivative of the momentum p equals the *Newtonian force*, and so here the first Hamilton Equation means that the force on the particle equals the rate at which it loses potential energy with respect to changes in x , its location. (Force equals the negative gradient of potential energy.)

The time-derivative of q here means the velocity: the second Hamilton Equation here means that the particle's velocity equals the derivative of its kinetic energy with respect to its momentum. (For the derivative with respect to p of $p^2/2m$ equals $p/m = mv/m = v$.)

Using Hamilton's equations

1. First write out the Lagrangian $L = T - V$. Express T and V as though you were going to use Lagrange's equation.
2. Calculate the momenta by differentiating the Lagrangian with respect to velocity.
3. Express the velocities in terms of the momenta by inverting the expressions in step (2).
4. Calculate the Hamiltonian using the usual definition, $\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$. Substitute for the velocities using the results in step (3).
5. Apply Hamilton's equations.

Notes

Hamilton's equations are appealing in view of their beautiful simplicity and (slightly *broken*) symmetry. They have been analyzed under almost every imaginable angle of view, from basic physics up to symplectic geometry. A lot is known about solutions of these equations, yet the exact general case solution of the equations of motion cannot be given explicitly for a system of more than two massive point particles. The finding of conserved quantities plays an important role in the search for solutions or information about their nature. In models with an infinite number of degrees of freedom, this is of course even more complicated. An interesting and promising area of research is the study of integrable systems, where an infinite number of independent conserved quantities can be constructed.

Deriving Hamilton's equations

We can derive Hamilton's equations by looking at how the Lagrangian changes as you change the time and the positions and velocities of particles

$$d\mathcal{L} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} dq_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t} dt.$$

Now the generalized momenta were defined as $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ and Lagrange's equations tell us that

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

We can rearrange this to get

$$\frac{\partial \mathcal{L}}{\partial q_i} = \dot{p}_i$$

and substitute the result into the variation of the Lagrangian

$$d\mathcal{L} = \sum_i [\dot{p}_i dq_i + p_i d\dot{q}_i] + \frac{\partial \mathcal{L}}{\partial t} dt.$$

We can rewrite this as

$$d\mathcal{L} = \sum_i [\dot{p}_i dq_i + d(p_i \dot{q}_i) - \dot{q}_i dp_i] + \frac{\partial \mathcal{L}}{\partial t} dt$$

and rearrange again to get

$$d \left(\sum_i p_i \dot{q}_i - \mathcal{L} \right) = \sum_i [-\dot{p}_i dq_i + \dot{q}_i dp_i] - \frac{\partial \mathcal{L}}{\partial t} dt.$$

The term on the left-hand side is just the Hamiltonian that we have defined before, so we find that

$$d\mathcal{H} = \sum_i [-\dot{p}_i dq_i + \dot{q}_i dp_i] - \frac{\partial \mathcal{L}}{\partial t} dt = \sum_i \left[\frac{\partial \mathcal{H}}{\partial q_i} dq_i + \frac{\partial \mathcal{H}}{\partial p_i} dp_i \right] + \frac{\partial \mathcal{H}}{\partial t} dt$$

where the second equality holds because of the definition of the partial derivatives. Associating terms from both sides of the equation above yields Hamilton's equations

$$\frac{\partial \mathcal{H}}{\partial q_j} = -\dot{p}_j, \quad \frac{\partial \mathcal{H}}{\partial p_j} = \dot{q}_j, \quad \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}.$$

As a reformulation of Lagrangian mechanics

Starting with Lagrangian mechanics, the equations of motion are based on generalized coordinates

$$\{q_j | j = 1, \dots, N\}$$

and matching generalized velocities

$$\{\dot{q}_j | j = 1, \dots, N\}$$

We write the Lagrangian as

$$\mathcal{L}(q_j, \dot{q}_j, t)$$

with the subscripted variables understood to represent all N variables of that type. Hamiltonian mechanics aims to replace the generalized velocity variables with generalized

momentum variables, also known as *conjugate momenta*. By doing so, it is possible to handle certain systems, such as aspects of quantum mechanics, that would otherwise be even more complicated.

For each generalized velocity, there is one corresponding conjugate momentum, defined as:

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j}$$

In Cartesian coordinates, the generalized momenta are precisely the physical linear momenta. In circular polar coordinates, the generalized momentum corresponding to the angular velocity is the physical angular momentum. For an arbitrary choice of generalized coordinates, it may not be possible to obtain an intuitive interpretation of the conjugate momenta.

One thing which is not too obvious in this coordinate dependent formulation is that different generalized coordinates are really nothing more than different coordinatizations of the same symplectic manifold.

The *Hamiltonian* is the Legendre transform of the Lagrangian:

$$\mathcal{H}(q_j, p_j, t) = \sum_i \dot{q}_i p_i - \mathcal{L}(q_j, \dot{q}_j, t)$$

If the transformation equations defining the generalized coordinates are independent of t , and the Lagrangian is a sum of products of functions (in the generalised coordinates) which are homogeneous of order 0, 1 or 2, then it can be shown that H is equal to the total energy $E = T + V$.

Each side in the definition of $\mathcal{H}$ produces a differential:

$$\begin{aligned} d\mathcal{H} &= \sum_i \left[\left(\frac{\partial \mathcal{H}}{\partial q_i} \right) dq_i + \left(\frac{\partial \mathcal{H}}{\partial p_i} \right) dp_i \right] + \left(\frac{\partial \mathcal{H}}{\partial t} \right) dt \\ &= \sum_i \left[\dot{q}_i dp_i + p_i d\dot{q}_i - \left(\frac{\partial \mathcal{L}}{\partial q_i} \right) dq_i - \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) d\dot{q}_i \right] - \left(\frac{\partial \mathcal{L}}{\partial t} \right) dt \end{aligned}$$

Substituting the previous definition of the conjugate momenta into this equation and matching coefficients, we obtain the equations of motion of Hamiltonian mechanics, known as the canonical equations of Hamilton:

$$\frac{\partial \mathcal{H}}{\partial q_j} = -\dot{p}_j, \quad \frac{\partial \mathcal{H}}{\partial p_j} = \dot{q}_j, \quad \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

Hamilton's equations are first-order differential equations, and thus easier to solve than Lagrange's equations, which are second-order. However, the steps leading to the equations of motion are more onerous than in Lagrangian mechanics - beginning with the generalized coordinates and the Lagrangian, we must calculate the Hamiltonian, express each generalized velocity in terms of the conjugate momenta, and replace the generalized velocities in the Hamiltonian with the conjugate momenta. All in all, there is little labor saved from solving a problem with Hamiltonian mechanics rather than Lagrangian mechanics. Ultimately, it will produce the same solution as Lagrangian mechanics and Newton's laws of motion.

The principal appeal of the Hamiltonian approach is that it provides the groundwork for deeper results in the theory of classical mechanics.

Geometry of Hamiltonian systems

A Hamiltonian system may be understood as a fiber bundle E over time R , with the fibers E_t , $t \in R$ being the position space. The Lagrangian is thus a function on the jet bundle J over E ; taking the fiberwise Legendre transform of the Lagrangian produces a function on the dual bundle over time whose fiber at t is the cotangent space T^*E_t , which comes equipped with a natural symplectic form, and this latter function is the Hamiltonian.

Generalization to quantum mechanics through Poisson bracket

The Hamilton's equations above work well for classical mechanics, but not for quantum mechanics, since the differential equations discussed assume that one can specify the exact position and momentum of the particle simultaneously at any point in time. However, the equations can be further generalized to then be extended to apply to quantum mechanics as well as to classical mechanics, through the deformation of the Poisson algebra over p and q to the algebra of Moyal brackets.

Specifically, the more general form of the Hamilton's equation reads

$$\frac{df}{dt} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}$$

where f is some function of p and q , and H is the Hamiltonian. To find out the rules for evaluating a Poisson bracket without resorting to differential equations, see Lie algebra; a Poisson bracket is the name for the Lie bracket in a Poisson algebra. These Poisson brackets can then be extended to Moyal brackets comporting to an **inequivalent** Lie algebra, as proven by H Groenewold, and thereby describe quantum mechanical diffusion in phase space (See the uncertainty principle and Weyl quantization). This more algebraic approach not only permits ultimately extending probability distributions in phase space to Wigner quasi-probability distributions, but, at the mere Poisson bracket classical setting, also provides more power in helping analyze the relevant conserved quantities in a system.

Mathematical formalism

Any smooth real-valued function H on a symplectic manifold can be used to define a Hamiltonian system. The function H is known as the **Hamiltonian** or the **energy function**. The symplectic manifold is then called the phase space. The Hamiltonian induces a special vector field on the symplectic manifold, known as the symplectic vector field.

The symplectic vector field, also called the Hamiltonian vector field, induces a Hamiltonian flow on the manifold. The integral curves of the vector field are a one-parameter family of transformations of the manifold; the parameter of the curves is commonly called the **time**. The time evolution is given by symplectomorphisms. By Liouville's theorem, each symplectomorphism preserves the volume form on the phase space. The collection of symplectomorphisms induced by the Hamiltonian flow is commonly called the **Hamiltonian mechanics** of the Hamiltonian system.

The Hamiltonian vector field also induces a special operation, the Poisson bracket. The Poisson bracket acts on functions on the symplectic manifold, thus giving the space of functions on the manifold the structure of a Lie algebra.

In particular, given a function f

$$\frac{d}{dt}f = \frac{\partial}{\partial t}f + \{f, \mathcal{H}\}.$$

If we have a probability distribution, ρ , then (since the phase space velocity $(\dot{p}_i, \dot{q}_i)$ has zero divergence, and probability is conserved) its convective derivative can be shown to be zero and so

$$\frac{\partial}{\partial t}\rho = -\{\rho, \mathcal{H}\}.$$

This is called Liouville's theorem. Every smooth function G over the symplectic manifold generates a one-parameter family of symplectomorphisms and if $\{G, H\} = 0$, then G is conserved and the symplectomorphisms are symmetry transformations.

A Hamiltonian may have multiple conserved quantities G_i . If the symplectic manifold has dimension $2n$ and there are n functionally independent conserved quantities G_i which are in involution (i.e., $\{G_i, G_j\} = 0$), then the Hamiltonian is Liouville integrable. The Liouville–Arnol'd theorem says that locally, any Liouville integrable Hamiltonian can be transformed via a symplectomorphism in a new Hamiltonian with the conserved quantities G_i as coordinates; the new coordinates are called *action-angle coordinates*. The transformed Hamiltonian depends only on the G_i , and hence the equations of motion have the simple form

$$\dot{G}_i = 0, \quad \dot{\phi}_i = F(G),$$

for some function F (Arnol'd et al., 1988). There is an entire field focusing on small deviations from integrable systems governed by the KAM theorem.

The integrability of Hamiltonian vector fields is an open question. In general, Hamiltonian systems are chaotic; concepts of measure, completeness, integrability and stability are poorly defined. At this time, the study of dynamical systems is primarily qualitative, and not a quantitative science.

Riemannian manifolds

An important special case consists of those Hamiltonians that are quadratic forms, that is, Hamiltonians that can be written as

$$\mathcal{H}(q, p) = \frac{1}{2}\langle p, p \rangle_q$$

where $\langle \cdot, \cdot \rangle_q$ is a cometric on the fiber T_q^*Q , the cotangent space to the point q in the configuration space. This Hamiltonian consists entirely of the kinetic term.

If one considers a Riemannian manifold or a pseudo-Riemannian manifold, so that one has an invertible, non-degenerate metric, then the cometric is given simply as the inverse of the metric. The solutions to the Hamilton–Jacobi equations for this Hamiltonian are then the same as the geodesics on the manifold. In particular, the Hamiltonian flow in this case is the same thing as the geodesic flow. The existence of such solutions, and the completeness of the set of solutions, are discussed in detail in the article on geodesics. See also Geodesics as Hamiltonian flows.

Sub-Riemannian manifolds

When the cometric is degenerate, then it is not invertible. In this case, one does not have a Riemannian manifold, as one does not have a metric. However, the Hamiltonian still exists. In the case where the cometric is degenerate at every point q of the configuration space manifold Q , so that the rank of the cometric is less than the dimension of the manifold Q , one has a sub-Riemannian manifold.

The Hamiltonian in this case is known as a **sub-Riemannian Hamiltonian**. Every such Hamiltonian uniquely determines the cometric, and vice-versa. This implies that every sub-Riemannian manifold is uniquely determined by its sub-Riemannian Hamiltonian, and that the converse is true: every sub-Riemannian manifold has a unique sub-Riemannian Hamiltonian. The existence of sub-Riemannian geodesics is given by the Chow-Rashevskii theorem.

The continuous, real-valued Heisenberg group provides a simple example of a sub-Riemannian manifold. For the Heisenberg group, the Hamiltonian is given by

$$\mathcal{H}(x, y, z, p_x, p_y, p_z) = \frac{1}{2} (p_x^2 + p_y^2).$$

p_z is not involved in the Hamiltonian.

Poisson algebras

Hamiltonian systems can be generalized in various ways. Instead of simply looking at the algebra of smooth functions over a symplectic manifold, Hamiltonian mechanics can be formulated on general commutative unital real Poisson algebras. A state is a continuous linear functional on the Poisson algebra (equipped with some suitable topology) such that for any element A of the algebra, A^2 maps to a nonnegative real number.

A further generalization is given by Nambu dynamics.

Charged particle in an electromagnetic field

A good illustration of Hamiltonian mechanics is given by the Hamiltonian of a charged particle in an electromagnetic field. In Cartesian coordinates (i.e. $q_i = x_i$), the Lagrangian of a non-relativistic classical particle in an electromagnetic field is (in SI Units):

$$\mathcal{L} = \sum_i \frac{1}{2} m \dot{x}_i^2 + \sum_i e \dot{x}_i A_i - e\phi,$$

where e is the electric charge of the particle (not necessarily the electron charge), ϕ is the electric scalar potential, and the A_i are the components of the magnetic vector potential (these may be modified through a gauge transformation).

The generalized momenta may be derived by:

$$p_j = \frac{\partial L}{\partial \dot{x}_j} = m \dot{x}_j + e A_j.$$

Rearranging, we may express the velocities in terms of the momenta, as:

$$\dot{x}_j = \frac{p_j - e A_j}{m}.$$

If we substitute the definition of the momenta, and the definitions of the velocities in terms of the momenta, into the definition of the Hamiltonian given above, and then simplify and rearrange, we get:

$$\mathcal{H} = \sum_i \dot{x}_i p_i - \mathcal{L} = \sum_i \frac{(p_i - eA_i)^2}{2m} + e\phi.$$

This equation is used frequently in quantum mechanics.

Relativistic charged particle in an electromagnetic field

The Lagrangian for a relativistic charged particle is given by:

$$\mathcal{L}[t] = -mc^2 \sqrt{1 - \frac{\dot{\vec{x}}[t]^2}{c^2}} - e\phi[\vec{x}[t], t] + e\dot{\vec{x}}[t] \cdot \vec{A}[\vec{x}[t], t].$$

Thus the particle's canonical (total) momentum is

$$\vec{P}[t] = \frac{\partial \mathcal{L}[t]}{\partial \dot{\vec{x}}[t]} = \frac{m\dot{\vec{x}}[t]}{\sqrt{1 - \frac{\dot{\vec{x}}[t]^2}{c^2}}} + e\vec{A}[\vec{x}[t], t],$$

that is, the sum of the kinetic momentum and the potential momentum.

Solving for the velocity, we get

$$\dot{\vec{x}}[t] = \frac{\vec{P}[t] - e\vec{A}[\vec{x}[t], t]}{\sqrt{m^2 + \frac{1}{c^2} (\vec{P}[t] - e\vec{A}[\vec{x}[t], t])^2}}.$$

So the Hamiltonian is

$$\mathcal{H}[t] = \dot{\vec{x}}[t] \cdot \vec{P}[t] - \mathcal{L}[t] = c\sqrt{m^2 c^2 + (\vec{P}[t] - e\vec{A}[\vec{x}[t], t])^2} + e\phi[\vec{x}[t], t].$$

From this we get the force equation (equivalent to the Euler-Lagrange equation)

$$\dot{\vec{P}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} = e(\vec{\nabla} \vec{A}) \cdot \dot{\vec{x}} - e\vec{\nabla} \phi$$

from which one can derive

$$\frac{d}{dt} \left(\frac{m\dot{\vec{x}}}{\sqrt{1 - \frac{\dot{\vec{x}}^2}{c^2}}} \right) = e\vec{E} + e\dot{\vec{x}} \times \vec{B}.$$

An equivalent expression for the Hamiltonian as function of the relativistic (kinetic) momentum, $\vec{p} = \gamma m \dot{\vec{x}}[t]$, is

$$\mathcal{H}[t] = \dot{\vec{x}}[t] \cdot \vec{p}[t] + \frac{mc^2}{\gamma} + e\phi[\vec{x}[t], t] = \gamma mc^2 + e\phi[\vec{x}[t], t] = E + V.$$

This has the advantage that $\vec{p}$ can be measured experimentally whereas $\vec{P}$ cannot. Notice that the Hamiltonian (total energy) can be viewed as the sum of the relativistic energy (kinetic+rest), $E = \gamma mc^2$, plus the potential energy, $V = e\phi$.

Hamilton's principle applied to deformable bodies

Hamilton's principle is an important variational principle in elastodynamics. As opposed to a system composed of rigid bodies, deformable bodies have an infinite number of degrees of freedom and occupy continuous regions of space; consequently, the state of the system is described by using continuous functions of space and time. The extended Hamilton Principle for such bodies is given by

$$\int_{t_1}^{t_2} [\delta W_e + \delta T - \delta U] dt = 0$$

where T is the kinetic energy, U is the elastic energy, W_e is the work done by external loads on the body, and t_1, t_2 the initial and final times. If the system is conservative, the work done by external forces may be derived from a scalar potential V . In this case,

$$\delta \int_{t_1}^{t_2} [T - (U + V)] dt = 0$$

This is called Hamilton's Principle and it is invariant under coordinate transformations.

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See also

- Hamiltonian (quantum mechanics)
- Lagrangian mechanics
- Canonical transformation
- Classical mechanics
- Dynamical systems
- Quantum mechanics
- Maxwell's equations
- Field theory
- Hamilton-Jacobi equation

Noether's theorem

Noether's (first) theorem states that any differentiable symmetry of the action of a physical system has a corresponding conservation law. The action of a physical system is an integral of a so-called Lagrangian function, from which the system's behavior can be determined by the principle of least action. This seminal theorem was proven by Emmy Noether in 1915 and published in 1918.^[1]

Noether's theorem has become a fundamental tool of modern theoretical physics and the calculus of variations. Noether's theorem allows a far-reaching generalization of earlier work on constants of motion in Lagrangian and Hamiltonian mechanics. Noether's theorem does not apply to systems that cannot be modeled with a Lagrangian; for example, dissipative systems with continuous symmetries need not have a corresponding conservation law.

For illustration, if a physical system behaves the same regardless of how it is oriented in space, its Lagrangian is rotationally symmetric; from this symmetry, Noether's theorem shows the angular momentum of the system must be conserved. The physical system itself need not be symmetric; a jagged asteroid tumbling in space conserves angular momentum despite its asymmetry – it is the laws of motion that are symmetric. As another example, if a physical experiment has the same outcome regardless of place or time (having the same outcome, say, in Cleveland on Tuesday and Samaria on Wednesday), then its Lagrangian is symmetric under continuous translations in space and time; by Noether's theorem, these symmetries account for the conservation laws of linear momentum and energy within this system, respectively.

Noether's theorem is profoundly important, both because of the insight it gives into conservation laws, and also as a practical calculational tool. It allows researchers to determine the conserved quantities from the observed symmetries of a physical system. Conversely, it allows researchers to consider whole classes of hypothetical Lagrangians to describe a physical system. For illustration, suppose that a new field is discovered that conserves a quantity X . Using Noether's theorem, the types of Lagrangians that conserve X because of a continuous symmetry can be determined, and then their fitness judged by other criteria.

There are numerous different versions of Noether's theorem, with varying degrees of generality. The original version only applied to ordinary differential equations (particles) and not partial differential equations (fields). The original versions also assume that the Lagrangian only depends upon the first derivative, while later versions generalize the theorem to Lagrangians depending on the n^{th} derivative. There is also a quantum version of this theorem, known as the Ward-Takahashi identity. Generalizations of Noether's theorem to superspaces also exist.

Informal statement of the theorem

All fine technical points aside, Noether's theorem can be stated informally

If a system has a continuous symmetry property, then there are corresponding quantities whose values are conserved in time.^[2]

A more sophisticated version of the theorem states that:

To every differentiable symmetry generated by local actions, there corresponds a conserved current.

The word "symmetry" in the above statement refers more precisely to the covariance of the form that a physical law takes with respect to a one-dimensional Lie group of transformations satisfying certain technical criteria. The conservation law of a physical quantity is usually expressed as a continuity equation.

The formal proof of the theorem uses only the condition of invariance to derive an expression for a current associated with a conserved physical quantity. The conserved quantity is called the *Noether charge* and the flow carrying that 'charge' is called the *Noether current*. The Noether current is defined up to a solenoidal vector field.

Historical context

A conservation law states that some quantity X describing a system remains constant throughout its motion; expressed mathematically, the rate of change of X (its derivative with respect to time) is zero:

$$\frac{dX}{dt} = 0.$$

Such quantities are said to be conserved; they are often called constants of motion, although motion *per se* need not be involved, just evolution in time. For example, if the energy of a system is conserved, its energy is constant at all times, which imposes a constraint on the system's motion and may help to solve for it. Aside from the insight that such constants of motion give into the nature of a system, they are a useful calculational tool; for example, an approximate solution can be corrected by finding the nearest state that satisfies the necessary conservation laws.

The earliest constants of motion discovered were momentum and energy, which were proposed in the 17th century by René Descartes and Gottfried Leibniz on the basis of collision experiments, and refined by subsequent researchers. Isaac Newton was the first to enunciate the conservation of momentum in its modern form, and showed that it was a consequence of Newton's third law; interestingly, conservation of momentum still holds even in situations when Newton's third law is incorrect. Modern physics has revealed that the conservation laws of momentum and energy are only approximately true, but their modern refinements – the conservation of four-momentum in special relativity and the zero divergence of the stress-energy tensor in general relativity – are rigorously true within the limits of those theories. The conservation of angular momentum, a generalization to rotating rigid bodies, likewise holds in modern physics. Another important conserved quantity, discovered in studies of the celestial mechanics of astronomical bodies, was the Laplace-Runge-Lenz vector.

In the late 18th and early 19th centuries, physicists developed more systematic methods for discovering conserved quantities. A major advance came in 1788 with the development of Lagrangian mechanics, which is related to the principle of least action. In this approach,

the state of the system can be described by any type of generalized coordinates $\mathbf{q}$; the laws of motion need not be expressed in a Cartesian coordinate system, as was customary in Newtonian mechanics. The action is defined as the time integral I of a function known as the Lagrangian L

$$I = \int L(\mathbf{q}, \dot{\mathbf{q}}, t) dt$$

where the dot over $\mathbf{q}$ signifies the rate of change of the coordinates $\mathbf{q}$

$$\dot{\mathbf{q}} = \frac{d\mathbf{q}}{dt}$$

Hamilton's principle states that the physical path $\mathbf{q}(t)$ – the one truly taken by the system – is a path for which infinitesimal variations in that path cause no change in I , at least up to first order. This principle results in the Euler-Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) = \frac{\partial L}{\partial \mathbf{q}}$$

Thus, if one of the coordinates, say q_k , does not appear in the Lagrangian, the right-hand side of the equation is zero, and the left-hand side shows that

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = \frac{dp_k}{dt} = 0$$

where the conserved momentum p_k is defined as the left-hand quantity in parentheses. The absence of the coordinate q_k from the Lagrangian implies that the Lagrangian is unaffected by changes or transformations of q_k ; the Lagrangian is invariant, and is said to exhibit a kind of symmetry. This is the seed idea from which Noether's theorem was born.

Several alternative methods for finding conserved quantities were developed in the 19th century, especially by William Rowan Hamilton. For example, he developed a theory of canonical transformations that allowed researchers to change coordinates so that coordinates disappeared from the Lagrangian, resulting in conserved quantities. Another approach and perhaps the most efficient for finding conserved quantities is the Hamilton-Jacobi equation.

Mathematical expression

The essence of Noether's theorem is the following: Imagine that the action I defined above is invariant under small perturbations (warpings) of the time variable t and the generalized coordinates $\mathbf{q}$; (in a notation commonly used by physicists) we write

$$t \rightarrow t' = t + \delta t$$

$$\mathbf{q} \rightarrow \mathbf{q}' = \mathbf{q} + \delta \mathbf{q}$$

where the perturbations δt and $\delta \mathbf{q}$ are both small but variable. For generality, assume that there might be several such symmetry transformations of the action, say, N ; we may use an index $r=1, 2, 3, \dots, N$ to keep track of them. Then a generic perturbation can be written as a linear sum of the individual types of perturbations

$$\delta t = \sum_r \epsilon_r T_r$$

$$\delta \mathbf{q} = \sum_r \epsilon_r \mathbf{Q}_r$$

Using these definitions, Emmy Noether showed that the N quantities

$$\left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{q}} - L \right) T_r - \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \mathbf{Q}_r$$

are conserved, i.e., are constants of motion; this is a simple version of Noether's theorem.

Examples

For illustration, consider a Lagrangian that does not depend on time, i.e., that is invariant (symmetric) under changes $t \rightarrow t + \delta t$, without any change in the coordinates $\mathbf{q}$. In this case, $N=1$, $T=1$ and $\mathbf{Q} = 0$; the corresponding conserved quantity is the total energy H ^[3]

$$H = \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{q}} - L$$

Similarly, consider a Lagrangian that does not depend on a coordinate q_k , i.e., that is invariant (symmetric) under changes $q_k \rightarrow q_k + \delta q_k$. In that case, $N=1$, $T = 0$, and $Q_k=1$; the conserved quantity is the corresponding momentum p_k ^[4]

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

In special and general relativity, these apparently separate conservation laws are aspects of a single conservation law, that of the stress-energy tensor,^[5] that is derived in the next section.

The conservation of the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is slightly more complicated to derive, but analogous to its linear momentum counterpart.^[6] It is assumed that the symmetry of the Lagrangian is rotational, i.e., that the Lagrangian does not depend on the absolute orientation of the physical system in space. For concreteness, assume that the Lagrangian does not change under small rotations of an angle $\delta\theta$ about an axis $\mathbf{n}$; such a rotation transforms the Cartesian coordinates by the equation

$$\mathbf{r} \rightarrow \mathbf{r} + \delta\theta \mathbf{n} \times \mathbf{r}$$

Since time is not being transformed, T equals zero. Taking $\delta\theta$ as the ε parameter and the Cartesian coordinates $\mathbf{r}$ as the generalized coordinates $\mathbf{q}$, the corresponding $\mathbf{Q}$ variables are given by

$$\mathbf{Q} = \mathbf{n} \times \mathbf{r}$$

Then Noether's theorem states that the following quantity is conserved

$$\frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \mathbf{Q}_r = \mathbf{p} \cdot (\mathbf{n} \times \mathbf{r}) = \mathbf{n} \cdot (\mathbf{r} \times \mathbf{p}) = \mathbf{n} \cdot \mathbf{L}$$

In other words, the component of the angular momentum $\mathbf{L}$ along the $\mathbf{n}$ axis is conserved. If $\mathbf{n}$ is arbitrary, i.e., if the system is insensitive to any rotation, then every component of $\mathbf{L}$ is conserved; in short, angular momentum is conserved.

Field-theory version

Although useful in its own right, the version of her theorem just given was a special case of the general version she derived in 1915. To give the flavor of the general theorem, a version of the Noether theorem for continuous fields in four-dimensional space-time is now given. Since field theory problems are more common in modern physics than mechanics problems, this field-theory version is the most commonly used version of Noether's theorem.

Let there be a set of differentiable fields ϕ_k defined over all space and time; for example, the temperature $T(\mathbf{x}, t)$ would be representative of such a field, being a number defined at every place and time. The principle of least action can be applied to such fields, but the action is now an integral over space and time

$$I = \int L(\phi, \partial_\mu \phi, x^\mu) d^4x$$

(the theorem can actually be further generalized to the case where the Lagrangian depends on up to the n^{th} derivative using jet bundles)

Let the action be invariant under certain transformations of the space-time coordinates x^μ and the fields ϕ_k

$$x^\mu \rightarrow x^\mu + \delta x^\mu$$

$$\phi \rightarrow \phi + \delta \phi$$

where the transformations can be indexed by $r = 1, 2, 3, \dots, N$

$$\delta x^\mu = \epsilon_r X_r^\mu$$

$$\delta \phi = \epsilon_r \Psi_r$$

For such systems, Noether's theorem states that there are N conserved current densities

$$j_r^\nu = - \left(\frac{\partial L}{\partial \phi_{,\nu}} \right) \cdot \Psi_r + \sum_\sigma \left[\left(\frac{\partial L}{\partial \phi_{,\nu}} \right) \cdot \phi_{,\sigma} - L \delta_\sigma^\nu \right] X_r^\sigma$$

In such cases, the conservation law is expressed in a four-dimensional way

$$\sum_\nu \frac{\partial j^\nu}{\partial x^\nu} = 0$$

which expresses the idea that the amount of a conserved quantity within a sphere cannot change unless some of it flows out of the sphere. For example, electric charge is conserved; the amount of charge within a sphere cannot change unless some of the charge leaves the sphere.

For illustration, consider a physical system of fields that behaves the same under translations in time and space, as considered above; in other words, the fields do not depend on the absolute position in space and time. In that case, $N=4$, one for each dimension of space and time. Since only the positions in space-time are being warped, not the fields, the Ψ are all zero and the X_μ^ν equal the Kronecker delta δ_μ^ν , where we have used μ instead of r for the index. In that case, Noether's theorem corresponds to the conservation law for the stress-energy tensor $T_\mu^{\nu[5]}$

$$T_\mu^\nu = \sum_\sigma \left[\left(\frac{\partial L}{\partial \phi_{,\nu}} \right) \cdot \phi_{,\sigma} - L \delta_\sigma^\nu \right] \delta_\mu^\sigma = \left(\frac{\partial L}{\partial \phi_{,\nu}} \right) \cdot \phi_{,\mu} - L \delta_\mu^\nu$$

The conservation of electric charge can be derived by considering transformations of the fields themselves.^[7] In quantum mechanics, the probability amplitude $\psi(\mathbf{x})$ of finding a particle at a point $\mathbf{x}$ is a complex field, because it ascribes a complex number to every point in space and time. The probability amplitude itself is physically unmeasurable; only the probability $p = |\psi|^2$ is directly measurable. Therefore, the system is invariant under transformations of the ψ field and its complex conjugate field ψ^* that leave $|\psi|^2$ unchanged, such as

$$\psi = e^{i\theta} \psi, \quad \psi^* = e^{-i\theta} \psi^*$$

In the limit when θ becomes infinitesimally small (60), it may be taken as the ε , and the Ψ are equal to $i\psi$ and $-i\psi^*$, respectively. A specific example is the Klein-Gordon equation, the relativistically correct version of the Schrödinger equation for spinless particles, which has the Lagrangian density

$$L = \psi_{,\nu} \psi_{,\mu}^* \eta^{\nu\mu} + m^2 \psi \psi^*.$$

In this case, Noether's theorem states that the conserved current equals

$$j^\nu = i \left(\frac{\partial \psi}{\partial x^\mu} \psi^* - \frac{\partial \psi^*}{\partial x^\mu} \psi \right) \eta^{\nu\mu}$$

which, when multiplied by the charge, equals the electric current density. This transformation was first noted by Hermann Weyl and is one of the fundamental gauge symmetries of modern physics.

Derivations

One independent variable

Consider the simplest case, a system with one independent variable, time. Suppose the dependent variables $\mathbf{q}$ are such that the action integral

$$I = \int_{t_1}^{t_2} L[\mathbf{q}[t], \dot{\mathbf{q}}[t], t] dt$$

is invariant under brief infinitesimal variations in the dependent variables. In other words, they satisfy the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}} [t] = \frac{\partial L}{\partial \mathbf{q}} [t].$$

And suppose that the integral is invariant under a continuous symmetry. Mathematically such a symmetry is represented as a flow, ϕ , which acts on the variables as follows

$$t \rightarrow t' = t + \epsilon T$$

$$\mathbf{q}[t] \rightarrow \mathbf{q}'[t'] = \phi[\mathbf{q}[t], \epsilon] = \phi[\mathbf{q}[t' - \epsilon T], \epsilon]$$

where ϵ is a real variable indicating the amount of flow and T is a real constant (which could be zero) indicating how much the flow shifts time.

$$\dot{\mathbf{q}}[t] \rightarrow \dot{\mathbf{q}}'[t'] = \frac{d}{dt} \phi[\mathbf{q}[t], \epsilon] = \frac{\partial \phi}{\partial \mathbf{q}} [\mathbf{q}[t' - \epsilon T], \epsilon] \dot{\mathbf{q}}[t' - \epsilon T].$$

The action integral flows to

$$\begin{aligned} I'[\epsilon] &= \int_{t_1 + \epsilon T}^{t_2 + \epsilon T} L[\mathbf{q}'[t'], \dot{\mathbf{q}}'[t'], t'] dt' \\ &= \int_{t_1 + \epsilon T}^{t_2 + \epsilon T} L[\phi[\mathbf{q}[t' - \epsilon T], \epsilon], \frac{\partial \phi}{\partial \mathbf{q}} [\mathbf{q}[t' - \epsilon T], \epsilon] \dot{\mathbf{q}}[t' - \epsilon T], t'] dt' \end{aligned}$$

which may be regarded as a function of ϵ . Calculating the derivative at $\epsilon = 0$ and using the symmetry, we get

$$\begin{aligned} 0 &= \frac{dI'}{d\epsilon} [0] = L[\mathbf{q}[t_2], \dot{\mathbf{q}}[t_2], t_2] T - L[\mathbf{q}[t_1], \dot{\mathbf{q}}[t_1], t_1] T + \\ &\quad \int_{t_1}^{t_2} \frac{\partial L}{\partial \mathbf{q}} \left(-\frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}} T + \frac{\partial \phi}{\partial \epsilon} \right) + \frac{\partial L}{\partial \dot{\mathbf{q}}} \left(-\frac{\partial^2 \phi}{(\partial \mathbf{q})^2} \dot{\mathbf{q}}^2 T + \frac{\partial^2 \phi}{\partial \epsilon \partial \mathbf{q}} \dot{\mathbf{q}} - \frac{\partial \phi}{\partial \mathbf{q}} \ddot{\mathbf{q}} T \right) dt. \end{aligned}$$

Notice that the Euler-Lagrange equations imply

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}} T \right) &= \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}} \right) \frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}} T + \frac{\partial L}{\partial \dot{\mathbf{q}}} \left(\frac{d}{dt} \frac{\partial \phi}{\partial \mathbf{q}} \right) \dot{\mathbf{q}} T + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \mathbf{q}} \ddot{\mathbf{q}} T \\ &= \frac{\partial L}{\partial \mathbf{q}} \frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}} T + \frac{\partial L}{\partial \dot{\mathbf{q}}} \left(\frac{\partial^2 \phi}{(\partial \mathbf{q})^2} \dot{\mathbf{q}} \right) \dot{\mathbf{q}} T + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \mathbf{q}} \ddot{\mathbf{q}} T. \end{aligned}$$

Substituting this into the previous equation, one gets

$$\begin{aligned} 0 &= \frac{dI'}{d\epsilon} [0] = L[\mathbf{q}[t_2], \dot{\mathbf{q}}[t_2], t_2] T - L[\mathbf{q}[t_1], \dot{\mathbf{q}}[t_1], t_1] T - \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}}[t_2] T + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}}[t_1] T + \\ &\quad \int_{t_1}^{t_2} \frac{\partial L}{\partial \mathbf{q}} \frac{\partial \phi}{\partial \epsilon} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial^2 \phi}{\partial \epsilon \partial \mathbf{q}} \dot{\mathbf{q}} dt. \end{aligned}$$

Again using the Euler-Lagrange equations we get

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \epsilon} \right) = \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}} \right) \frac{\partial \phi}{\partial \epsilon} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial^2 \phi}{\partial \epsilon \partial \mathbf{q}} \dot{\mathbf{q}} = \frac{\partial L}{\partial \mathbf{q}} \frac{\partial \phi}{\partial \epsilon} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial^2 \phi}{\partial \epsilon \partial \mathbf{q}} \dot{\mathbf{q}}.$$

Substituting this into the previous equation, one gets

$$\begin{aligned} 0 &= L[\mathbf{q}[t_2], \dot{\mathbf{q}}[t_2], t_2] T - L[\mathbf{q}[t_1], \dot{\mathbf{q}}[t_1], t_1] T - \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}}[t_2] T + \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}}[t_1] T + \\ &\quad \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \epsilon} [t_2] - \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \epsilon} [t_1]. \end{aligned}$$

From which one can see that

$$\left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \mathbf{q}} \dot{\mathbf{q}} - L \right) T - \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \epsilon}$$

is a constant of the motion, i.e. a conserved quantity. Since $\phi[\mathbf{q}, 0] = \mathbf{q}$, we get $\frac{\partial \phi}{\partial \mathbf{q}} = 1$ and

so the conserved quantity simplifies to

$$\left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \dot{\mathbf{q}} - L \right) T - \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{\partial \phi}{\partial \epsilon}.$$

To avoid excessive complication of the formulas, this derivation assumed that the flow does not change as time passes. The same result can be obtained in the more general case.

Field-theoretic derivation

Noether's theorem may also be derived for tensor fields ϕ^A where the index A ranges over the various components of the various tensor fields. These field quantities are functions defined over a four-dimensional space whose points are labeled by coordinates x^μ where the index μ ranges over time ($\mu=0$) and three spatial dimensions ($\mu=1,2,3$). These four coordinates are the independent variables; and the values of the fields at each event are the dependent variables. Under an infinitesimal transformation, the variation in the coordinates is written

$$x^\mu \rightarrow \xi^\mu = x^\mu + \delta x^\mu$$

whereas the transformation of the field variables is expressed as

$$\phi^A \rightarrow \alpha^A(\xi^\mu) = \phi^A(x^\mu) + \delta \phi^A(x^\mu).$$

By this definition, the field variations $\delta \phi^A$ result from two factors: intrinsic changes in the field themselves and changes in coordinates, since the transformed field α^A depends on the transformed coordinates ξ^μ . To isolate the intrinsic changes, the field variation at a single point x^μ may be defined

$$\alpha^A(x^\mu) = \phi^A(x^\mu) + \bar{\delta}\phi^A(x^\mu).$$

If the coordinates are changed, the boundary of the region of space-time over which the Lagrangian is being integrated also changes; the original boundary and its transformed version are denoted as Ω and Ω' , respectively.

Noether's theorem begins with the assumption that a specific transformation of the coordinates and field variables does not change the action, which is defined as the integral of the Lagrangian over the given region of spacetime. Expressed mathematically, this assumption may be written as

$$\int_{\Omega'} L(\alpha^A, \alpha^A_{,\nu}, \xi^\mu) d^4\xi - \int_{\Omega} L(\phi^A, \phi^A_{,\nu}, x^\mu) d^4x = 0$$

where the comma subscript indicates a partial derivative with respect to the coordinate(s) that follows the comma, e.g.

$$\phi^A_{,\sigma} = \frac{\partial \phi^A}{\partial x^\sigma}.$$

Since ξ is a dummy variable of integration, and since the change in the boundary Ω is infinitesimal by assumption, the two integrals may be combined using the four-dimensional version of the divergence theorem into the following form

$$\int_{\Omega} \left\{ \left[L(\alpha^A, \alpha^A_{,\nu}, x^\mu) - L(\phi^A, \phi^A_{,\nu}, x^\mu) \right] + \frac{\partial}{\partial x^\sigma} \left[L(\phi^A, \phi^A_{,\nu}, x^\mu) \delta x^\sigma \right] \right\} d^4x = 0.$$

The difference in Lagrangians can be written to first-order in the infinitesimal variations as

$$\left[L(\alpha^A, \alpha^A_{,\nu}, x^\mu) - L(\phi^A, \phi^A_{,\nu}, x^\mu) \right] = \frac{\partial L}{\partial \phi^A} \bar{\delta}\phi^A + \frac{\partial L}{\partial \phi^A_{,\sigma}} \bar{\delta}\phi^A_{,\sigma}.$$

However, because the variations are defined at the same point as described above, the variation and the derivative can be done in reverse order; they commute

$$\bar{\delta}\phi^A_{,\sigma} = \bar{\delta} \frac{\partial \phi^A}{\partial x^\sigma} = \frac{\partial}{\partial x^\sigma} (\bar{\delta}\phi^A).$$

Using the Euler-Lagrange field equations

$$\frac{\partial}{\partial x^\sigma} \left(\frac{\partial L}{\partial \phi^A_{,\sigma}} \right) = \frac{\partial L}{\partial \phi^A}$$

the difference in Lagrangians can be written neatly as

$$\left[L(\alpha^A, \alpha^A_{,\nu}, x^\mu) - L(\phi^A, \phi^A_{,\nu}, x^\mu) \right] = \frac{\partial}{\partial x^\sigma} \left(\frac{\partial L}{\partial \phi^A_{,\sigma}} \right) \bar{\delta}\phi^A + \frac{\partial L}{\partial \phi^A_{,\sigma}} \bar{\delta}\phi^A_{,\sigma} = \frac{\partial}{\partial x^\sigma} \left(\frac{\partial L}{\partial \phi^A_{,\sigma}} \bar{\delta}\phi^A \right).$$

Thus, the change in the action can be written as

$$\int_{\Omega} \frac{\partial}{\partial x^\sigma} \left\{ \frac{\partial L}{\partial \phi^A_{,\sigma}} \bar{\delta}\phi^A + L(\phi^A, \phi^A_{,\nu}, x^\mu) \delta x^\sigma \right\} d^4x = 0.$$

Since this holds for any region Ω , the integrand must be zero

$$\frac{\partial}{\partial x^\sigma} \left\{ \frac{\partial L}{\partial \phi^A_{,\sigma}} \bar{\delta}\phi^A + L(\phi^A, \phi^A_{,\nu}, x^\mu) \delta x^\sigma \right\} = 0.$$

For any combination of the various symmetry transformations, the perturbation can be written

$$\begin{aligned} \delta x^\mu &= \epsilon X^\mu \\ \delta \phi^A &= \epsilon \Psi^A = \bar{\delta}\phi^A + \epsilon \mathcal{L}_X \phi^A \end{aligned}$$

where $\mathcal{L}_X \phi^A$ is the Lie derivative of ϕ^A in the X^μ direction. When ϕ^A is a scalar or $X^\mu_{,\nu} = 0$,

$$\mathcal{L}_X \phi^A = \frac{\partial \phi^A}{\partial x^\mu} X^\mu.$$

These equations imply that the field variation taken at one point equals

$$\bar{\delta} \phi^A = \epsilon \Psi^A - \epsilon \mathcal{L}_X \phi^A.$$

Differentiating the above divergence with respect to ϵ at $\epsilon=0$ and changing the sign yields the conservation law

$$\frac{\partial}{\partial x^\sigma} j^\sigma = 0$$

where the conserved current equals

$$j^\sigma = \left[\frac{\partial L}{\partial \phi^A_{,\sigma}} \mathcal{L}_X \phi^A - L X^\sigma \right] - \left(\frac{\partial L}{\partial \phi^A_{,\sigma}} \right) \Psi^A.$$

Manifold/fiber bundle derivation

Suppose we have an n -dimensional manifold, M and a target manifold T . Let $\mathcal{C}$ be the configuration space of smooth functions from M to T . (More generally, we can have smooth sections of a fiber bundle over M .)

Examples of this M in physics include:

- In classical mechanics, in the Hamiltonian formulation, M is the one-dimensional manifold $\mathbf{R}$, representing time and the target space is the cotangent bundle of space of generalized positions.
- In field theory, M is the spacetime manifold and the target space is the set of values the fields can take at any given point. For example, if there are m real-valued scalar fields, $\phi_1, \dots, \phi_m$, then the target manifold is $\mathbf{R}^m$. If the field is a real vector field, then the target manifold is isomorphic to $\mathbf{R}^3$.

Now suppose there is a functional

$$\mathcal{S} : \mathcal{C} \rightarrow \mathbb{R},$$

called the action. (Note that it takes values into $\mathbb{R}$, rather than $\mathbb{C}$; this is for physical reasons, and doesn't really matter for this proof.)

To get to the usual version of Noether's theorem, we need additional restrictions on the action. We assume $\mathcal{S}[\phi]$ is the integral over M of a function

$$\mathcal{L}(\phi, \partial_\mu \phi, x)$$

called the Lagrangian, depending on ϕ , its derivative and the position. In other words, for ϕ in $\mathcal{C}$

$$\mathcal{S}[\phi] = \int_M \mathcal{L}[\phi(x), \partial_\mu \phi(x), x] d^n x.$$

Suppose we are given boundary conditions, ie., a specification of the value of ϕ at the boundary if M is compact, or some limit on ϕ as x approaches ∞ . Then the subspace of $\mathcal{C}$ consisting of functions ϕ such that all functional derivatives of $\mathcal{S}$ at ϕ are zero, that is:

$$\frac{\delta \mathcal{S}[\phi]}{\delta \phi(x)} \approx 0$$

and that ϕ satisfies the given boundary conditions, is the subspace of on shell solutions. (See principle of stationary action)

Now, suppose we have an infinitesimal transformation on $\mathcal{C}$, generated by a functional derivation, Q such that

$$Q \left[\int_N \mathcal{L} d^n x \right] \approx \int_{\partial N} f^\mu [\phi(x), \partial\phi, \partial\partial\phi, \dots] ds_\mu$$

for all compact submanifolds N or in other words,

$$Q[\mathcal{L}(x)] \approx \partial_\mu f^\mu(x)$$

for all x , where we set $\mathcal{L}(x) = \mathcal{L}[\phi(x), \partial_\mu\phi(x), x]$.

If this holds on shell and off shell, we say Q generates an off-shell symmetry. If this only holds on shell, we say Q generates an on-shell symmetry. Then, we say Q is a generator of a one parameter symmetry Lie group.

Now, for any N , because of the Euler-Lagrange theorem, on shell (and only on-shell), we have

$$\begin{aligned} Q \left[\int_N \mathcal{L} d^n x \right] &= \int_N \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right] Q[\phi] d^n x + \int_{\partial N} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} Q[\phi] ds_\mu \\ &\approx \int_{\partial N} f^\mu ds_\mu. \end{aligned}$$

Since this is true for any N , we have

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} Q[\phi] - f^\mu \right] \approx 0.$$

But this is the continuity equation for the current J^μ defined by:^[8]

$$J^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} Q[\phi] - f^\mu,$$

which is called the **Noether current** associated with the symmetry. The continuity equation tells us that if we integrate this current over a space-like slice, we get a conserved quantity called the Noether charge (provided, of course, if M is noncompact, the currents fall off sufficiently fast at infinity).

Comments

Noether's theorem is really a reflection of the relation between the boundary conditions and the variational principle. Assuming no boundary terms in the action, Noether's theorem implies that

$$\int_{\partial N} J^\mu ds_\mu \approx 0.$$

Noether's theorem is an on shell theorem. The quantum analog of Noether's theorem are the Ward-Takahashi identities.

Generalization to Lie algebras

Suppose say we have two symmetry derivations Q_1 and Q_2 . Then, $[Q_1, Q_2]$ is also a symmetry derivation. Let's see this explicitly. Let's say

$$Q_1[\mathcal{L}] \approx \partial_\mu f_1^\mu$$

and

$$Q_2[\mathcal{L}] \approx \partial_\mu f_2^\mu$$

Then,

$$[Q_1, Q_2][\mathcal{L}] = Q_1[Q_2[\mathcal{L}]] - Q_2[Q_1[\mathcal{L}]] \approx \partial_\mu f_{12}^\mu$$

where $f_{12}^\mu = Q_1[f_2^\mu] - Q_2[f_1^\mu]$. So,

$$j_{12}^\mu = \left(\frac{\partial}{\partial(\partial_\mu \phi)} \mathcal{L} \right) (Q_1[Q_2[\phi]] - Q_2[Q_1[\phi]]) - f_{12}^\mu.$$

This shows we can (trivially) extend Noether's theorem to larger Lie algebras.

Generalization of the proof

This applies to *any* local symmetry derivation Q satisfying $QS \approx 0$, and also to more general local functional differentiable actions, including ones where the Lagrangian depends on higher derivatives of the fields. Let ε be any arbitrary smooth function of the spacetime (or time) manifold such that the closure of its support is disjoint from the boundary. ε is a test function. Then, because of the variational principle (which does *not* apply to the boundary, by the way), the derivation distribution q generated by $q[\varepsilon][\Phi(x)] = \varepsilon(x)Q[\Phi(x)]$ satisfies $q[\varepsilon][S] \approx 0$ for any ε , or more compactly, $q(x)[S] \approx 0$ for all x not on the boundary (but remember that $q(x)$ is a shorthand for a derivation *distribution*, not a derivation parametrized by x in general). This is the generalization of Noether's theorem.

To see how the generalization related to the version given above, assume that the action is the spacetime integral of a Lagrangian that only depends on ϕ and its first derivatives. Also, assume

$$Q[\mathcal{L}] \approx \partial_\mu f^\mu$$

Then,

$$\begin{aligned} q[\varepsilon][S] &= \int q[\varepsilon][\mathcal{L}] d^n x \\ &= \int \left\{ \left(\frac{\partial}{\partial \phi} \mathcal{L} \right) \varepsilon Q[\phi] + \left[\frac{\partial}{\partial(\partial_\mu \phi)} \mathcal{L} \right] \partial_\mu (\varepsilon Q[\phi]) \right\} d^n x \\ &= \int \left\{ \varepsilon Q[\mathcal{L}] - \partial_\mu \varepsilon \left[\frac{\partial}{\partial(\partial_\mu \phi)} \mathcal{L} \right] Q[\phi] \right\} d^n x \\ &\approx \int \varepsilon \partial_\mu \left\{ f^\mu - \left[\frac{\partial}{\partial(\partial_\mu \phi)} \mathcal{L} \right] Q[\phi] \right\} d^n x \end{aligned}$$

for all ε .

More generally, if the Lagrangian depends on higher derivatives, then

$$\partial_\mu \left[f^\mu - \left[\frac{\partial}{\partial(\partial_\mu \phi)} \mathcal{L} \right] Q[\phi] - 2 \left[\frac{\partial}{\partial(\partial_\mu \partial_\nu \phi)} \mathcal{L} \right] \partial_\nu Q[\phi] + \partial_\nu \left[\left[\frac{\partial}{\partial(\partial_\mu \partial_\nu \phi)} \mathcal{L} \right] Q[\phi] \right] - \dots \right] \approx 0.$$

Examples

Example 1: Conservation of energy

Looking at the specific case of a Newtonian particle of mass m , coordinate x , moving under the influence of a potential V , coordinatized by time t . The action, S , is:

$$\begin{aligned} S[x] &= \int L[x(t), \dot{x}(t)] dt \\ &= \int \left(\frac{m}{2} \sum_{i=1}^3 \dot{x}_i^2 - V(x(t)) \right) dt \end{aligned}$$

Consider the generator of time translations $Q = \partial/\partial t$. In other words, $Q[x(t)] = \dot{x}(t)$. Note that x has an explicit dependence on time, whilst V does not; consequently:

$$Q[L] = m \sum_i \dot{x}_i \ddot{x}_i - \sum_i \frac{\partial V(x)}{\partial x_i} \dot{x}_i = \frac{d}{dt} \left[\frac{m}{2} \sum_i \dot{x}_i^2 - V(x) \right]$$

so we can set

$$f = \frac{m}{2} \sum_i \dot{x}_i^2 - V(x).$$

Then,

$$\begin{aligned} j &= \sum_{i=1}^3 \frac{\partial L}{\partial \dot{x}_i} Q[x_i] - f \\ &= m \sum_i \dot{x}_i^2 - \left[\frac{m}{2} \sum_i \dot{x}_i^2 - V(x) \right] \\ &= \frac{m}{2} \sum_i \dot{x}_i^2 + V(x). \end{aligned}$$

The right hand side is the energy and Noether's theorem states that $\dot{j} = 0$ (i.e. the principle of conservation of energy is a consequence of invariance under time translations). More generally, if the Lagrangian does not depend explicitly on time, the quantity

$$\sum_{i=1}^3 \frac{\partial L}{\partial \dot{x}_i} \dot{x}_i - L$$

(called the Hamiltonian) is conserved.

Example 2: Conservation of center of momentum

Still considering 1-dimensional time, let

$$\begin{aligned} S[\vec{x}] &= \int \mathcal{L}[\vec{x}(t), \dot{\vec{x}}(t)] dt \\ &= \int \left[\sum_{\alpha=1}^N \frac{m_{\alpha}}{2} (\dot{\vec{x}}_{\alpha})^2 - \sum_{\alpha < \beta} V_{\alpha\beta}(\vec{x}_{\beta} - \vec{x}_{\alpha}) \right] dt \end{aligned}$$

i.e. N Newtonian particles where the potential only depends pairwise upon the relative displacement.

For $\vec{Q}$, let's consider the generator of Galilean transformations (i.e. a change in the frame of reference). In other words,

$$Q_i[x_{\alpha}^j(t)] = t \delta_i^j.$$

Note that

$$\begin{aligned} Q_i[\mathcal{L}] &= \sum_{\alpha} m_{\alpha} \dot{x}_{\alpha}^i - \sum_{\alpha < \beta} \partial_i V_{\alpha\beta}(\vec{x}_{\beta} - \vec{x}_{\alpha})(t - t) \\ &= \sum_{\alpha} m_{\alpha} \dot{x}_{\alpha}^i. \end{aligned}$$

This has the form of $\frac{d}{dt} \sum_{\alpha} m_{\alpha} x_{\alpha}^i$ so we can set

$$\vec{f} = \sum_{\alpha} m_{\alpha} \vec{x}_{\alpha}.$$

Then,

$$\begin{aligned} \vec{j} &= \sum_{\alpha} \left(\frac{\partial}{\partial \dot{\vec{x}}_{\alpha}} \mathcal{L} \right) \cdot \vec{Q}[\vec{x}_{\alpha}] - \vec{f} \\ &= \sum_{\alpha} (m_{\alpha} \dot{\vec{x}}_{\alpha} t - m_{\alpha} \vec{x}_{\alpha}) \\ &= \vec{P}t - M\vec{x}_{CM} \end{aligned}$$

where $\vec{P}$ is the total momentum, M is the total mass and $\vec{x}_{CM}$ is the center of mass. Noether's theorem states:

$$\dot{\vec{j}} = 0 \Rightarrow \vec{P} - M\dot{\vec{x}}_{CM} = 0.$$

Example 3: Conformal transformation

Both examples 1 and 2 are over a 1-dimensional manifold (time). An example involving spacetime is a conformal transformation of a massless real scalar field with a quartic potential in (3 + 1)-Minkowski spacetime.

$$\begin{aligned} \mathcal{S}[\phi] &= \int \mathcal{L}[\phi(x), \partial_{\mu}\phi(x)] d^4x \\ &= \int \left(\frac{1}{2} \partial^{\mu}\phi \partial_{\mu}\phi - \lambda\phi^4 \right) d^4x \end{aligned}$$

For Q , consider the generator of a spacetime rescaling. In other words,

$$Q[\phi(x)] = x^{\mu} \partial_{\mu} \phi(x) + \phi(x).$$

The second term on the right hand side is due to the "conformal weight" of ϕ . Note that

$$Q[\mathcal{L}] = \partial^{\mu}\phi (\partial_{\mu}\phi + x^{\nu} \partial_{\mu} \partial_{\nu} \phi + \partial_{\mu}\phi) - 4\lambda\phi^3 (x^{\mu} \partial_{\mu}\phi + \phi).$$

This has the form of

$$\partial_{\mu} \left[\frac{1}{2} x^{\mu} \partial^{\nu} \phi \partial_{\nu} \phi - \lambda x^{\mu} \phi^4 \right] = \partial_{\mu} (x^{\mu} \mathcal{L})$$

(where we have performed a change of dummy indices) so set

$$f^{\mu} = x^{\mu} \mathcal{L}.$$

Then,

$$\begin{aligned} j^{\mu} &= \left[\frac{\partial}{\partial (\partial_{\mu}\phi)} \mathcal{L} \right] Q[\phi] - f^{\mu} \\ &= \partial^{\mu}\phi (x^{\nu} \partial_{\nu} \phi + \phi) - x^{\mu} \left(\frac{1}{2} \partial^{\nu} \phi \partial_{\nu} \phi - \lambda\phi^4 \right). \end{aligned}$$

Noether's theorem states that $\partial_\mu j^\mu = 0$ (as one may explicitly check by substituting the Euler-Lagrange equations into the left hand side).

(Aside: If one tries to find the Ward-Takahashi analog of this equation, one runs into a problem because of anomalies.)

Applications

Application of Noether's theorem allows physicists to gain powerful insights into any general theory in physics, by just analyzing the various transformations that would make the form of the laws involved invariant. For example:

- the invariance of physical systems with respect to spatial translation (in other words, that the laws of physics do not vary with locations in space) gives the law of conservation of linear momentum;
- invariance with respect to rotation gives the law of conservation of angular momentum;
- invariance with respect to time translation gives the well-known law of conservation of energy

In quantum field theory, the analog to Noether's theorem, the Ward-Takahashi identities, yields further conservation laws, such as the conservation of electric charge from the invariance with respect to a change in the phase factor of the complex field of the charged particle and the associated gauge of the electric potential and vector potential.

The Noether charge is also used in calculating the entropy of stationary black holes^[9].

See also

- Charge (physics)
- Invariant (physics)

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Rigged Hilbert space

In mathematics, a **rigged Hilbert space** (**Gelfand triple**, **nested Hilbert space**, **equipped Hilbert space**) is a construction designed to link the distribution and square-integrable aspects of functional analysis. Such spaces were introduced to study spectral theory in the broad sense. They can bring together the 'bound state' (eigenvector) and 'continuous spectrum', in one place.

Motivation

Since a function such as

$$x \mapsto e^{ix},$$

which is in an obvious sense an eigenvector of the differential operator

$$-i \frac{d}{dx}$$

on the real line $\mathbf{R}$, is not square-integrable for the usual Borel measure on $\mathbf{R}$, this requires some way of stepping outside the strict confines of the Hilbert space theory. This was supplied by the apparatus of Schwartz distributions, and a *generalized eigenfunction* theory was developed in the years after 1950.

Functional analysis approach

The concept of rigged Hilbert space places this idea in abstract functional-analytic framework. Formally, a rigged Hilbert space consists of a Hilbert space H , together with a subspace Φ which carries a finer topology, that is one for which the natural inclusion

$$\Phi \subseteq H$$

is continuous. It is no loss to assume that Φ is dense in H for the Hilbert norm. We consider the inclusion of dual spaces H^* in Φ^* . The latter, dual to Φ in its 'test function' topology, is realised as a space of distributions or generalised functions of some sort, and the linear functionals on the subspace Φ of type

$$\phi \mapsto \langle v, \phi \rangle$$

for v in H are faithfully represented as distributions (because we assume Φ dense).

Now by applying the Riesz representation theorem we can identify H^* with H . Therefore the definition of *rigged Hilbert space* is in terms of a sandwich:

$$\Phi \subseteq H \subseteq \Phi^*.$$

The most significant examples are for which Φ is a nuclear space; this comment is an abstract expression of the idea that Φ consists of test functions and Φ^* of the corresponding distributions.

Formal definition (Gelfand triple)

A **rigged Hilbert space** is a pair (H, Φ) with H a Hilbert space, Φ a dense subspace, such that Φ is given a topological vector space structure for which the inclusion map i is continuous. Identifying H with its dual space H^* , the adjoint to i is the map $i^* : H = H^* \rightarrow \Phi^*$. The duality pairing between Φ and Φ^* has to be compatible with the inner product on H : $\langle u, v \rangle_{\Phi \times \Phi^*} = (u, v)_H$ whenever $u \in \Phi \subset H$ and $v \in H = H^* \subset \Phi^*$.

The specific triple $\{\Phi, H, \Phi^*\}$ is often named the "Gelfand triple" (after the Ukrainian mathematician Israel Gelfand).

Note that even though Φ is isomorphic to Φ^* if Φ is a Hilbert space in its own right, this isomorphism is *not* the same as the composition of the inclusion i with its adjoint i^*

$$i^* i : \Phi \subset H = H^* \rightarrow \Phi^*.$$

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Kac-Moody algebra

In mathematics, a **Kac-Moody algebra** is a Lie algebra, usually infinite-dimensional, that can be defined by generators and relations through a generalized Cartan matrix. Kac-Moody algebras are named after Victor Kac and Robert Moody, who independently discovered them. These algebras form a generalization of finite-dimensional semisimple Lie algebras, and many properties related to structure of the Lie algebra, its root system, irreducible representations, connection to flag manifolds have natural analogues in the Kac-Moody setting. A class of Kac-Moody algebras called **affine Lie algebras** is of particular importance in mathematics and theoretical physics, especially conformal field theory and the theory of exactly solvable models. Kac discovered an elegant proof of certain combinatorial identities, Macdonald identities, which is based on the representation theory of affine Kac-Moody algebras. Garland and Lepowski demonstrated that Rogers-Ramanujan identities can be derived in a similar fashion.

Definition

A Kac-Moody algebra is given by the following:

1. An n by n generalized Cartan matrix $C = (c_{ij})$ of rank r .
2. A vector space $\mathfrak{h}$ over the complex numbers of dimension $2n - r$.
3. A set of n linearly independent elements of $\mathfrak{h}$ and a set of n linearly independent elements α_i^* of the dual space, such that $\alpha_i^*(\alpha_j) = c_{ij}$. The are known as **coroots**, while the α_i^* are known as **roots**.

The Kac-Moody algebra is the Lie algebra $\mathfrak{g}$ defined by generators e_i and f_i and the elements of $\mathfrak{h}$ and relations

- for $i \neq j$
- $[e_i, x] = \alpha_i^*(x)e_i$, for $x \in \mathfrak{h}$
- $[f_i, x] = -\alpha_i^*(x)f_i$, for $x \in \mathfrak{h}$
- for $x, x' \in \mathfrak{h}$
- $\text{ad}(e_i)^{1-c_{ij}}(e_j) = 0$
- $\text{ad}(f_i)^{1-c_{ij}}(f_j) = 0$

where $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$, $\text{ad}(x)(y) = [x, y]$ is the adjoint representation of $\mathfrak{g}$.

A real (possibly infinite-dimensional) Lie algebra is also considered a Kac-Moody algebra if its complexification is a Kac-Moody algebra.

Interpretation

$\mathfrak{h}$ is a Cartan subalgebra of the Kac-Moody algebra.

If g is an element of the Kac-Moody algebra such that

$$\forall x \in \mathfrak{h} [g, x] = \omega(x)g$$

where ω is an element of $\mathfrak{h}^*$, then g is said to have weight ω . The Kac-Moody algebra can be diagonalized into weight eigenvectors. The Cartan subalgebra h has weight zero, e_i has weight α_i^* and f_i has weight $-\alpha_i^*$. If the Lie bracket of two weight eigenvectors is nonzero, then its weight is the sum of their weights. The condition for $i \neq j$ simply means the α_i^* are simple roots.

Types of Kac-Moody algebras

Properties of a Kac-Moody algebra are controlled by the algebraic properties of its generalized Cartan matrix C . In order to classify Kac-Moody algebras, it is enough to consider the case of an *indecomposable* matrix C , that is, assume that there is no decomposition of the set of indices I into a disjoint union of non-empty subsets I_1 and I_2 such that $C_{ij} = 0$ for all i in I_1 and j in I_2 . Any decomposition of the generalized Cartan matrix leads to the direct sum decomposition of the corresponding Kac-Moody algebra:

$$\mathfrak{g}(C) \simeq \mathfrak{g}(C_1) \oplus \mathfrak{g}(C_2),$$

where the two Kac-Moody algebras in the right hand side are associated with the submatrices of C corresponding to the index sets I_1 and I_2 .

An important subclass of Kac-Moody algebras corresponds to *symmetrizable* generalized Cartan matrices C , which can be decomposed as DS , where D is a diagonal matrix with positive integer entries and S is a symmetric matrix. Under the assumptions that C is symmetrizable and indecomposable, the Kac-Moody algebras are divided into three classes:

- A positive definite matrix S gives rise to a finite-dimensional simple Lie algebra.
- A positive semidefinite matrix S gives rise to an infinite-dimensional Kac-Moody algebra of **affine type**, or an affine Lie algebra.
- An indefinite matrix S gives rise to a Kac-Moody algebra of **indefinite type**.
- Since the diagonal entries of C and S are positive, S cannot be negative definite or negative semidefinite.

Symmetrizable indecomposable generalized Cartan matrices of finite and affine type have been completely classified. They correspond to Dynkin diagrams and affine Dynkin diagrams. Very little is known about the Kac-Moody algebras of indefinite type. Among those, the main focus has been on the (generalized) Kac-Moody algebras of **hyperbolic type**, for which the matrix S is indefinite, but for each proper subset of I , the corresponding submatrix is positive definite or positive semidefinite. Such matrices have rank at most 10 and have also been completely determined.

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See also

- Weyl-Kac character formula
- Generalized Kac-Moody algebra

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Operator algebra

In functional analysis, an **operator algebra** is an algebra of continuous linear operators on a topological vector space with the multiplication given by the composition of mappings. Although it is usually classified as a branch of functional analysis, it has direct applications to representation theory, differential geometry, quantum statistical mechanics and quantum field theory.

Such algebras can be used to study arbitrary sets of operators with little algebraic relation *simultaneously*. From this point of view, operator algebras can be regarded as a generalization of spectral theory of a single operator. In general operator algebras are non-commutative rings.

An operator algebra is typically required to be closed in a specified operator topology inside the algebra of the whole continuous linear operators. In particular, it is a set of operators with both algebraic and topological closure properties. In some discipline such properties are axiomized and algebras with certain topological structure become the subject of the research.

Though algebras of operators are studied in various context (for example, algebras of pseudo-differential operators acting on spaces of distributions), the term *operator algebra* is usually used in reference to algebras of bounded operators on a Banach space or, even more specially in reference to algebras of operators on a separable Hilbert space, endowed with the operator norm topology.

In the case of operators on a Hilbert space, the adjoint map on operators gives a natural involution which provides an additional algebraic structure which can be imposed on the algebra. In this context, the best studied examples are self-adjoint operator algebras, meaning that they are closed under taking adjoints. These include C*-algebras and von Neumann algebras. C*-algebras can be easily characterized abstractly by a condition relating the norm, involution and multiplication. Such abstractly defined C*-algebras can be

identified to a certain closed subalgebra of the algebra of the continuous linear operators on a suitable Hilbert space. A similar result holds for von Neumann algebras.

Commutative self-adjoint operator algebras can be regarded as the algebra of complex valued continuous functions on a locally compact space, or that of measurable functions on a standard measurable space. Thus, general operator algebras are often regarded as a noncommutative generalizations of these algebras, or the structure of the *base space* on which the functions are defined. This point of view is elaborated as the philosophy of noncommutative geometry, which tries to study various non-classical and/or pathological objects by noncommutative operator algebras.

Examples of operator algebras which are not self-adjoint include:

- nest algebras
- many commutative subspace lattice algebras
- many limit algebras

See also

- Topologies on the set of operators on a Hilbert space

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Clifford algebra

In mathematics, **Clifford algebras** are a type of associative algebra. They can be thought of as one of the possible generalizations of the complex numbers and quaternions. The theory of Clifford algebras is intimately connected with the theory of quadratic forms and orthogonal transformations. Clifford algebras have important applications in a variety of fields including geometry and theoretical physics. They are named for the English geometer William Kingdon Clifford.

Introduction and basic properties

Specifically, a Clifford algebra is a unital associative algebra which contains and is generated by a vector space V equipped with a quadratic form Q . The Clifford algebra $C\ell(V, Q)$ is the "freest" algebra generated by V subject to the condition^[1]

$$v^2 = Q(v) \text{ for all } v \in V.$$

If the characteristic of the ground field K is not 2, then one can rewrite this fundamental identity in the form

$$uv + vu = 2\langle u, v \rangle \text{ for all } u, v \in V,$$

where $\langle u, v \rangle = \frac{1}{2}(Q(u + v) - Q(u) - Q(v))$ is the symmetric bilinear form associated to Q . This idea of "freest" or "most general" algebra subject to this identity can be formally expressed through the notion of a universal property (see below).

Quadratic forms and Clifford algebras in characteristic 2 form an exceptional case. In particular, if $\text{char } K = 2$ it is not true that a quadratic form is determined by its symmetric

bilinear form, or that every quadratic form admits an orthogonal basis. Many of the statements in this article include the condition that the characteristic is not 2, and are false if this condition is removed.

As quantization of exterior algebra

Clifford algebras are closely related to exterior algebras. In fact, if $Q = 0$ then the Clifford algebra $C\ell(V, Q)$ is just the exterior algebra $\Lambda(V)$. For nonzero Q there exists a canonical *linear* isomorphism between $\Lambda(V)$ and $C\ell(V, Q)$ whenever the ground field K does not have characteristic two. That is, they are naturally isomorphic as vector spaces, but with different multiplications (in the case of characteristic two, they are still isomorphic as vector spaces, just not naturally). Clifford multiplication is strictly richer than the exterior product since it makes use of the extra information provided by Q .

More precisely, Clifford algebras may be thought of as *quantizations* (cf. quantization (physics), Quantum group) of the exterior algebra, in the same way that the Weyl algebra is a quantization of the symmetric algebra.

Weyl algebras and Clifford algebras admit a further structure of a $*$ -algebra, and can be unified as even and odd terms of a superalgebra, as discussed in CCR and CAR algebras.

Universal property and construction

Let V be a vector space over a field K , and let $Q : V \rightarrow K$ be a quadratic form on V . In most cases of interest the field K is either $\mathbf{R}$, $\mathbf{C}$ or a finite field.

A Clifford algebra $C\ell(V, Q)$ is a unital associative algebra over K together with a linear map $i : V \rightarrow C\ell(V, Q)$ satisfying $i(v)^2 = Q(v)1$ for all $v \in V$, defined by the following universal property: Given any associative algebra A over K and any linear map $j : V \rightarrow A$ such that

$$j(v)^2 = Q(v)1 \text{ for all } v \in V$$

(where 1 denotes the multiplicative identity of A), there is a unique algebra homomorphism $f : C\ell(V, Q) \rightarrow A$ such that the following diagram commutes (i.e. such that $f \circ i = j$):

$$\begin{array}{ccc} V & \xrightarrow{i} & C\ell(V, Q) \\ & \searrow j & \downarrow f \\ & & A \end{array}$$

Working with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ instead of Q (in characteristic not 2), the requirement on j is

$$j(v)j(w) + j(w)j(v) = 2\langle v, w \rangle \text{ for all } v, w \in V.$$

A Clifford algebra as described above always exists and can be constructed as follows: start with the most general algebra that contains V , namely the tensor algebra $T(V)$, and then enforce the fundamental identity by taking a suitable quotient. In our case we want to take the two-sided ideal I_Q in $T(V)$ generated by all elements of the form

$$v \otimes v - Q(v)1 \text{ for all } v \in V$$

and define $C\ell(V, Q)$ as the quotient

$$C\ell(V, Q) = T(V)/I_Q.$$

It is then straightforward to show that $C\ell(V, Q)$ contains V and satisfies the above universal property, so that $C\ell$ is unique up to a unique isomorphism; thus one speaks of "the" Clifford

algebra $C\ell(V, Q)$. It also follows from this construction that i is injective. One usually drops the i and considers V as a linear subspace of $C\ell(V, Q)$.

The universal characterization of the Clifford algebra shows that the construction of $C\ell(V, Q)$ is *functorial* in nature. Namely, $C\ell$ can be considered as a functor from the category of vector spaces with quadratic forms (whose morphisms are linear maps preserving the quadratic form) to the category of associative algebras. The universal property guarantees that linear maps between vector spaces (preserving the quadratic form) extend uniquely to algebra homomorphisms between the associated Clifford algebras.

Basis and dimension

If the dimension of V is n and $\{e_1, \dots, e_n\}$ is a basis of V , then the set

$$\{e_{i_1}e_{i_2}\cdots e_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n \text{ and } 0 \leq k \leq n\}$$

is a basis for $C\ell(V, Q)$. The empty product ($k = 0$) is defined as the multiplicative identity element. For each value of k there are $\binom{n}{k}$ basis elements, so the total dimension of the Clifford algebra is

$$\dim C\ell(V, Q) = \sum_{k=0}^n \binom{n}{k} = 2^n.$$

Since V comes equipped with a quadratic form, there is a set of privileged bases for V : the orthogonal ones. An orthogonal basis is one such that

$$\langle e_i, e_j \rangle = 0 \quad i \neq j.$$

where $\langle \cdot, \cdot \rangle$ is the symmetric bilinear form associated to Q . The fundamental Clifford identity implies that for an orthogonal basis

$$e_i e_j = -e_j e_i \quad i \neq j.$$

This makes manipulation of orthogonal basis vectors quite simple. Given a product $e_{i_1}e_{i_2}\cdots e_{i_k}$ of *distinct* orthogonal basis vectors, one can put them into standard order by including an overall sign corresponding to the number of flips needed to correctly order them (i.e. the signature of the ordering permutation).

If the characteristic is not 2 then an orthogonal basis for V exists, and one can easily extend the quadratic form on V to a quadratic form on all of $C\ell(V, Q)$ by requiring that distinct elements $e_{i_1}e_{i_2}\cdots e_{i_k}$ are orthogonal to one another whenever the $\{e_i\}$'s are orthogonal. Additionally, one sets

$$Q(e_{i_1}e_{i_2}\cdots e_{i_k}) = Q(e_{i_1})Q(e_{i_2})\cdots Q(e_{i_k}).$$

The quadratic form on a scalar is just $Q(\lambda) = \lambda^2$. Thus, orthogonal bases for V extend to orthogonal bases for $C\ell(V, Q)$. The quadratic form defined in this way is actually independent of the orthogonal basis chosen (a basis-independent formulation will be given later).

Examples: real and complex Clifford algebras

The most important Clifford algebras are those over real and complex vector spaces equipped with nondegenerate quadratic forms.

Every nondegenerate quadratic form on a finite-dimensional real vector space is equivalent to the standard diagonal form:

$$Q(v) = v_1^2 + \cdots + v_p^2 - v_{p+1}^2 - \cdots - v_{p+q}^2$$

where $n = p + q$ is the dimension of the vector space. The pair of integers (p, q) is called the signature of the quadratic form. The real vector space with this quadratic form is often denoted $\mathbf{R}^{p,q}$. The Clifford algebra on $\mathbf{R}^{p,q}$ is denoted $C\!\!\!\!/\!\!\!_{p,q}(\mathbf{R})$. The symbol $C\!\!\!\!/\!\!\!_n(\mathbf{R})$ means either $C\!\!\!\!/\!\!\!_{n,0}(\mathbf{R})$ or $C\!\!\!\!/\!\!\!_{0,n}(\mathbf{R})$ depending on whether the author prefers positive definite or negative definite spaces.

A standard orthonormal basis $\{e_i\}$ for $\mathbf{R}^{p,q}$ consists of $n = p + q$ mutually orthogonal vectors, p of which have norm $+1$ and q of which have norm -1 . The algebra $C\!\!\!\!/\!\!\!_{p,q}(\mathbf{R})$ will therefore have p vectors which square to $+1$ and q vectors which square to -1 .

Note that $C\!\!\!\!/\!\!\!_{0,0}(\mathbf{R})$ is naturally isomorphic to $\mathbf{R}$ since there are no nonzero vectors. $C\!\!\!\!/\!\!\!_{0,1}(\mathbf{R})$ is a two-dimensional algebra generated by a single vector e_1 which squares to -1 , and therefore is isomorphic to $\mathbf{C}$, the field of complex numbers. The algebra $C\!\!\!\!/\!\!\!_{0,2}(\mathbf{R})$ is a four-dimensional algebra spanned by $\{1, e_1, e_2, e_1 e_2\}$. The latter three elements square to -1 and all anticommute, and so the algebra is isomorphic to the quaternions $\mathbf{H}$. The next algebra in the sequence is $C\!\!\!\!/\!\!\!_{0,3}(\mathbf{R})$ is an 8-dimensional algebra isomorphic to the direct sum $\mathbf{H} \oplus \mathbf{H}$ called split-biquaternions.

One can also study Clifford algebras on complex vector spaces. Every nondegenerate quadratic form on a complex vector space is equivalent to the standard diagonal form

$$Q(z) = z_1^2 + z_2^2 + \cdots + z_n^2$$

where $n = \dim V$, so there is essentially only one Clifford algebra in each dimension. We will denote the Clifford algebra on $\mathbf{C}^n$ with the standard quadratic form by $C\!\!\!\!/\!\!\!_n(\mathbf{C})$. One can show that the algebra $C\!\!\!\!/\!\!\!_n(\mathbf{C})$ may be obtained as the complexification of the algebra $C\!\!\!\!/\!\!\!_{p,q}(\mathbf{R})$ where $n = p + q$:

$$C\ell_n(\mathbf{C}) \cong C\ell_{p,q}(\mathbf{R}) \otimes \mathbf{C} \cong C\ell(\mathbf{C}^{p+q}, Q \otimes \mathbf{C}).$$

Here Q is the real quadratic form of signature (p, q) . Note that the complexification does not depend on the signature. The first few cases are not hard to compute. One finds that

$$C\!\!\!\!/\!\!\!_0(\mathbf{C}) = \mathbf{C}$$

$$C\!\!\!\!/\!\!\!_1(\mathbf{C}) = \mathbf{C} \oplus \mathbf{C}$$

$$C\!\!\!\!/\!\!\!_2(\mathbf{C}) = M_2(\mathbf{C})$$

where $M_2(\mathbf{C})$ denotes the algebra of 2×2 matrices over $\mathbf{C}$.

It turns out that every one of the algebras $C\!\!\!\!/\!\!\!_{p,q}(\mathbf{R})$ and $C\!\!\!\!/\!\!\!_n(\mathbf{C})$ is isomorphic to a matrix algebra over $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ or to a direct sum of two such algebras. For a complete classification of these algebras see classification of Clifford algebras.

Properties

Relation to the exterior algebra

Given a vector space V one can construct the exterior algebra $\Lambda(V)$, whose definition is independent of any quadratic form on V . It turns out that if F does not have characteristic 2 then there is a natural isomorphism between $\Lambda(V)$ and $C\!\!\!\!/\!(V, Q)$ considered as vector spaces (and there exists an isomorphism in characteristic two, which may not be natural). This is an algebra isomorphism if and only if $Q = 0$. One can thus consider the Clifford algebra $C\!\!\!\!/\!(V, Q)$ as an enrichment (or more precisely, a quantization, cf. the Introduction) of the exterior algebra on V with a multiplication that depends on Q (one can still define the exterior product independent of Q).

The easiest way to establish the isomorphism is to choose an *orthogonal* basis $\{e_i\}$ for V and extend it to an orthogonal basis for $C\!\!\!\!/\!(V, Q)$ as described above. The map $C\!\!\!\!/\!(V, Q) \rightarrow \Lambda(V)$ is determined by

$$e_{i_1} e_{i_2} \cdots e_{i_k} \mapsto e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}.$$

Note that this only works if the basis $\{e_i\}$ is orthogonal. One can show that this map is independent of the choice of orthogonal basis and so gives a natural isomorphism.

If the characteristic of K is 0, one can also establish the isomorphism by antisymmetrizing. Define functions $f_k : V \times \cdots \times V \rightarrow C\!\!\!\!/\!(V, Q)$ by

$$f_k(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) v_{\sigma(1)} \cdots v_{\sigma(k)}$$

where the sum is taken over the symmetric group on k elements. Since f_k is alternating it induces a unique linear map $\Lambda^k(V) \rightarrow C\!\!\!\!/\!(V, Q)$. The direct sum of these maps gives a linear map between $\Lambda(V)$ and $C\!\!\!\!/\!(V, Q)$. This map can be shown to be a linear isomorphism, and it is natural.

A more sophisticated way to view the relationship is to construct a filtration on $C\!\!\!\!/\!(V, Q)$. Recall that the tensor algebra $T(V)$ has a natural filtration: $F^0 \subset F^1 \subset F^2 \subset \cdots$ where F^k contains sums of tensors with rank $\leq k$. Projecting this down to the Clifford algebra gives a filtration on $C\!\!\!\!/\!(V, Q)$. The associated graded algebra

$$Gr_F C\ell(V, Q) = \bigoplus_k F^k / F^{k-1}$$

is naturally isomorphic to the exterior algebra $\Lambda(V)$. Since the associated graded algebra of a filtered algebra is always isomorphic to the filtered algebra as filtered vector spaces (by choosing complements of F^k in F^{k+1} for all k), this provides an isomorphism (although not a natural one) in any characteristic, even two.

Grading

In the following, assume that the characteristic is not 2.^[2]

Clifford algebras are $\mathbf{Z}_2$ -graded algebra (also known as superalgebras). Indeed, the linear map on V defined by $v \mapsto -v$ (reflection through the origin) preserves the quadratic form Q and so by the universal property of Clifford algebras extends to an algebra automorphism

$$\alpha : C\!\!\!\!/\!(V, Q) \rightarrow C\!\!\!\!/\!(V, Q).$$

Since α is an involution (i.e. it squares to the identity) one can decompose $C\!\!\!\!/\!(V, Q)$ into positive and negative eigenspaces

$$C\ell(V, Q) = C\ell^0(V, Q) \oplus C\ell^1(V, Q)$$

where $C\ell^i(V, Q) = \{x \in C\!\!\!\!/\!(V, Q) \mid \alpha(x) = (-1)^i x\}$. Since α is an automorphism it follows that

$$C\ell^i(V, Q)C\ell^j(V, Q) = C\ell^{i+j}(V, Q)$$

where the superscripts are read modulo 2. This gives $C\!\!\!\!/\!(V, Q)$ the structure of a $\mathbf{Z}_2$ -graded algebra. The subspace $C\ell^0(V, Q)$ forms a subalgebra of $C\!\!\!\!/\!(V, Q)$, called the *even subalgebra*. The subspace $C\ell^1(V, Q)$ is called the *odd part* of $C\!\!\!\!/\!(V, Q)$ (it is not a subalgebra). The $\mathbf{Z}_2$ -grading plays an important role in the analysis and application of Clifford algebras. The automorphism α is called the *main involution* or *grade involution*.

Remark. In characteristic not 2 the underlying vector space of $C\!\!\!\!/\!(V, Q)$ inherits a $\mathbf{Z}$ -grading from the canonical isomorphism with the underlying vector space of the exterior algebra $\Lambda(V)$. It is important to note, however, that this is a *vector space grading only*. That is, Clifford multiplication does not respect the $\mathbf{Z}$ -grading, only the $\mathbf{Z}_2$ -grading: for instance if $Q(v) \neq 0$, then $v \in C\ell^1(V, Q)$, but $v^2 \in C\ell^0(V, Q)$, not in $C\ell^2(V, Q)$. Happily, the gradings are related in the natural way: $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$. Further, the Clifford algebra is $\mathbf{Z}$ -filtered: $C\ell^{\leq i}(V, Q) \cdot C\ell^{\leq j}(V, Q) \subset C\ell^{\leq i+j}(V, Q)$. The *degree* of a Clifford number usually refers to the degree in the $\mathbf{Z}$ -grading. Elements which are pure in the $\mathbf{Z}_2$ -grading are simply said to be even or odd.

The even subalgebra $C\ell^0(V, Q)$ of a Clifford algebra is itself a Clifford algebra^[3]. If V is the orthogonal direct sum of a vector a of norm $Q(a)$ and a subspace U , then $C\ell^0(V, Q)$ is isomorphic to $C\!\!\!\!/\!(U, -Q(a)Q)$, where $-Q(a)Q$ is the form Q restricted to U and multiplied by $-Q(a)$. In particular over the reals this implies that

$$C\ell_{p,q}^0(\mathbb{R}) \cong C\ell_{p,q-1}(\mathbb{R}) \text{ for } q > 0, \text{ and}$$

$$C\ell_{p,q}^0(\mathbb{R}) \cong C\ell_{q,p-1}(\mathbb{R}) \text{ for } p > 0.$$

In the negative-definite case this gives an inclusion $C\!\!\!\!/\!_{0,n-1}(\mathbf{R}) \subset C\!\!\!\!/\!_{0,n}(\mathbf{R})$ which extends the sequence

$$\mathbf{R} \subset \mathbf{C} \subset \mathbf{H} \subset \mathbf{H} \oplus \mathbf{H} \subset \dots$$

Likewise, in the complex case, one can show that the even subalgebra of $C\!\!\!\!/\!_n(\mathbf{C})$ is isomorphic to $C\!\!\!\!/\!_{n-1}(\mathbf{C})$.

Antiautomorphisms

In addition to the automorphism α , there are two antiautomorphisms which play an important role in the analysis of Clifford algebras. Recall that the tensor algebra $T(V)$ comes with an antiautomorphism that reverses the order in all products:

$$v_1 \otimes v_2 \otimes \cdots \otimes v_k \mapsto v_k \otimes \cdots \otimes v_2 \otimes v_1.$$

Since the ideal I_Q is invariant under this reversal, this operation descends to an antiautomorphism of $C\!\!\!\!/(V,Q)$ called the *transpose* or *reversal* operation, denoted by x^t . The transpose is an antiautomorphism: $(xy)^t = y^t x^t$. The transpose operation makes no use of the $\mathbf{Z}_2$ -grading so we define a second antiautomorphism by composing α and the transpose. We call this operation *Clifford conjugation* denoted $\bar{x}$

$$\bar{x} = \alpha(x^t) = \alpha(x)^t.$$

Of the two antiautomorphisms, the transpose is the more fundamental.^[4]

Note that all of these operations are involutions. One can show that they act as ± 1 on elements which are pure in the $\mathbf{Z}$ -grading. In fact, all three operations depend only on the degree modulo 4. That is, if x is pure with degree k then

$$\alpha(x) = \pm x \quad x^t = \pm x \quad \bar{x} = \pm x$$

where the signs are given by the following table:

$k \bmod 4$	0	1	2	3	
$\alpha(x)$	+	−	+	−	$(-1)^k$
x^t	+	+	−	−	$(-1)^{k(k-1)/2}$
$\bar{x}$	+	−	−	+	$(-1)^{k(k+1)/2}$

The Clifford scalar product

When the characteristic is not 2 the quadratic form Q on V can be extended to a quadratic form on all of $C\!\!\!\!/(V,Q)$ as explained earlier (which we also denoted by Q). A basis independent definition is

$$Q(x) = \langle x^t x \rangle$$

where $\langle a \rangle$ denotes the scalar part of a (the grade 0 part in the $\mathbf{Z}$ -grading). One can show that

$$Q(v_1 v_2 \cdots v_k) = Q(v_1) Q(v_2) \cdots Q(v_k)$$

where the v_i are elements of V — this identity is *not* true for arbitrary elements of $C\!\!\!\!/(V,Q)$.

The associated symmetric bilinear form on $C\!\!\!\!/(V,Q)$ is given by

$$\langle x, y \rangle = \langle x^t y \rangle.$$

One can check that this reduces to the original bilinear form when restricted to V . The bilinear form on all of $C\!\!\!\!/(V,Q)$ is nondegenerate if and only if it is nondegenerate on V .

It is not hard to verify that the transpose is the adjoint of left/right Clifford multiplication with respect to this inner product. That is,

$$\begin{aligned} \langle ax, y \rangle &= \langle x, a^t y \rangle, \text{ and} \\ \langle xa, y \rangle &= \langle x, ya^t \rangle. \end{aligned}$$

Structure of Clifford algebras

In this section we assume that the vector space V is finite dimensional and that the bilinear form of Q is non-singular. A central simple algebra over K is a matrix algebra over a (finite dimensional) division algebra with center K . For example, the central simple algebras over the reals are matrix algebras over either the reals or the quaternions.

- If V has even dimension then $C\Box(V, Q)$ is a central simple algebra over K .
- If V has even dimension then $C\Box^0(V, Q)$ is a central simple algebra over a quadratic extension of K or a sum of two isomorphic central simple algebras over K .
- If V has odd dimension then $C\Box(V, Q)$ is a central simple algebra over a quadratic extension of K or a sum of two isomorphic central simple algebras over K .
- If V has odd dimension then $C\Box^0(V, Q)$ is a central simple algebra over K .

The structure of Clifford algebras can be worked out explicitly using the following result. Suppose that U has even dimension and a non-singular bilinear form with discriminant d , and suppose that V is another vector space with a quadratic form. The Clifford algebra of $U+V$ is isomorphic to the tensor product of the Clifford algebras of U and $(-1)^{\dim(U)/2}dV$, which is the space V with its quadratic form multiplied by $(-1)^{\dim(U)/2}d$. Over the reals, this implies in particular that

$$\begin{aligned} Cl_{p+2,q}(\mathbb{R}) &= M_2(\mathbb{R}) \otimes Cl_{q,p}(\mathbb{R}) \\ Cl_{p+1,q+1}(\mathbb{R}) &= M_2(\mathbb{R}) \otimes Cl_{p,q}(\mathbb{R}) \\ Cl_{p,q+2}(\mathbb{R}) &= \mathbb{H} \otimes Cl_{q,p}(\mathbb{R}) \end{aligned}$$

These formulas can be used to find the structure of all real Clifford algebras; see the classification of Clifford algebras.

Notably, the Morita equivalence class of a Clifford algebra (its representation theory: the equivalence class of the category of modules over it) depends only on the signature $p - q \pmod 8$. This is an algebraic form of Bott periodicity.

The Clifford group Γ

In this section we assume that V is finite dimensional and the quadratic form Q is nondegenerate.

The invertible elements of the Clifford algebra act on it by twisted conjugation: conjugation by x maps $y \mapsto xy\alpha(x)^{-1}$.

The Clifford group Γ is defined to be the set of invertible elements x that *stabilize vectors*, meaning that

$$xv\alpha(x)^{-1} \in V$$

for all v in V .

This formula also defines an action of the Clifford group on the vector space V that preserves the norm Q , and so gives a homomorphism from the Clifford group to the orthogonal group. The Clifford group contains all elements r of V of nonzero norm, and these act on V by the corresponding reflections that take v to $v - \langle v, r \rangle r / Q(r)$ (In characteristic 2 these are called orthogonal transvections rather than reflections.)

The Clifford group Γ is the disjoint union of two subsets Γ^0 and Γ^1 , where Γ^i is the subset of elements of degree i . The subset Γ^0 is a subgroup of index 2 in Γ .

If V is a finite dimensional real vector space with positive definite (or negative definite) quadratic form then the Clifford group maps onto the orthogonal group of V with respect to the form (by the Cartan-Dieudonné theorem) and the kernel consists of the nonzero elements of the field K . This leads to exact sequences

$$1 \rightarrow K^* \rightarrow \Gamma \rightarrow O_V(K) \rightarrow 1,$$

$$1 \rightarrow K^* \rightarrow \Gamma^0 \rightarrow SO_V(K) \rightarrow 1.$$

Over other fields or with indefinite forms, the map is not in general onto, and the failure is captured by the spinor norm.

Spinor norm

In arbitrary characteristic, the spinor norm Q is defined on the Clifford group by

$$Q(x) = x^t x$$

It is a homomorphism from the Clifford group to the group K^* of non-zero elements of K . It coincides with the quadratic form Q of V when V is identified with a subspace of the Clifford algebra. Several authors define the spinor norm slightly differently, so that it differs from the one here by a factor of -1 , 2 , or -2 on Γ^1 . The difference is not very important in characteristic other than 2 .

The nonzero elements of K have spinor norm in the group K^{*2} of squares of nonzero elements of the field K . So when V is finite dimensional and non-singular we get an induced map from the orthogonal group of V to the group K^*/K^{*2} , also called the spinor norm. The spinor norm of the reflection of a vector r has image $Q(r)$ in K^*/K^{*2} , and this property uniquely defines it on the orthogonal group. This gives exact sequences:

$$1 \rightarrow \{\pm 1\} \rightarrow \text{Pin}_V(K) \rightarrow O_V(K) \rightarrow K^*/K^{*2},$$

$$1 \rightarrow \{\pm 1\} \rightarrow \text{Spin}_V(K) \rightarrow SO_V(K) \rightarrow K^*/K^{*2}.$$

Note that in characteristic 2 the group $\{\pm 1\}$ has just one element.

From the point of view of Galois cohomology of algebraic groups, the spinor norm is a connecting homomorphism on cohomology. Writing μ_2 for the algebraic group of square roots of 1 (over a field of characteristic not 2 it is roughly the same as a two-element group with trivial Galois action), the short exact sequence

$$1 \rightarrow \mu_2 \rightarrow \text{Pin}_V \rightarrow O_V \rightarrow 1$$

yields a long exact sequence on cohomology, which begins

$$1 \rightarrow H^0(\mu_2; K) \rightarrow H^0(\text{Pin}_V; K) \rightarrow H^0(O_V; K) \rightarrow H^1(\mu_2; K)$$

The 0 th Galois cohomology group of an algebraic group with coefficients in K is just the group of K -valued points: $H^0(G; K) = G(K)$, and $H^1(\mu_2; K) \cong K^*/K^{*2}$, which recovers the previous sequence

$$1 \rightarrow \{\pm 1\} \rightarrow \text{Pin}_V(K) \rightarrow O_V(K) \rightarrow K^*/K^{*2},$$

where the spinor norm is the connecting homomorphism $H^0(O_V; K) \rightarrow H^1(\mu_2; K)$.

Spin and Pin groups

In this section we assume that V is finite dimensional and its bilinear form is non-singular. (If K has characteristic 2 this implies that the dimension of V is even.)

The Pin group $Pin_V(K)$ is the subgroup of the Clifford group Γ of elements of spinor norm 1, and similarly the Spin group $Spin_V(K)$ is the subgroup of elements of Dickson invariant 0 in $Pin_V(K)$. When the characteristic is not 2, these are the elements of determinant 1. The Spin group usually has index 2 in the Pin group.

Recall from the previous section that there is a homomorphism from the Clifford group onto the orthogonal group. We define the special orthogonal group to be the image of Γ^0 . If K does not have characteristic 2 this is just the group of elements of the orthogonal group of determinant 1. If K does have characteristic 2, then all elements of the orthogonal group have determinant 1, and the special orthogonal group is the set of elements of Dickson invariant 0.

There is a homomorphism from the Pin group to the orthogonal group. The image consists of the elements of spinor norm 1 $\square K^*/K^{*2}$. The kernel consists of the elements $+1$ and -1 , and has order 2 unless K has characteristic 2. Similarly there is a homomorphism from the Spin group to the special orthogonal group of V .

In the common case when V is a positive or negative definite space over the reals, the spin group maps onto the special orthogonal group, and is simply connected when V has dimension at least 3. Further the kernel of this homomorphism consists of 1 and -1. So in this case the spin group, $Spin(n)$, is a double cover of $SO(n)$. Please note, however, that the simple connectedness of the spin group is not true in general: if V is $R^{p,q}$ for p and q both at least 2 then the spin group is not simply connected. In this case the algebraic group $Spin_{p,q}$ is simply connected as an algebraic group, even though its group of real valued points $Spin_{p,q}(R)$ is not simply connected. This is a rather subtle point, which completely confused the authors of at least one standard book about spin groups.

Spinors

Clifford algebras $C\!\!\!\!/\!\!\!\!/_{p,q}(\mathbf{C})$, with $p+q=2n$ even, are matrix algebras which have a complex representation of dimension 2^n . By restricting to the group $Pin_{p,q}(\mathbf{R})$ we get a complex representation of the Pin group of the same dimension, called the spin representation. If we restrict this to the spin group $Spin_{p,q}(\mathbf{R})$ then it splits as the sum of two *half spin representations* (or *Weyl representations*) of dimension 2^{n-1} .

If $p+q=2n+1$ is odd then the Clifford algebra $C\!\!\!\!/\!\!\!\!/_{p,q}(\mathbf{C})$ is a sum of two matrix algebras, each of which has a representation of dimension 2^n , and these are also both representations of the Pin group $Pin_{p,q}(\mathbf{R})$. On restriction to the spin group $Spin_{p,q}(\mathbf{R})$ these become isomorphic, so the spin group has a complex spinor representation of dimension 2^n .

More generally, spinor groups and pin groups over any field have similar representations whose exact structure depends on the structure of the corresponding Clifford algebras: whenever a Clifford algebra has a factor that is a matrix algebra over some division algebra, we get a corresponding representation of the pin and spin groups over that division algebra. For examples over the reals see the article on spinors.

Real spinors

To describe the real spin representations, one must know how the spin group sits inside its Clifford algebra. The Pin group, $Pin_{p,q}$ is the set of invertible elements in $Cl_{p,q}$ which can be written as a product of unit vectors:

$$Pin_{p,q} = \{v_1 v_2 \dots v_r \mid \forall i, \|v_i\| = \pm 1\}.$$

Comparing with the above concrete realizations of the Clifford algebras, the Pin group corresponds to the products of arbitrarily many reflections: it is a cover of the full orthogonal group $O(p,q)$. The Spin group consists of those elements of $Pin_{p,q}$ which are products of an even number of unit vectors. Thus by the Cartan-Dieudonné theorem $Spin$ is a cover of the group of proper rotations $SO(p,q)$.

Let $\alpha : C\!\!\!\!l_{p,q} \rightarrow C\!\!\!\!l_{p,q}$ be the automorphism which is given by $-Id$ acting on pure vectors. Then in particular, $Spin_{p,q}$ is the subgroup of $Pin_{p,q}$ whose elements are fixed by α . Let

$$Cl_{p,q}^0 = \{x \in Cl_{p,q} \mid \alpha(x) = x\}.$$

(These are precisely the elements of even degree in $C\!\!\!\!l_{p,q}$.) Then the spin group lies within $C\!\!\!\!l_{p,q}^0$.

The irreducible representations of $C\!\!\!\!l_{p,q}$ restrict to give representations of the pin group. Conversely, since the pin group is generated by unit vectors, all of its irreducible representations are induced in this manner. Thus the two representations coincide. For the same reasons, the irreducible representations of the spin coincide with the irreducible representations of $C\!\!\!\!l_{p,q}^0$.

To classify the pin representations, one need only appeal to the classification of Clifford algebras. To find the spin representations (which are representations of the even subalgebra), one can first make use of either of the isomorphisms (see above)

$$C\!\!\!\!l_{p,q}^0 \approx C\!\!\!\!l_{p,q-1}, \text{ for } q > 0$$

$$C\!\!\!\!l_{p,q}^0 \approx C\!\!\!\!l_{q,p-1}, \text{ for } p > 0$$

and realize a spin representation in signature (p,q) as a pin representation in either signature $(p,q-1)$ or $(q,p-1)$.

Applications

Differential geometry

One of the principal applications of the exterior algebra is in differential geometry where it is used to define the bundle of differential forms on a smooth manifold. In the case of a (pseudo-)Riemannian manifold, the tangent spaces come equipped with a natural quadratic form induced by the metric. Thus, one can define a Clifford bundle in analogy with the exterior bundle. This has a number of important applications in Riemannian geometry. Perhaps more importantly is the link to a spin manifold, its associated spinor bundle and spin^c manifolds.

Physics

Clifford algebras have numerous important applications in physics. Physicists usually consider a Clifford algebra to be an algebra spanned by matrices $\gamma_1, \dots, \gamma_n$ called Dirac matrices which have the property that

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\eta_{ij}$$

where η is the matrix of a quadratic form of signature (p, q) — typically $(1, 3)$ when working in Minkowski space. These are exactly the defining relations for the Clifford algebra $Cl_{1,3}(C)$ (up to an unimportant factor of 2), which by the classification of Clifford algebras is isomorphic to the algebra of 4 by 4 complex matrices.

The Dirac matrices were first written down by Paul Dirac when he was trying to write a relativistic first-order wave equation for the electron, and give an explicit isomorphism from the Clifford algebra to the algebra of complex matrices. The result was used to define the Dirac equation and introduce the Dirac operator. The entire Clifford algebra shows up in quantum field theory in the form of Dirac field bilinears.

Computer Vision

Recently, Clifford algebras have been applied in the problem of action recognition and classification in computer vision. Rodriguez et al. ^[5] propose a Clifford embedding to generalize traditional MACH filters to video (3D spatiotemporal volume), and vector-valued data such as optical flow. Vector-valued data is analyzed using the Clifford Fourier transform. Based on these vectors action filters are synthesized in the Clifford Fourier domain and recognition of actions is performed using Clifford Correlation. The authors demonstrate the effectiveness of the Clifford embedding by recognizing actions typically performed in classic feature films and sports broadcast television.

See also

- Algebra of physical space, APS
- Classification of Clifford algebras
- Clifford module
- Gamma matrices
- Exterior algebra
- Geometric algebra
- Spin group
- Spinor
- Paravector
- Cayley-Dickson algebra
- spinor bundle
- Dirac operator
- Clifford analysis
- spin structure
- quaternion
- octonion
- complex spin structure

Footnotes

- [1] Mathematicians who work with real Clifford algebras and prefer positive definite quadratic forms (especially those working in index theory) sometimes use a different choice of sign in the fundamental Clifford identity. That is, they take $v^2 = -Q(v)$. One must replace Q with $-Q$ in going from one convention to the other.
- [2] Thus the group algebra $\mathbf{K}[\mathbf{Z}/2]$ is semisimple and the Clifford algebra splits into eigenspaces of the main involution.
- [3] We are still assuming that the characteristic is not 2.
- [4] The opposite is true when uses the alternate $(-)$ sign convention for Clifford algebras: it is the conjugate which is more important. In general, the meanings of conjugation and transpose are interchanged when passing from one sign convention to the other. For example, in the convention used here the inverse of a vector is given by $v^{-1} = v^t/Q(v)$ while in the $(-)$ convention it is given by $v^{-1} = \bar{v}/Q(v)$.
- [5] Rodriguez, Mikel; Shah, M (2008). "Action MACH: A Spatio-Temporal Maximum Average Correlation Height Filter for Action Classification". *Computer Vision and Pattern Recognition (CVPR)*.

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External links

- Planetmath entry on Clifford algebras (<http://planetmath.org/encyclopedia/CliffordAlgebra2.html>)
- A history of Clifford algebras (<http://members.fortunecity.com/jonhays/clifhistory.htm>) (unverified)
- John Baez on Clifford algebras (<http://www.math.ucr.edu/home/baez/octonions/node6.html>)

Von Neumann algebra

In mathematics, a **von Neumann algebra** or **W*-algebra** is a *-algebra of bounded operators on a Hilbert space that is closed in the weak operator topology and contains the identity operator. They were originally introduced by John von Neumann, motivated by the study of single operators, group representations, ergodic theory and quantum mechanics. His double commutant theorem shows that the analytic definition is equivalent to a purely algebraic definition as an algebra of symmetries.

Two basic examples of von Neumann algebras are as follows. The ring $L^\infty(\mathbf{R})$ of essentially bounded measurable functions on the real line is a commutative von Neumann algebra, which acts by pointwise multiplication on the Hilbert space $L^2(\mathbf{R})$ of square integrable functions. The algebra $B(H)$ of all bounded operators on a Hilbert space H is a von Neumann algebra, non-commutative if the Hilbert space has dimension at least 2.

Von Neumann algebras were first studied by von Neumann (1929); he and Francis Murray developed the basic theory, under the original name of **rings of operators**, in a series of papers written in the 1930s and 1940s (*F.J. Murray & J. von Neumann 1936, 1937, 1943; J. von Neumann 1938, 1940, 1943, 1949*), reprinted in the collected works of von Neumann (1961).

Introductory accounts of von Neumann algebras are given in the online notes of Jones (2003) and Wassermann (1991) and the books by Dixmier (1981), Schwartz (1967), Blackadar (2005) and Sakai (1971). The three volume work by Takesaki (1979) gives an encyclopedic account of the theory. The book by Connes (1994) discusses more advanced topics.

Definitions

There are three common ways to define von Neumann algebras.

The first and most common way is to define them as weakly closed *-algebras of bounded operators (on a Hilbert space) containing the identity. In this definition the weak (operator) topology can be replaced by almost any common topology other than the norm topology (a notable exception is the Banach space weak topology), in particular by the strong, ultrastrong or ultraweak operator topologies. The *-algebras of bounded operators that are closed in the norm topology are C*-algebras, so in particular any von Neumann algebra is a C*-algebra.

The second definition is that a von Neumann algebra is a subset of the bounded operators closed under * and equal to its double commutant, or equivalently the commutant of some subset closed under *. The von Neumann double commutant theorem (*von Neumann 1929*) says that the first two definitions are equivalent.

The first two definitions describe a von Neumann algebras concretely as a set of operators acting on some given Hilbert space. Sakai (1971) showed that von Neumann algebras can also be defined abstractly as C*-algebras that have a predual; in other words the von Neumann algebra, considered as a Banach space, is the dual of some other Banach space called the predual. The predual of a von Neumann algebra is in fact unique up to isomorphism. Some authors use "von Neumann algebra" for the algebras together with a Hilbert space action, and "W*-algebra" for the abstract concept, so a von Neumann algebra is a W*-algebra together with a Hilbert space and a suitable faithful unital action on the

Hilbert space. The concrete and abstract definitions of a von Neumann algebra are similar to the concrete and abstract definitions of a C^* -algebra, which can be defined either as norm-closed $*$ algebras of operators on a Hilbert space, or as Banach $*$ -algebras such that $\|a a^*\| = \|a\| \|a^*\|$.

Terminology

Some of the terminology in von Neumann algebra theory can be confusing, and the terms often have different meanings outside the subject.

- A **factor** is a von Neumann algebra with trivial center, i.e. a center consisting only of scalar operators.
- A **finite** von Neumann algebra is one which is the direct integral of finite factors. Similarly, **properly infinite** von Neumann algebras are the direct integral of properly infinite factors.
- A von Neumann algebra that acts on a separable Hilbert space is called **separable**. Note that such algebras are rarely separable in the norm topology.
- The von Neumann algebra **generated** by a set of bounded operators on a Hilbert space is the smallest von Neumann algebra containing all those operators.
- The **tensor product** of two von Neumann algebras acting on two Hilbert spaces is defined to be the von Neumann algebra generated by their algebraic tensor product, considered as operators on the Hilbert space tensor product of the Hilbert spaces.

Commutative von Neumann algebras

Main article: Abelian von Neumann algebra

The relationship between commutative von Neumann algebras and measure spaces is analogous to that between commutative C^* -algebras and locally compact Hausdorff spaces. Every commutative von Neumann algebra is isomorphic to $L^\infty(X)$ for some measure space (X, μ) and conversely, for every σ -finite measure space X , the $*$ algebra $L^\infty(X)$ is a von Neumann algebra.

Due to this analogy, the theory of von Neumann algebras has been called noncommutative measure theory, while the theory of C^* -algebras is sometimes called noncommutative topology.

Projections

Operators E in a von Neumann algebra for which $E = EE = E^*$ are called **projections**; they are exactly the operators which give an orthogonal projection of H onto some closed subspace. A subspace of the Hilbert space H is said to **belong to** the von Neumann algebra M if it is the image of some projection in M . Informally these are the closed subspaces that can be described using elements of M , or that M "knows" about. The closure of the image of any operator in M , or the kernel of any operator in M belong to M , and the closure of the image of any subspace belonging to M under an operator of M also belongs to M . There is a 1:1 correspondence between projections of M and subspaces that belong to it.

The basic theory of projections was worked out by Murray & von Neumann (1936). Two subspaces belonging to M are called (**Murray-von Neumann**) **equivalent** if there is a partial isometry mapping the first isomorphically onto the other that is an element of the von Neumann algebra (informally, if M "knows" that the subspaces are isomorphic). This

induces a natural equivalence relation on projections by defining E to be equivalent to F if the corresponding subspaces are equivalent, or in other words if there is a partial isometry of H that maps the image of E isometrically to the image of F and is an element of the von Neumann algebra. Another way of stating this is that E is equivalent to F if $E = uu^*$ and $F = u^*u$ for some partial isometry u in M .

The equivalence relation $\sim$ thus defined is additive in the following sense: Suppose $E_1 \sim F_1$ and $E_2 \sim F_2$. If $E_1 \sqcap E_2$ and $F_1 \sqcap F_2$, then $E_1 + E_2 \sim F_1 + F_2$. This is not true in general if one requires unitary equivalence in the definition of $\sim$, i.e. if we say E is equivalent to F if $u^*Eu = F$ for some unitary u .

The subspaces belonging to M are partially ordered by inclusion, and this induces a partial order $\leq$ of projections. There is also a natural partial order on the set of *equivalence classes* of projections, induced by the partial order $\leq$ of projections. If M is a factor, $\leq$ is a total order on equivalence classes of projections, described in the section on traces below.

A projection (or subspace belonging to M) E is said to be *finite* if there is no projection $F < E$ that is equivalent to E . For example, all finite-dimensional projections (or subspaces) are finite (since isometries between Hilbert spaces leave the dimension fixed), but the identity operator on an infinite-dimensional Hilbert space is not finite in the von Neumann algebra of all bounded operators on it, since it is isometrically isomorphic to a proper subset of itself. However it is possible for infinite dimensional subspaces to be finite.

Orthogonal projections are noncommutative analogues of indicator functions in $L^\infty(\mathbf{R})$. $L^\infty(\mathbf{R})$ is the $\|\cdot\|_\infty$ -closure of the subspace generated by the indicator functions. Similarly, a von Neumann algebra is generated by its projections; this is a consequence of the spectral theorem for self-adjoint operators.

Factors

A von Neumann algebra N whose center consists only of multiples of the identity operator is called a **factor**. von Neumann (1949) showed that every von Neumann algebra on a separable Hilbert space is isomorphic to a direct integral of factors. This decomposition is essentially unique. Thus, the problem of classifying isomorphism classes of von Neumann algebras on separable Hilbert spaces can be reduced to that of classifying isomorphism classes of factors.

Murray & von Neumann (1936) showed that every factor has one of 3 types as described below. The type classification can be extended to von Neumann algebras that are not factors, and a von Neumann algebra is of type X if it can be decomposed as a direct integral of type X factors; for example, every commutative von Neumann algebra has type I_1 . Every von Neumann algebra can be written uniquely as a sum of von Neumann algebras of types I, II, and III.

There are several other ways to divide factors into classes that are sometimes used:

- A factor is called **discrete** (or occasionally **tame**) if it has type I, and **continuous** (or occasionally **wild**) if it has type II or III.
- A factor is called **semifinite** if it has type I or II, and **purely infinite** if it has type III.
- A factor is called **finite** if the projection 1 is finite and **properly infinite** otherwise. Factors of types I and II may be either finite or properly infinite, but factors of type III are always properly infinite.

Type I factors

A factor is said to be of **type I** if there is a minimal projection $E \neq 0$, i.e. a projection E such that there is no other projection F with $0 < F < E$. Any factor of type I is isomorphic to the von Neumann algebra of *all* bounded operators on some Hilbert space; since there is one Hilbert space for every cardinal number, isomorphism classes of factors of type I correspond exactly to the cardinal numbers. Since many authors consider von Neumann algebras only on separable Hilbert spaces, it is customary to call the bounded operators on a Hilbert space of finite dimension n a factor of type I_n , and the bounded operators on a separable infinite-dimensional Hilbert space, a factor of type I_∞ .

Type II factors

A factor is said to be of **type II** if there are no minimal projections but there are non-zero finite projections. This implies that every projection E can be halved in the sense that there are equivalent projections F and G such that $E = F + G$. If the identity operator in a type II factor is finite, the factor is said to be of type II_1 ; otherwise, it is said to be of type II_∞ . The best understood factors of type II are the hyperfinite type II_1 factor and the hyperfinite type II_∞ factor, found by Murray & von Neumann (1936). These are the unique hyperfinite factors of types II_1 and II_∞ ; there are an uncountable number of other factors of these types that are the subject of intensive study. Murray & von Neumann (1937) proved the fundamental result that a factor of type II_1 has a unique finite tracial state, and the set of traces of projections is $[0,1]$.

A factor of type II_∞ has a semifinite trace, unique up to rescaling, and the set of traces of projections is $[0,\infty]$. The set of real numbers λ such that there is an automorphism rescaling the trace by a factor of λ is called the **fundamental group** of the type II_∞ factor.

The tensor product of a factor of type II_1 and an infinite type I factor has type II_∞ , and conversely any factor of type II_∞ can be constructed like this. The **fundamental group** of a type II_1 factor is defined to be the fundamental group of its tensor product with the infinite (separable) factor of type I. For many years it was an open problem to find a type II factor whose fundamental group was not the group of all positive reals, but Connes then showed that the von Neumann group algebra of a countable discrete group with Kazhdan's property T (the trivial representation is isolated in the dual space), such as $SL_3(\mathbf{Z})$, has a countable fundamental group. Subsequently Sorin Popa showed that the fundamental group can be trivial for certain groups, including the semidirect product of $\mathbf{Z}^2$ by $SL_2(\mathbf{Z})$.

An example of a type II_1 factor is the von Neumann group algebra of a countable infinite discrete group such that every non-trivial conjugacy class is infinite. McDuff (1969) found an uncountable family of such groups with non-isomorphic von Neumann group algebras, thus showing the existence of uncountably many different separable type II_1 factors.

Type III factors

Lastly, **type III** factors are factors that do not contain any nonzero finite projections at all. In their first paper Murray & von Neumann (1936) were unable decide whether or not they existed; the first examples were later found by von Neumann (1940). Since the identity operator is always infinite in those factors, they were sometimes called type III_∞ in the past, but recently that notation has been superseded by the notation III_λ , where λ is a real number in the interval $[0,1]$. More precisely, if the Connes spectrum (of its modular group) is 1 then the factor is of type III_0 , if the Connes spectrum is all integral powers of λ for

$0 < \lambda < 1$, then the type is III_λ , and if the Connes spectrum is all positive reals then the type is III_1 . (The Connes spectrum is a closed subgroup of the positive reals, so these are the only possibilities.) The only trace on type III factors takes value ∞ on all non-zero positive elements, and any two non-zero projections are equivalent. At one time type III factors were considered to be intractable objects, but Tomita-Takesaki theory has led to a good structure theory. In particular, any type III factor can be written in a canonical way as the crossed product of a type II_∞ factor and the real numbers.

The predual

Any von Neumann algebra M has a **predual** M_* , which is the Banach space of all ultraweakly continuous linear functionals on M . As the name suggests, M is (as a Banach space) the dual of its predual. The predual is unique in the sense that any other Banach space whose dual is M is canonically isomorphic to M_* . Sakai (1971) showed that the existence of a predual characterizes von Neumann algebras among C^* algebras.

The definition of the predual given above seems to depend on the choice of Hilbert space that M acts on, as this determines the ultraweak topology. However the predual can also be defined without using the Hilbert space that M acts on, by defining it to be the space generated by all positive **normal** linear functionals on M . (Here "normal" means that it preserves suprema when applied to increasing nets of self adjoint operators; or equivalently to increasing sequences of projections.)

The predual M_* is a closed subspace of the dual M^* (which consists of all norm-continuous linear functionals on M) but is generally smaller. The proof that M_* is (usually) not the same as M^* is nonconstructive and uses the axiom of choice in an essential way; it is very hard to exhibit explicit elements of M^* that are not in M_* . For example, exotic positive linear forms on the von Neumann algebra $l^\infty(Z)$ are given by free ultrafilters; they correspond to exotic $*$ -homomorphisms into C and describe the Stone-Cech compactification of Z .

Examples:

1. The predual of the von Neumann algebra $L^\infty(\mathbf{R})$ of essentially bounded functions on $\mathbf{R}$ is the Banach space $L^1(\mathbf{R})$ of integrable functions.
2. The predual of the von Neumann algebra $B(H)$ of bounded operators on a Hilbert space H is the Banach space of all trace class operators with the trace norm $\|A\| = \text{Tr}(|A|)$. The Banach space of trace class operators is itself the dual of the C^* -algebra of compact operators (which is not a von Neumann algebra).

Weights, states, and traces

Weights and their special cases states and traces are discussed in detail in (Takesaki 1979).

- A **weight** ω on a von Neumann algebra is a linear map from the set of positive elements (those of the form a^*a) to $[0, \infty]$.
- A **positive linear functional** is a weight with $\omega(1)$ finite (or rather the extension of ω to the whole algebra by linearity).
- A **state** is a weight with $\omega(1) = 1$.
- A **trace** is a weight with $\omega(aa^*) = \omega(a^*a)$ for all a .
- A **tracial state** is a trace with $\omega(1) = 1$.

Any factor has a trace such that the trace of a non-zero projection is non-zero and the trace of a projection is infinite if and only if the projection is infinite. Such a trace is unique up to rescaling. For factors that are separable or finite, two projections are equivalent if and only if they have the same trace. The type of a factor can be read off from the possible values of this trace as follows:

- Type I_n : $0, x, 2x, \dots, nx$ for some positive x (usually normalized to be $1/n$ or 1).
- Type I_∞ : $0, x, 2x, \dots, \infty$ for some positive x (usually normalized to be 1).
- Type II_1 : $[0, x]$ for some positive x (usually normalized to be 1).
- Type II_∞ : $[0, \infty]$.
- Type III: $0, \infty$

If a von Neumann algebra acts on a Hilbert space containing a norm 1 vector v , then the functional $a \rightarrow (av, v)$ is a normal state. This construction can be reversed to give an action on a Hilbert space from a normal state: this is the GNS construction for normal states.

Modules over a factor

Given an abstract separable factor, one can ask for a classification its modules, meaning the separable Hilbert spaces that it acts on. The answer is given as follows: every such module H can be given an M -dimension $\dim_M(H)$ (not its dimension as a complex vector space) such that modules are isomorphic if and only if they have the same dimension. The M -dimension is additive, and a module is isomorphic to a subspace of another module if and only if it has smaller or equal M -dimension.

A module is called **standard** if it has a cyclic separating vector. Each factor has a standard representation, which is unique up to isomorphism. The standard representation has an antilinear involution J such that $JMJ = M'$. For finite factors the standard module is given by the GNS construction applied to the unique normal tracial state and the M -dimension is normalized so that the standard module has M -dimension 1, while for infinite factors the standard module is the module with M -dimension equal to ∞ .

The possible M -dimensions of modules are given as follows:

- Type I_n (n finite): The M -dimension can be any of $0/n, 1/n, 2/n, 3/n, \dots, \infty$. The standard module has M -dimension 1 (and complex dimension n^2 .)
- Type I_∞ : The M -dimension can be any of $0, 1, 2, 3, \dots, \infty$. The standard representation of $B(H)$ is $H \otimes H$; its M -dimension is ∞ .
- Type II_1 : The M -dimension can be anything in $[0, \infty]$. It is normalized so that the standard module has M -dimension 1. The M -dimension is also called the **coupling constant** of the module H .
- Type II_∞ : The M -dimension can be anything in $[0, \infty]$. There is in general no canonical way to normalize it; the factor may have outer automorphisms multiplying the M -dimension by constants. The standard representation is the one with M -dimension ∞ .
- Type III: The M -dimension can be 0 or ∞ . Any two non-zero modules are isomorphic, and all non-zero modules are standard.

Amenable von Neumann algebras

Connes (1976) and others proved that the following conditions on a von Neumann algebra M on a separable Hilbert space H are all equivalent:

- M is **hyperfinite** or **AFD** or **approximately finite dimensional** or **approximately finite**: this means the algebra contains an ascending sequence of finite dimensional subalgebras with dense union. (Warning: some authors use "hyperfinite" to mean "AFD and finite".)
- M is **amenable**: this means that the derivations of M with values in a normal dual Banach bimodule are all inner.
- M has Schwartz's **property P**: for any bounded operator T on H the weak operator closed convex hull of the elements uTu^* contains an element commuting with M .
- M is **semidiscrete**: this means the identity map from M to M is a weak pointwise limit of completely positive maps of finite rank.
- M has **property E** or the **Hakeda-Tomiyama extension property**: this means that there is a projection of norm 1 from bounded operators on H to M .
- M is **injective**: any completely positive linear map from any self adjoint closed subspace containing 1 of any unital C^* -algebra A to M can be extended to a completely positive map from A to M .

There is no generally accepted term for the class of algebras above; Connes has suggested that **amenable** should be the standard term.

The amenable factors have been classified: there is a unique one of each of the types I_n , I_∞ , II_1 , II_∞ , III_λ , for $0 < \lambda \leq 1$, and the ones of type III_0 correspond to certain ergodic flows. (For type III_0 calling this a classification is a little misleading, as it is known that there is no easy way to classify the corresponding ergodic flows.) The ones of type I and II_1 were classified by Murray & von Neumann (1943), and the remaining ones were classified by Connes (1976), except for the type III_0 case which was completed by Haagerup.

All amenable factors can be constructed using the **group-measure space construction** of Murray and von Neumann for a single ergodic transformation. In fact they are precisely the factors arising as crossed products by free ergodic actions of $\mathbb{Z}$ or $\mathbb{Z}_n$ on abelian von Neumann algebras $L^\infty(X)$. Type I factors occur when the measure space X is atomic and the action transitive. When X is diffuse or non-atomic, it is equivalent to $[0,1]$ as a measure space. Type II factors occur when X admits an equivalent finite (II_1) or infinite (II_∞) measure, invariant under $\mathbb{Z}$. Type III factors occur in the remaining cases where there is no invariant measure, but only an invariant measure class: these factors are called **Krieger factors**.

Tensor products of von Neumann algebras

The Hilbert space tensor product of two Hilbert spaces is the completion of their algebraic tensor product. One can define a tensor product of von Neumann algebras (a completion of the algebraic tensor product of the algebras considered as rings), which is again a von Neumann algebra, and act on the tensor product of the corresponding Hilbert spaces. The tensor product of two finite algebras is finite, and the tensor product of an infinite algebra and a non-zero algebra is infinite. The type of the tensor product of two von Neumann algebras (I, II, or III) is the maximum of their types. The **commutation theorem for**

tensor products states that

$$(M \otimes N)' = M' \otimes N'$$

(where M' denotes the commutant of M).

The tensor product of an infinite number of von Neumann algebras, if done naively, is usually a ridiculously large non-separable algebra. Instead von Neumann (1938) showed that one should choose a state on each of the von Neumann algebras, use this to define a state on the algebraic tensor product, which can be used to produce a Hilbert space and a (reasonably small) von Neumann algebra. Araki & Woods (1968) studied the case where all the factors are finite matrix algebras; these factors are called **Araki-Woods** factors or **ITPFI factors** (ITPFI stands for "infinite tensor product of finite type I factors"). The type of the infinite tensor product can vary dramatically as the states are changed; for example, the infinite tensor product of an infinite number of type I_2 factors can have any type depending on the choice of states. In particular Powers (1967) found an uncountable family of non-isomorphic hyperfinite type III_λ factors for $0 < \lambda < 1$, called **Powers factors**, by taking an infinite tensor product of type I_2 factors, each with the state given by :

$$x \mapsto \text{Tr} \begin{pmatrix} \frac{1}{\lambda+1} & 0 \\ 0 & \frac{\lambda}{\lambda+1} \end{pmatrix} x.$$

All hyperfinite von Neumann algebras not of type III_0 are isomorphic to Araki-Woods factors, but there are uncountably many of type III_0 that are not.

Bimodules and subfactors

A **bimodule** (or correspondence) is a Hilbert space H with module actions of two commuting von Neumann algebras. Bimodules have a much richer structure than that of modules. Any bimodule over two factors always gives a subfactor since one of the factors is always contained in the commutant of the other. There is also a subtle relative tensor product operation due to Connes on bimodules. The theory of subfactors, initiated by Vaughan Jones, reconciles these two seemingly different points of view.

Bimodules are also important for the von Neumann group algebra M of a discrete group Γ . Indeed if V is any unitary representation of Γ , then, regarding Γ as the diagonal subgroup of $\Gamma \times \Gamma$, the corresponding induced representation on $l^2(\Gamma, V)$ is naturally a bimodule for two commuting copies of M . Important representation theoretic properties of Γ can be formulated entirely in terms of bimodules and therefore make sense for the von Neumann algebra itself. For example Connes and Jones gave a definition of an analogue of Kazhdan's Property T for von Neumann algebras in this way.

Non-amenable factors

von Neumann algebras of type I are always amenable, but for the other types there are an uncountable number of different non-amenable factors, which seem very hard to classify, or even distinguish from each other. Nevertheless Voiculescu has shown that the class of non-amenable factors coming from the group-measure space construction is **disjoint** from the class coming from group von Neumann algebras of free groups. Later Narutaka Ozawa proved that group von Neumann algebras of hyperbolic groups yield prime type II_1 factors, i.e. ones that cannot be factored as tensor products of type II_1 factors, a result first proved by Leeming Ge for free group factors using Voiculescu's free entropy. Popa's work on fundamental groups of non-amenable factors represents another significant advance. The

theory of factors "beyond the hyperfinite" is rapidly expanding at present, with many new and surprising results; it has close links with rigidity phenomena in geometric group theory and ergodic theory.

Examples

- The essentially bounded functions on a σ -finite measure space form a commutative (type I_1) von Neumann algebra acting on the L^2 functions. For certain non- σ -finite measure spaces, usually considered pathological, $L^\infty(X)$ is not a von Neumann algebra; for example, the σ -algebra of measurable sets might be the countable-cocountable algebra on an uncountable set.
- The bounded operators on any Hilbert space form a von Neumann algebra, indeed a factor, of type I.
- If we have any unitary representation of a group G on a Hilbert space H then the bounded operators commuting with G form a von Neumann algebra G' , whose projections correspond exactly to the closed subspaces of H invariant under G . Equivalent subrepresentations correspond to equivalent projections in G' . The double commutant G'' of G is also a von Neumann algebra.
- The **von Neumann group algebra** of a discrete group G is the algebra of all bounded operators on $H = l^2(G)$ commuting with the action of G on H through right multiplication. One can show that this is the von Neumann algebra generated by the operators corresponding to multiplication from the left with an element $g \in G$. It is a factor (of type II_1) if every non-trivial conjugacy class of G is infinite (for example, a non-abelian free group), and is the hyperfinite factor of type II_1 if in addition G is a union of finite subgroups (for example, the group of all permutations of the integers fixing all but a finite number of elements).
- The tensor product of two von Neumann algebras, or of a countable number with states, is a von Neumann algebra as described in the section above.
- The crossed product of a von Neumann algebra by a discrete (or more generally locally compact) group can be defined, and is a von Neumann algebra. Special cases are the **group-measure space construction** of Murray and von Neumann and **Krieger factors**.
- The von Neumann algebras of a measurable equivalence relation and a measurable groupoid can be defined. These examples generalise von Neumann group algebras and the group-measure space construction.

See also

- By forgetting about the topology on a von Neumann algebra, we can consider it a (unital) $*$ -algebra, or just a ring. Von Neumann algebras are semihereditary: every finitely generated submodule of a projective module is itself projective. There have been several attempts to axiomatize the underlying rings of von Neumann algebras, including Baer $*$ -rings and AW $*$ algebras.
- The $*$ -algebra of affiliated operators of a finite von Neumann algebra is a von Neumann regular ring. (The von Neumann algebra itself is in general not von Neumann regular.)
- Von Neumann algebras have found applications in diverse areas of mathematics like knot theory, statistical mechanics, Quantum field theory, Local quantum physics, Free

probability, Noncommutative geometry, representation theory, geometry, and probability.

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C*-algebra

C*-algebras (pronounced "C-star") are an important area of research in functional analysis, a branch of mathematics. The prototypical example of a C*-algebra is a complex algebra A of linear operators on a complex Hilbert space with two additional properties:

- A is a topologically closed set in the norm topology of operators.
- A is closed under the operation of taking adjoints of operators.

It is generally believed that C*-algebras were first considered primarily for their use in quantum mechanics to model algebras of physical observables. This line of research began in an extremely rudimentary form with Werner Heisenberg's matrix mechanics and in a more mathematically developed form with Pascual Jordan around 1933. Subsequently John von Neumann attempted to establish a general framework for these algebras which culminated in a series of papers on rings of operators. These papers considered a special class of C*-algebras which are now known as von Neumann algebras.

Around 1943, the work of Israel Gelfand, Mark Naimark and Irving Segal yielded an abstract characterisation of C*-algebras making no reference to operators.

C*-algebras are now an important tool in the theory of unitary representations of locally compact groups, and are also used in algebraic formulations of quantum mechanics.

Abstract characterization

We begin with the abstract characterization of C*-algebras given in the 1943 paper by Gel'fand and Naimark.

A C*-algebra, A , is a Banach algebra over the field of complex numbers, together with a map, $*$: $A \rightarrow A$, called involution. The image of an element x of A under involution is written x^* . Involution has the following properties:

- For all x, y in A :

$$\begin{aligned}(x + y)^* &= x^* + y^* \\ (xy)^* &= y^* x^*.\end{aligned}$$

- For every λ in $\mathbf{C}$ and every x in A :

$$(\lambda x)^* = \bar{\lambda} x^*.$$

- For all x in A

$$(x^*)^* = x.$$

- The **C*-identity** holds for all x in A :

$$\|x^* x\| = \|x\|^2.$$

Note that the C* identity is equivalent to: for all x in A :

$$\|x x^*\| = \|x\|^2.$$

Thus any C*-algebra is automatically a **Banach*-algebra**. However, not every Banach*-algebra is a C*-algebra.

The C*-identity is a very strong requirement. For instance, together with the spectral radius formula, it implies the C*-norm is uniquely determined by the algebraic structure:

$$\|x\|^2 = \|x^* x\| = \sup\{|\lambda|, x^* x - \lambda 1 \text{ is not invertible}\}.$$

A bounded linear map, $\pi : A \rightarrow B$, between C*-algebras A and B is called a ***-homomorphism** if

- For x and y in A

$$\pi(xy) = \pi(x)\pi(y).$$

- For x in A

$$\pi(x^*) = \pi(x)^*.$$

In the case of C*-algebras, any *-homomorphism π between C*-algebras is non-expansive, i.e. bounded with norm ≤ 1 . Furthermore, an injective *-homomorphism between C*-algebras is isometric. These are consequences of the C*-identity.

A bijective *-homomorphism π is called a **C*-isomorphism**, in which case A and B are said to be **isomorphic**.

Examples

Finite-dimensional C*-algebras

The algebra $M_n(\mathbf{C})$ of n -by- n matrices over $\mathbf{C}$ becomes a C*-algebra if we consider matrices as operators on the Euclidean space, $\mathbf{C}^n$, and use the operator norm $\|.\|$ on matrices. The involution is given by the conjugate transpose. More generally, one can consider finite direct sums of matrix algebras. In fact, all finite dimensional C*-algebras are of this form. The self-adjoint requirement means finite-dimensional C*-algebras are semisimple, from which fact one can deduce the following theorem of Artin-Wedderburn type:

Theorem. A finite-dimensional C*-algebra, A , is canonically isomorphic to a finite direct sum

$$A = \bigoplus_{e \in \min A} Ae$$

where $\min A$ is the set of minimal nonzero self-adjoint central projections of A .

Each C*-algebra, Ae , is isomorphic (in a noncanonical way) to the full matrix algebra $M_{\dim(e)}(\mathbf{C})$. The finite family indexed on $\min A$ given by $\{\dim(e)\}_e$ is called the *dimension vector* of A . This vector uniquely determines the isomorphism class of a finite-dimensional C*-algebra.

C*-algebras of operators

The prototypical example of a C*-algebra is the algebra $B(H)$ of bounded (equivalently continuous) linear operators defined on a complex Hilbert space H ; here x^* denotes the adjoint operator of the operator $x : H \rightarrow H$. In fact, every C*-algebra, A , is *-isomorphic to a norm-closed adjoint closed subalgebra of $B(H)$ for a suitable Hilbert space, H ; this is the content of the Gelfand-Naimark theorem.

Commutative C*-algebras

Let X be a locally compact Hausdorff space. The space $C_0(X)$ of complex-valued continuous functions on X that *vanish at infinity* (defined in the article on local compactness) form a commutative C*-algebra $C_0(X)$ under pointwise multiplication and addition. The involution is pointwise conjugation. $C_0(X)$ has a multiplicative unit element if and only if X is compact. As does any C*-algebra, $C_0(X)$ has an approximate identity. In the case of $C_0(X)$ this is immediate: consider the directed set of compact subsets of X , and for each compact K let f_K be a function of compact support which is identically 1 on K . Such functions exist by the Tietze extension theorem which applies to locally compact Hausdorff spaces. $\{f_K\}_K$ is an approximate identity.

The Gelfand representation states that every commutative C*-algebra is *-isomorphic to the algebra $C_0(X)$, where X is the space of characters equipped with the weak* topology. Furthermore if $C_0(X)$ is isomorphic to $C_0(Y)$ as C*-algebras, it follows that X and Y are homeomorphic. This characterization is one of the motivations for the noncommutative topology and noncommutative geometry programs.

C*-algebras of compact operators

Let H be a separable infinite-dimensional Hilbert space. The algebra $K(H)$ of compact operators on H is a norm closed subalgebra of $B(H)$. It is also closed under involution; hence it is a C*-algebra.

Concrete C*-algebras of compact operators admit a characterization similar to Wedderburn's theorem for finite dimensional C*-algebras.

Theorem. If A is a C*-subalgebra of $K(H)$, then there exists Hilbert spaces $\{H_i\}_{i \in I}$ such that A is isomorphic to the following direct sum

$$\bigoplus_{i \in I} K(H_i),$$

where the (C*-)direct sum consists of elements (T_i) of the Cartesian product $\prod K(H_i)$ with $\|T_i\| \rightarrow 0$.

Though $K(H)$ does not have an identity element, a sequential approximate identity for $K(H)$ can be easily displayed. To be specific, H is isomorphic to the space of square summable sequences ℓ^2 ; we may assume that

$$H = \ell^2.$$

For each natural number n let H_n be the subspace of sequences of ℓ^2 which vanish for indices

$$k \geq n$$

and let

$$e_n$$

be the orthogonal projection onto H_n . The sequence $\{e_n\}_n$ is an approximate identity for $K(H)$.

$K(H)$ is a two-sided closed ideal of $B(H)$. For separable Hilbert spaces, it is the unique ideal. The quotient of $B(H)$ by $K(H)$ is the Calkin algebra.

C*-enveloping algebra

Given a B*-algebra A with an approximate identity, there is a unique (up to C*-isomorphism) C*-algebra $\mathbf{E}(A)$ and *-morphism π from A into $\mathbf{E}(A)$ which is universal, that is, every other B*-morphism $\pi' : A \rightarrow B$ factors uniquely through π . The algebra $\mathbf{E}(A)$ is called the **C*-enveloping algebra** of the B*-algebra A .

Of particular importance is the C*-algebra of a locally compact group G . This is defined as the enveloping C*-algebra of the group algebra of G . The C*-algebra of G provides context for general harmonic analysis of G in the case G is non-abelian. In particular, the dual of a locally compact group is defined to be the primitive ideal space of the group C*-algebra. See spectrum of a C*-algebra.

A complete locally m-convex *-algebra A is a topological *-algebra that is an inverse limit of Banach *-algebras. In representation theory of these algebras, the enveloping algebra of A provides a solution of the universal problem for continuous *-representations of A into bounded Hilbert space operators. This corresponds to the construction of the enveloping C*-algebra of a Banach *-algebra.^[1]

von Neumann algebras

von Neumann algebras, known as W^* algebras before the 1960s, are a special kind of C*-algebra. They are required to be closed in the weak operator topology, which is weaker than the norm topology. Their study is a specialized area of functional analysis.

Properties of C*-algebras

C*-algebras have a large number of properties that are technically convenient. These properties can be established by use the continuous functional calculus or by reduction to commutative C*-algebras. In the latter case, we can use the fact that the structure of these is completely determined by the Gelfand isomorphism.

- The set of elements of a C*-algebra A of the form x^*x forms a closed convex cone. This cone is identical to the elements of the form $x x^*$. Elements of this cone are called *non-negative* (or sometimes *positive*, even though this terminology conflicts with its use for elements of $\mathbf{R}$.)
- The set of self-adjoint elements of a C*-algebra A naturally has the structure of a partially ordered vector space; the ordering is usually denoted $\geq$. In this ordering, a self-adjoint element x of A satisfies $x \geq 0$ if and only if the spectrum of x is non-negative. Two self-adjoint elements x and y of A satisfy $x \geq y$ if $x - y \geq 0$.
- Any C*-algebra A has an approximate identity. In fact, there is a directed family $\{e_\lambda\}_{\lambda \in I}$ of self-adjoint elements of A such that

$$xe_\lambda \rightarrow x$$

$$0 \leq e_\lambda \leq e_\mu \leq 1 \quad \text{whenever } \lambda \leq \mu.$$

In case A is separable, A has a sequential approximate identity. More generally, A will have a sequential approximate identity if and only if A contains a **strictly positive element**, i.e. a positive element h such that hAh is dense in A .

- Using approximate identities, one can show that the algebraic quotient of a C*-algebra by a closed proper two-sided ideal, with the natural norm, is a C*-algebra.
- Similarly, a closed two-sided ideal of a C*-algebra is itself a C*-algebra.

Type for C*-algebras

A C*-algebra A is of type I if and only if for all non-degenerate representations π of A the von Neumann algebra $\pi(A)''$ (that is, the bicommutant of $\pi(A)$) is a type I von Neumann algebra. In fact it is sufficient to consider only factor representations, i.e. representations π for which $\pi(A)''$ is a factor.

A locally compact group is said to be of type I if and only if its group C*-algebra is type I.

However, if a C*-algebra has non-type I representations, then by results of James Glimm it also has representations of type II and type III. Thus for C*-algebras and locally compact groups, it is only meaningful to speak of type I and non type I properties.

C*-algebras and quantum field theory

In quantum field theory, one typically describes a physical system with a C*-algebra A with unit element; the self-adjoint elements of A (elements x with $x^* = x$) are thought of as the *observables*, the measurable quantities, of the system. A *state* of the system is defined as a positive functional on A (a $\mathbb{C}$ -linear map $\varphi : A \rightarrow \mathbb{C}$ with $\varphi(u u^*) \geq 0$ for all $u \in A$) such that $\varphi(1) = 1$. The expected value of the observable x , if the system is in state φ , is then $\varphi(x)$.

See Local quantum physics.

See also

- algebra
- associative algebra
- * algebra
- Hilbert C*-module
- K-theory

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Jones polynomial

In the mathematical field of knot theory, the **Jones polynomial** is a knot polynomial discovered by Vaughan Jones in 1983. Specifically, it is an invariant of an oriented knot or link which assigns to each oriented knot or link a Laurent polynomial in the variable $t^{1/2}$ with integer coefficients.

Definition by the bracket

Suppose we have an oriented link L , given as a knot diagram. We will define the Jones polynomial, $V(L)$, using Kauffman's bracket polynomial, which we denote by $\langle \rangle$. Note that here the bracket polynomial is a Laurent polynomial in the variable A with integer coefficients.

First, we define the auxiliary polynomial (also known as the normalized bracket polynomial) $X(L) = (-A^3)^{-w(L)} \langle L \rangle$, where $w(L)$ denotes the writhe of L in its given diagram. The writhe of a diagram is the number of positive crossings (L_+ in the figure below) minus the number of negative crossings (L_-). The writhe is not a knot invariant.

$X(L)$ is a knot invariant since it is invariant under changes of the diagram of L by the three Reidemeister moves. Invariance under type II and III Reidemeister moves follows from invariance of the bracket under those moves. The bracket polynomial is known to change by multiplication by $-A^{\pm 3}$ under a type I Reidemeister move. The definition of the X polynomial given above is designed to nullify this change, since the writhe changes appropriately by +1 or -1 under type I moves.

Now make the substitution $A = t^{-1/4}$ in $X(L)$ to get the Jones polynomial $V(L)$. This results in a Laurent polynomial with integer coefficients in the variable $t^{1/2}$.

Definition by braid representation

Jones' original formulation of his polynomial came from his study of operator algebras. In Jones' approach, it resulted from a kind of "trace" of a particular braid representation into an algebra which originally arose while studying certain models, e.g. the Potts model, in statistical mechanics.

Let a link L be given. A theorem of Alexander's states that it is the trace closure of a braid, say with n strands. Now define a representation ρ of the braid group on n strands, B_n , into the Temperley-Lieb algebra TL_n with coefficients in $\mathbb{Z}[A, A^{-1}]$ and $\delta = -A^2 - A^{-2}$. A standard braid generator σ_i gets sent to $Ae_i + A^{-1}1$, where $1, e_1, \dots, e_{n-1}$ are the standard generators of the Temperley-Lieb algebra. It can be checked easily that this defines a representation.

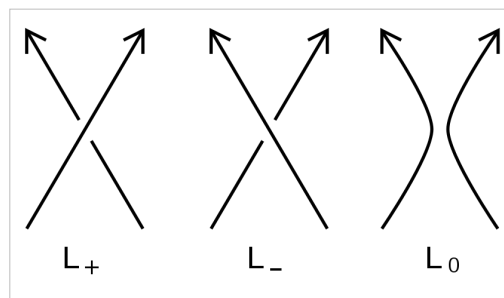
Take the braid word σ obtained previously from L and compute $\delta^{n-1} \text{tr} \rho(\sigma)$ where tr is the Markov trace. This gives $\langle L \rangle$, where $\langle \rangle$ is the bracket polynomial. This can be seen by considering, as Kauffman did, the Temperley-Lieb algebra as a particular diagram algebra. An advantage of this approach is that one can pick similar representations into other algebras, such as the R -matrix representations, leading to "generalized Jones invariants".

Properties

The Jones polynomial is characterized by the fact that it takes the value 1 on any diagram of the unknot and satisfies the following skein relation:

$$(t^{1/2} - t^{-1/2})V(L_0) = t^{-1}V(L_+) - tV(L_-)$$

where L_+ , L_- , and L_0 are three oriented link diagrams that are identical except in one small region where they differ by the crossing changes or smoothing shown in the figure below:



The definition of the Jones polynomial by the bracket makes it simple to show that for a knot K , the Jones polynomial of its mirror image is given by substitution of t^{-1} for t in $V(K)$. Thus, an **amphicheiral knot**, a knot equivalent to its mirror image, has palindromic entries in its Jones polynomial. See the article on skein relation for an example of a computation using these relations.

Link with Chern-Simons theory

As first shown by Edward Witten, the Jones polynomial of a given knot γ , can be obtained by considering Chern-Simons theory on the three-sphere with gauge group $SU(2)$, and computing the vacuum expectation value of a Wilson loop $W_F(\gamma)$, associated to γ , and the fundamental representation F of $SU(2)$.

Open problems

- Is there a nontrivial knot with Jones polynomial equal to that of the unknot? It is known that there are nontrivial *links* with Jones polynomial equal to that of the corresponding unlinks by the work of Morwen Thistlethwaite.

See also

- HOMFLY polynomial
- Khovanov homology

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External links

- Links with trivial Jones polynomial ^[2] by Morwen Thistlethwaite

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Subfactor

In the theory of von Neumann algebras, a **subfactor** of a factor M is a subalgebra that is a factor and contains 1. The theory of subfactors led to the discovery of the Jones polynomial in knot theory.

Index of a subfactor

Usually M is taken to be a factor of type II_1 , so that it has a finite trace. In this case every Hilbert space module H has a dimension $\dim_M(H)$ which is a non-negative real number or $+\infty$. The **index** $[M:N]$ of a subfactor N is defined to be $\dim_N(L^2(M))$. Here $L^2(M)$ is the representation of N obtained from the GNS construction of the trace of M .

The Jones index theorem

This states that if N is a subfactor of M (both of type II_1) then the index $[M:N]$ is either of the form $4 \cos(\pi/n)^2$ for $n = 3, 4, 5, \dots$, or is at least 4. All these values occur.

The first few values of $4 \cos(\pi/n)^2$ are 1, 2, $(3+\sqrt{5})/2=2.618\dots$, 3, $3.247\dots$, ...

The basic construction

Suppose that N is a subfactor of M , and that both are finite von Neumann algebras. The GNS construction produces a Hilbert space $L^2(M)$ acted on by M with a cyclic vector Ω . Let e_N be the projection onto the subspace $N\Omega$. Then M and e_N generate a new von Neumann algebra $\langle M, e_N \rangle$ acting on $L^2(M)$, containing M as a subfactor. The passage from the inclusion of N in M to the inclusion of M in $\langle M, e_N \rangle$ is called the **basic construction**.

If N and M are both factors of type II_1 and N has finite index in M then $\langle M, e_N \rangle$ is also of type II_1 . Moreover the inclusions have the same index: $[M:N] = [\langle M, e_N \rangle : M]$, and $\text{tr}_{\langle M, e_N \rangle}(e_N) = 1/[M:N]$.

The tower

Suppose that $M_{-1} \sqsubset M_0$ is an inclusion of type II_1 factors of finite index. By iterating the basic construction we get a tower of inclusions

$$M_{-1} \sqsubset M_0 \sqsubset M_1 \sqsubset M_2 \dots$$

where each $M_{n+1} = \langle M_n, e_{n+1} \rangle$ is generated by the previous algebra and a projection. The union of all these algebras has a tracial state tr whose restriction to each M_n is the tracial state, and so the closure of the union is another type II_1 von Neumann algebra M_∞ .

The algebra M_∞ contains a sequence of projections $e_1, e_2, e_3, \dots$, which satisfy the Temperley-Lieb relations at parameter $\lambda = 1/[M:N]$. Moreover, the algebra generated by the e_n is a C*-algebra in which the e_n are self-adjoint, and such that $\text{tr}(xe_n) = \lambda \text{tr}(x)$ when x is in the algebra generated by e_1 up to e_{n-1} . Whenever these extra conditions are satisfied, the algebra is called a Temperley-Lieb-Jones algebra at parameter λ . It can be shown to be unique up to *-isomorphism. It exists only when λ takes on those special values $4 \cos(\pi/n)^2$ for $n = 3, 4, 5, \dots$, or the values larger than 4.

Principal graphs

A subfactor of finite index $N \subseteq M$ is said to be **irreducible** if either of the following equivalent conditions are satisfied

- $L^2(M)$ is irreducible as an (N, M) bimodule;
- the relative commutant $N' \cap M$ is $\mathbb{C}$.

In this case $L^2(M)$ defines an (N, M) bimodule X as well as its conjugate (M, N) bimodule X^* . The relative tensor product, described in Jones (1983) and often called **Connes fusion** after a prior definition for general von Neumann algebras of Alain Connes, can be used to define new bimodules over (N, M) , (M, N) , (M, M) and (N, N) by decomposing the following tensor products into irreducible components:

$$X \otimes X^* \otimes \cdots \otimes X, X^* \otimes X \otimes \cdots \otimes X^*, X^* \otimes X \otimes \cdots \otimes X, X \otimes X^* \otimes \cdots \otimes X^*.$$

The irreducible (M, M) and (M, N) bimodules arising in this way form the vertices of the **principal graph**, a bipartite graph. The directed edges of these graphs describe the way an irreducible bimodule decomposes when tensored with X and X^* on the right. The **dual principal graph** is defined in a similar way using (N, N) and (N, M) bimodules.

Since any bimodule corresponds to the commuting actions of two factors, each factor is contained in the commutant of the other and therefore defines a subfactor. When the bimodule is irreducible, its dimension is defined to be the square root of the index of this subfactor. The dimension is extended additively to direct sums of irreducible bimodules. It is multiplicative with respect to Connes fusion.

The subfactor is said to have **finite depth** if the principal graph and its dual are finite, i.e. if only finitely many irreducible bimodules occur in these decompositions. In this case if M and N are hyperfinite, Sorin Popa showed that the inclusion $N \subseteq M$ is isomorphic to the model

$$(\mathbb{C} \otimes \text{End } X^* \otimes X \otimes X^* \otimes \cdots)'' \subseteq (\text{End } X \otimes X^* \otimes X \otimes X^* \otimes \cdots)'',$$

where the II_1 factors are obtained from the GNS construction with respect to the canonical trace.

Knot polynomials

The algebra generated by the elements e_n with the relations above is called the Temperley-Lieb algebra. This is a quotient of the group algebra of the braid group, so representations of the Temperley-Lieb algebra give representations of the braid group, which in turn often given invariants for knots.

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Quantum logic

In quantum mechanics, **quantum logic** is a set of rules for reasoning about propositions which takes the principles of quantum theory into account. This research area and its name originated in the 1936 paper by Garrett Birkhoff and John von Neumann, who were attempting to reconcile the apparent inconsistency of classical boolean logic with the facts concerning the measurement of complementary variables in quantum mechanics, such as position and momentum.

Quantum logic can be formulated either as a modified version of propositional logic or as a non-commutative and non-associative many-valued (MV) logic^{[1] [2] [3] [4] [5]}. It has some properties which clearly distinguish it from classical logic, most notably, the failure of the distributive law of propositional logic:

$$p \text{ and } (q \text{ or } r) = (p \text{ and } q) \text{ or } (p \text{ and } r),$$

where the symbols p , q and r are propositional variables. To illustrate why the distributive law fails, consider a particle moving on a line and let

$$p = \text{"the particle is moving to the right"}$$

$$q = \text{"the particle is in the interval } [-1,1]\text{"}$$

$$r = \text{"the particle is not in the interval } [-1,1]\text{"}$$

then the proposition " q or r " is true, so

$$p \text{ and } (q \text{ or } r) = p$$

On the other hand, the propositions " p and q " and " p and r " are both false, since they assert tighter restrictions on simultaneous values of position and momentum than is allowed by the uncertainty principle. So,

$$(p \text{ and } q) \text{ or } (p \text{ and } r) = \text{false}$$

Thus the distributive law fails.

Quantum logic has been proposed as the correct logic for propositional inference generally, most notably by the philosopher Hilary Putnam, at least at one point in his career. This thesis was an important ingredient in Putnam's paper *Is Logic Empirical?* in which he analysed the epistemological status of the rules of propositional logic. Putnam attributes the idea that anomalies associated to quantum measurements originate with anomalies in the logic of physics itself to the physicist David Finkelstein. It should be noted, however, that this idea had been around for some time and had been revived several years earlier by George Mackey's work on group representations and symmetry.

The more common view regarding quantum logic, however, is that it provides a formalism for relating observables, system preparation filters and states. In this view, the quantum

logic approach resembles more closely the C*-algebraic approach to quantum mechanics; in fact with some minor technical assumptions it can be subsumed by it. The similarities of the quantum logic formalism to a system of deductive logic may then be regarded more as a curiosity than as a fact of fundamental philosophical importance.

Introduction

In his classic treatise *Mathematical Foundations of Quantum Mechanics*, John von Neumann noted that projections on a Hilbert space can be viewed as propositions about physical observables. The set of principles for manipulating these quantum propositions was called *quantum logic* by von Neumann and Birkhoff. In his book (also called *Mathematical Foundations of Quantum Mechanics*) G. Mackey attempted to provide a set of axioms for this propositional system as an orthocomplemented lattice. Mackey viewed elements of this set as potential *yes or no questions* an observer might ask about the state of a physical system, questions that would be settled by some measurement. Moreover Mackey defined a physical observable in terms of these basic questions. Mackey's axiom system is somewhat unsatisfactory though, since it assumes that the partially ordered set is actually given as the orthocomplemented closed subspace lattice of a separable Hilbert space. Piron, Ludwig and others have attempted to give axiomatizations which do not require such explicit relations to the lattice of subspaces.

The remainder of this article assumes the reader is familiar with the spectral theory of self-adjoint operators on a Hilbert space. However, the main ideas can be understood using the finite-dimensional spectral theorem.

Projections as propositions

The so-called *Hamiltonian* formulations of classical mechanics have three ingredients: *states*, *observables* and *dynamics*. In the simplest case of a single particle moving in $\mathbf{R}^3$, the state space is the position-momentum space $\mathbf{R}^6$. We will merely note here that an observable is some real-valued function f on the state space. Examples of observables are position, momentum or energy of a particle. For classical systems, the value $f(x)$, that is the value of f for some particular system state x , is obtained by a process of measurement of f . The propositions concerning a classical system are generated from basic statements of the form

- Measurement of f yields a value in the interval $[a, b]$ for some real numbers a, b .

It follows easily from this characterization of propositions in classical systems that the corresponding logic is identical to that of some Boolean algebra of subsets of the state space. By logic in this context we mean the rules that relate set operations and ordering relations, such as de Morgan's laws. These are analogous to the rules relating boolean conjunctives and material implication in classical propositional logic. For technical reasons, we will also assume that the algebra of subsets of the state space is that of all Borel sets. The set of propositions is ordered by the natural ordering of sets and has a complementation operation. In terms of observables, the complement of the proposition $\{f \geq a\}$ is $\{f < a\}$.

We summarize these remarks as follows:

- The proposition system of a classical system is a lattice with a distinguished *orthocomplementation* operation: The lattice operations of *meet* and *join* are respectively

set intersection and set union. The orthocomplementation operation is set complement. Moreover this lattice is *sequentially complete*, in the sense that any sequence $\{E_i\}_i$ of elements of the lattice has a least upper bound, specifically the set-theoretic union:

$$\text{LUB}(\{E_i\}) = \bigcup_{i=1}^{\infty} E_i.$$

In the Hilbert space formulation of quantum mechanics as presented by von Neumann, a physical observable is represented by some (possibly unbounded) densely-defined self-adjoint operator A on a Hilbert space H . A has a spectral decomposition, which is a projection-valued measure E defined on the Borel subsets of $\mathbf{R}$. In particular, for any bounded Borel function f , the following equation holds:

$$f(A) = \int_{\mathbf{R}} f(\lambda) dE(\lambda).$$

In case f is the indicator function of an interval $[a, b]$, the operator $f(A)$ is a self-adjoint projection, and can be interpreted as the quantum analogue of the classical proposition

- Measurement of A yields a value in the interval $[a, b]$.

The propositional lattice of a quantum mechanical system

This suggests the following quantum mechanical replacement for the orthocomplemented lattice of propositions in classical mechanics. This is essentially Mackey's *Axiom VII*:

- The orthocomplemented lattice Q of propositions of a quantum mechanical system is the lattice of closed subspaces of a complex Hilbert space H where orthocomplementation of V is the orthogonal complement $V^\perp$.

Q is also sequentially complete: any pairwise disjoint sequence $\{V_i\}_i$ of elements of Q has a least upper bound. Here disjointness of W_1 and W_2 means W_2 is a subspace of $W_1^\perp$. The least upper bound of $\{V_i\}_i$ is the closed internal direct sum.

Henceforth we identify elements of Q with self-adjoint projections on the Hilbert space H .

The structure of Q immediately points to a difference with the partial order structure of a classical proposition system. In the classical case, given a proposition p , the equations

$$I = p \vee q$$

$$0 = p \wedge q$$

have exactly one solution, namely the set-theoretic complement of p . In these equations I refers to the atomic proposition which is identically true and 0 the atomic proposition which is identically false. In the case of the lattice of projections there are infinitely many solutions to the above equations.

Having made these preliminary remarks, we turn everything around and attempt to define observables within the projection lattice framework and using this definition establish the correspondence between self-adjoint operators and observables : A *Mackey observable* is a countably additive homomorphism from the orthocomplemented lattice of the Borel subsets of $\mathbf{R}$ to Q . To say the mapping φ is a countably additive homomorphism means that for any sequence $\{S_i\}_i$ of pairwise disjoint Borel subsets of $\mathbf{R}$, $\{\varphi(S_i)\}_i$ are pairwise orthogonal projections and

$$\varphi\left(\bigcup_{i=1}^{\infty} S_i\right) = \sum_{i=1}^{\infty} \varphi(S_i).$$

Theorem. There is a bijective correspondence between Mackey observables and densely-defined self-adjoint operators on H .

This is the content of the spectral theorem as stated in terms of spectral measures.

Statistical structure

Imagine a forensics lab which has some apparatus to measure the speed of a bullet fired from a gun. Under carefully controlled conditions of temperature, humidity, pressure and so on the same gun is fired repeatedly and speed measurements taken. This produces some distribution of speeds. Though we will not get exactly the same value for each individual measurement, for each cluster of measurements, we would expect the experiment to lead to the same distribution of speeds. In particular, we can expect to assign probability distributions to propositions such as $\{a \leq \text{speed} \leq b\}$. This leads naturally to propose that under controlled conditions of preparation, the measurement of a classical system can be described by a probability measure on the state space. This same statistical structure is also present in quantum mechanics.

A *quantum probability measure* is a function P defined on Q with values in $[0,1]$ such that $P(0)=0$, $P(I)=1$ and if $\{E_i\}_i$ is a sequence of pairwise orthogonal elements of Q then

$$P\left(\sum_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i).$$

The following highly non-trivial theorem is due to Andrew Gleason:

Theorem. Suppose H is a separable Hilbert space of complex dimension at least 3. Then for any quantum probability measure on Q there exists a unique trace class operator S such that

$$P(E) = \text{Tr}(SE)$$

for any self-adjoint projection E .

The operator S is necessarily non-negative (that is all eigenvalues are non-negative) and of trace 1. Such an operator is often called a *density operator*.

Physicists commonly regard a density operator as being represented by a (possibly infinite) density matrix relative to some orthonormal basis.

For more information on statistics of quantum systems, see quantum statistical mechanics.

Automorphisms

An *automorphism* of Q is a bijective mapping $\alpha:Q \rightarrow Q$ which preserves the orthocomplemented structure of Q , that is

$$\alpha\left(\sum_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \alpha(E_i)$$

for any sequence $\{E_i\}_i$ of pairwise orthogonal self-adjoint projections. Note that this property implies monotonicity of α . If P is a quantum probability measure on Q , then $E \rightarrow \alpha(E)$ is also a quantum probability measure on Q . By the Gleason theorem characterizing quantum probability measures quoted above, any automorphism α induces a mapping α^* on the density operators by the following formula:

$$\text{Tr}(\alpha^*(S)E) = \text{Tr}(S\alpha(E)).$$

The mapping α^* is bijective and preserves convex combinations of density operators. This means

$$\alpha^*(r_1 S_1 + r_2 S_2) = r_1 \alpha^*(S_1) + r_2 \alpha^*(S_2)$$

whenever $1 = r_1 + r_2$ and r_1, r_2 are non-negative real numbers. Now we use a theorem of Richard Kadison:

Theorem. Suppose β is a bijective map from density operators to density operators which is convexity preserving. Then there is an operator U on the Hilbert space which is either linear or conjugate-linear, preserves the inner product and is such that

$$\beta(S) = U S U^*$$

for every density operator S . In the first case we say U is unitary, in the second case U is anti-unitary.

Remark. This note is included for technical accuracy only, and should not concern most readers. The result quoted above is not directly stated in Kadison's paper, but can be reduced to it by noting first that β extends to a positive trace preserving map on the trace class operators, then applying duality and finally applying a result of Kadison's paper.

The operator U is not quite unique; if r is a complex scalar of modulus 1, then $r U$ will be unitary or anti-unitary if U is and will implement the same automorphism. In fact, this is the only ambiguity possible.

It follows that automorphisms of Q are in bijective correspondence to unitary or anti-unitary operators modulo multiplication by scalars of modulus 1. Moreover, we can regard automorphisms in two equivalent ways: as operating on states (represented as density operators) or as operating on Q .

Non-relativistic dynamics

In non-relativistic physical systems, there is no ambiguity in referring to time evolution since there is a global time parameter. Moreover an isolated quantum system evolves in a deterministic way: if the system is in a state S at time t then at time $s > t$, the system is in a state $F_{s,t}(S)$. Moreover, we assume

- The dependence is reversible: The operators $F_{s,t}$ are bijective.
- The dependence is homogeneous: $F_{s,t} = F_{s-t,0}$.
- The dependence is convexity preserving: That is, each $F_{s,t}(S)$ is convexity preserving.
- The dependence is weakly continuous: The mapping $\mathbf{R} \rightarrow \mathbf{R}$ given by $t \rightarrow \text{Tr}(F_{s,t}(S) E)$ is continuous for every E in Q .

By Kadison's theorem, there is a 1-parameter family of unitary or anti-unitary operators $\{U_t\}_t$ such that

$$F_{s,t}(S) = U_{s-t} S U_{s-t}^*$$

In fact,

Theorem. Under the above assumptions, there is a strongly continuous 1-parameter group of unitary operators $\{U_t\}_t$ such that the above equation holds.

Note that it easily follows from uniqueness from Kadison's theorem that

$$U_{t+s} = \sigma(t, s) U_t U_s$$

where $\sigma(t,s)$ has modulus 1. Now the square of an anti-unitary is a unitary, so that all the U_t are unitary. The remainder of the argument shows that $\sigma(t,s)$ can be chosen to be 1 (by modifying each U_t by a scalar of modulus 1.)

Pure states

A convex combinations of statistical states S_1 and S_2 is a state of the form $S = p_1 S_1 + p_2 S_2$ where p_1, p_2 are non-negative and $p_1 + p_2 = 1$. Considering the statistical state of system as specified by lab conditions used for its preparation, the convex combination S can be regarded as the state formed in the following way: toss a biased coin with outcome probabilities p_1, p_2 and depending on outcome choose system prepared to S_1 or S_2 .

Density operators form a convex set. The convex set of density operators has extreme points; these are the density operators given by a projection onto a one-dimensional space. To see that any extreme point is such a projection, note that by the spectral theorem S can be represented by a diagonal matrix; since S is non-negative all the entries are non-negative and since S has trace 1, the diagonal entries must add up to 1. Now if it happens that the diagonal matrix has more than one non-zero entry it is clear that we can express it as a convex combination of other density operators.

The extreme points of the set of density operators are called pure states. If S is the projection on the 1-dimensional space generated by a vector ψ of norm 1 then

$$\text{Tr}(SE) = \langle E\psi | \psi \rangle$$

for any E in Q . In physics jargon, if

$$S = |\psi\rangle\langle\psi|,$$

where ψ has norm 1, then

$$\text{Tr}(SE) = \langle \psi | E | \psi \rangle.$$

Thus pure states can be identified with *rays* in the Hilbert space H .

The measurement process

Consider a quantum mechanical system with lattice Q which is in some statistical state given by a density operator S . This essentially means an ensemble of systems specified by a repeatable lab preparation process. The result of a cluster of measurements intended to determine the truth value of proposition E , is just as in the classical case, a probability distribution of truth values **T** and **F**. Say the probabilities are p for **T** and $q = 1 - p$ for **F**. By the previous section $p = \text{Tr}(SE)$ and $q = \text{Tr}(S(I-E))$.

Perhaps the most fundamental difference between classical and quantum systems is the following: regardless of what process is used to determine E immediately after the measurement the system will be in one of two statistical states:

- If the result of the measurement is **T**

$$\frac{1}{\text{Tr}(ES)} ESE.$$

- If the result of the measurement is **F**

$$\frac{1}{\text{Tr}((I-E)S)} (I-E)S(I-E).$$

(We leave to the reader the handling of the degenerate cases in which the denominators may be 0.) We now form the convex combination of these two ensembles using the relative

frequencies p and q . We thus obtain the result that the measurement process applied to a statistical ensemble in state S yields another ensemble in statistical state:

$$M_E(S) = ESE + (I - E)S(I - E).$$

We see that a pure ensemble becomes a mixed ensemble after measurement. Measurement, as described above, is a special case of quantum operations.

Limitations

Quantum logic derived from propositional logic provides a satisfactory foundation for a theory of reversible quantum processes. Examples of such processes are the covariance transformations relating two frames of reference, such as change of time parameter or the transformations of special relativity. Quantum logic also provides a satisfactory understanding of density matrices. Quantum logic can be stretched to account for some kinds of measurement processes corresponding to answering yes-no questions about the state of a quantum system. However, for more general kinds of measurement operations (that is quantum operations), a more complete theory of filtering processes is necessary. Such an approach is provided by the consistent histories formalism. On the other hand, quantum logics derived from MV-logic extend its range of applicability to irreversible quantum processes and/or 'open' quantum systems.

In any case, these quantum logic formalisms must be generalized in order to deal with super-geometry (which is needed to handle Fermi-fields) and non-commutative geometry (which is needed in string theory and quantum gravity theory). Both of these theories use a partial algebra with an "integral" or "trace". The elements of the partial algebra are not observables; instead the "trace" yields "greens functions" which generate scattering amplitudes. One thus obtains a local S-matrix theory (see D. Edwards).

Since around 1978 the Flato school (see F. Bayen) has been developing an alternative to the quantum logics approach called deformation quantization (see Weyl quantization).

In 2004, Prakash Panangaden described how to capture the kinematics of quantum causal evolution using System BV, a deep inference logic originally developed for use in structural proof theory.[6] Alessio Guglielmi, Lutz Straßburger, and Richard Blute have also done work in this area.[7]

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See also

- Mathematical formulation of quantum mechanics
- Multi-valued logic
- Quasi-set theory
- HPO formalism (An approach to temporal quantum logic)
- Quantum field theory

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External links

- Stanford Encyclopedia of Philosophy entry on Quantum Logic and Probability Theory (<http://plato.stanford.edu/entries/qt-quantlog/>)

Quantum group

In mathematics and theoretical physics, **quantum groups** are certain noncommutative algebras that first appeared in the theory of quantum integrable systems, and which were then formalized by Vladimir Drinfel'd and Michio Jimbo. There is no single, all-encompassing definition of quantum group.

In Drinfeld's approach, quantum groups arise as Hopf algebras depending on an auxiliary parameter q or h , which become universal enveloping algebras of a certain Lie algebra, frequently semisimple or affine, when $q = 1$ or $h = 0$. Distinct but related objects, also called quantum groups, are deformations of the algebra of functions on a semisimple algebraic group or a compact Lie group.

Since the discovery of quantum groups, it has become fashionable to introduce the attribute *quantum* into the names of many other mathematical objects, such as quantum plane^[1] or quantum Grassmannian. They may also be loosely referred to as aspects of "quantum groups".

Intuitive meaning

The discovery of quantum groups was quite unexpected, since it was known for a long time that compact groups and semisimple Lie algebras are "rigid" objects, in other words, cannot be deformed. One of the ideas behind quantum groups is that if we consider in some sense equivalent but larger structure, namely a group algebra or a universal enveloping algebra, then it *can* be deformed, although the deformation will no longer remain a group or enveloping algebra. More precisely, deformation can be accomplished within the category of Hopf algebras that are not required to be either commutative or cocommutative. One can think of the deformed object as an algebra of functions on a "noncommutative space", in the spirit of Alain Connes' noncommutative geometry. This intuition, however, came after particular classes of quantum groups had already proved their usefulness in the study of the quantum Yang-Baxter equation and quantum inverse scattering method developed by the Leningrad School (Ludwig Faddeev, Leon Takhtajan, Evgenii Sklyanin, Nicolai Reshetikhin and others) and related work by the Japanese School.^[2]

Drinfel'd-Jimbo type quantum groups

One type of objects commonly called a "quantum group" appeared in the work of Vladimir Drinfel'd and Michio Jimbo as a deformation of the universal enveloping algebra of a semisimple Lie algebra or, more generally, a Kac-Moody algebra, in the category of Hopf algebras. The resulting algebra has additional structure, making it into a quasitriangular Hopf algebra.

Let $A = (a_{ij})$ be the Cartan matrix of the Kac-Moody algebra, and let q be a nonzero complex number distinct from 1, then the quantum group, $U_q(G)$, where G is the Lie algebra whose Cartan matrix is A , is defined as the unital associative algebra with generators k_λ (where λ is an element of the weight lattice, i.e. $2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \in \mathbb{Z}$ for all i), and e_i and f_i (for simple roots, α_i), subject to the following relations:

- $k_0 = 1$,
 - $k_\lambda k_\mu = k_{\lambda+\mu}$,
-

- $k_\lambda e_i k_\lambda^{-1} = q^{(\lambda, \alpha_i)} e_i,$
- $k_\lambda f_i k_\lambda^{-1} = q^{-(\lambda, \alpha_i)} f_i,$
- $[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$
- $\sum_{n=0}^{1-a_{ij}} (-1)^n \frac{[1-a_{ij}]_{q_i}!}{[1-a_{ij}-n]_{q_i}! [n]_{q_i}!} e_i^n e_j e_i^{1-a_{ij}-n} = 0, \text{ for } i \neq j,$
- $\sum_{n=0}^{1-a_{ij}} (-1)^n \frac{[1-a_{ij}]_{q_i}!}{[1-a_{ij}-n]_{q_i}! [n]_{q_i}!} f_i^n f_j f_i^{1-a_{ij}-n} = 0, \text{ for } i \neq j,$

where $k_i = k_{\alpha_i}$, $q_i = q^{\frac{1}{2}(\alpha_i, \alpha_i)}$, $[0]_{q_i}! = 1$, $[n]_{q_i}! = \prod_{m=1}^n [m]_{q_i}$ for all positive integers n , and

$[m]_{q_i} = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}$. These are the q -factorial and q -number, respectively, the q -analogs of the ordinary factorial. The last two relations above are the q -Serre relations, the deformations of the Serre relations.

In the limit as $q \rightarrow 1$, these relations approach the relations for the universal enveloping algebra $U_q(G)$, where $k_\lambda \rightarrow 1$ and $\frac{k_\lambda - k_{-\lambda}}{q - q^{-1}} \rightarrow t_\lambda$ as $q \rightarrow 1$, where the element, t_λ , of the Cartan subalgebra satisfies $(t_\lambda, h) = \lambda(h)$ for all h in the Cartan subalgebra. There are various coassociative coproducts under which the quantum groups are Hopf algebras, for example,

- $\Delta_1(k_\lambda) = k_\lambda \otimes k_\lambda$, $\Delta_1(e_i) = 1 \otimes e_i + e_i \otimes k_i$, $\Delta_1(f_i) = k_i^{-1} \otimes f_i + f_i \otimes 1$,
- $\Delta_2(k_\lambda) = k_\lambda \otimes k_\lambda$, $\Delta_2(e_i) = k_i^{-1} \otimes e_i + e_i \otimes 1$, $\Delta_2(f_i) = 1 \otimes f_i + f_i \otimes k_i$,
- $\Delta_3(k_\lambda) = k_\lambda \otimes k_\lambda$, $\Delta_3(e_i) = k_i^{-\frac{1}{2}} \otimes e_i + e_i \otimes k_i^{\frac{1}{2}}$, $\Delta_3(f_i) = k_i^{-\frac{1}{2}} \otimes f_i + f_i \otimes k_i^{\frac{1}{2}}$, where the set of generators has been extended, if required, to include k_λ for λ which is expressible as the sum of an element of the weight lattice and half an element of the root lattice,

along with the reverse coproducts $T \circ \Delta$, where $T : U_q(G) \otimes U_q(G) \rightarrow U_q(G) \otimes U_q(G)$ is given by $T(x \otimes y) = y \otimes x$, i.e.

- $\Delta_4(k_\lambda) = k_\lambda \otimes k_\lambda$, $\Delta_4(e_i) = k_i \otimes e_i + e_i \otimes 1$, $\Delta_4(f_i) = 1 \otimes f_i + f_i \otimes k_i^{-1}$, where $\Delta_4 = T \circ \Delta_1$,
- $\Delta_5(k_\lambda) = k_\lambda \otimes k_\lambda$, $\Delta_5(e_i) = 1 \otimes e_i + e_i \otimes k_i^{-1}$, $\Delta_5(f_i) = k_i \otimes f_i + f_i \otimes 1$, where $\Delta_5 = T \circ \Delta_2$,
- $\Delta_6(k_\lambda) = k_\lambda \otimes k_\lambda$, $\Delta_6(e_i) = k_i^{\frac{1}{2}} \otimes e_i + e_i \otimes k_i^{-\frac{1}{2}}$, $\Delta_6(f_i) = k_i^{\frac{1}{2}} \otimes f_i + f_i \otimes k_i^{-\frac{1}{2}}$, where $\Delta_6 = T \circ \Delta_3$.

The counit on $U_q(A)$ is the same for all these coproducts: $\epsilon(k_\lambda) = 1$, $\epsilon(e_i) = 0$, $\epsilon(f_i) = 0$, and the respective antipodes for the above coproducts are given by

- $S_1(k_\lambda) = k_{-\lambda}$, $S_1(e_i) = -e_i k_i^{-1}$, $S_1(f_i) = -k_i f_i$,
- $S_2(k_\lambda) = k_{-\lambda}$, $S_2(e_i) = -k_i e_i$, $S_2(f_i) = -f_i k_i^{-1}$,
- $S_3(k_\lambda) = k_{-\lambda}$, $S_3(e_i) = -q_i e_i$, $S_3(f_i) = -q_i^{-1} f_i$,
- $S_4(k_\lambda) = k_{-\lambda}$, $S_4(e_i) = -k_i^{-1} e_i$, $S_4(f_i) = -f_i k_i$,
- $S_5(k_\lambda) = k_{-\lambda}$, $S_5(e_i) = -e_i k_i$, $S_5(f_i) = -k_i^{-1} f_i$,
- $S_6(k_\lambda) = k_{-\lambda}$, $S_6(e_i) = -q_i^{-1} e_i$, $S_6(f_i) = -q_i f_i$.

Alternatively, the quantum group $U_q(G)$ can be regarded as an algebra over the field $\mathbb{C}(q)$, the field of all rational functions of an indeterminate q over $\mathbb{C}$.

Similarly, the quantum group $U_q(G)$ can be regarded as an algebra over the field $\mathbb{Q}(q)$, the field of all rational functions of an indeterminate q over $\mathbb{Q}$ (see below in the section on quantum groups at $q = 0$).

Representation theory

Just as there are many different types of representations for Kac-Moody algebras and their universal enveloping algebras, so there are many different types of representation for quantum groups.

As is the case for all Hopf algebras, $U_q(G)$ has an adjoint representation on itself as a module, with the action being given by $\text{Ad}_x \cdot y = \sum_{(x)} x_{(1)} y S(x_{(2)})$, where $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$.

Case 1: q is not a root of unity

One important type of representation is a weight representation, and the corresponding module is called a weight module. A weight module is a module with a basis of weight vectors. A weight vector is a nonzero vector v such that $k_\lambda \cdot v = d_\lambda v$ for all λ , where d_λ are complex numbers for all weights λ such that

- $d_0 = 1$,
- $d_\lambda d_\mu = d_{\lambda+\mu}$, for all weights λ and μ .

A weight module is called integrable if the actions of e_i and f_i are locally nilpotent (*i.e.* for any vector v in the module, there exists a positive integer k , possibly dependent on v , such that $e_i^k \cdot v = f_i^k \cdot v = 0$ for all i). In the case of integrable modules, the complex numbers d_λ associated with a weight vector satisfy $d_\lambda = c_\lambda q^{(\lambda, \nu)}$, where ν is an element of the weight lattice, and c_λ are complex numbers such that

- $c_0 = 1$,
- $c_\lambda c_\mu = c_{\lambda+\mu}$, for all weights λ and μ ,
- $c_{2\alpha_i} = 1$ for all i .

Of special interest are highest weight representations, and the corresponding highest weight modules. A highest weight module is a module generated by a weight vector v , subject to $k_\lambda \cdot v = d_\lambda v$ for all weights λ , and $e_i \cdot v = 0$ for all i . Similarly, a quantum group can have a lowest weight representation and lowest weight module, *i.e.* a module generated by a weight vector v , subject to $k_\lambda \cdot v = d_\lambda v$ for all weights λ , and $f_i \cdot v = 0$ for all i .

Define a vector v to have weight ν if $k_\lambda \cdot v = q^{(\lambda, \nu)} v$ for all λ in the weight lattice.

If G is a Kac-Moody algebra, then in any irreducible highest weight representation of $U_q(G)$, with highest weight ν , the multiplicities of the weights are equal to their multiplicities in an irreducible representation of $U(G)$ with equal highest weight. If the highest weight is dominant and integral (a weight μ is dominant and integral if μ satisfies the condition that $2(\mu, \alpha_i)/(\alpha_i, \alpha_i)$ is a non-negative integer for all i), then the weight spectrum of the irreducible representation is invariant under the Weyl group for G , and the representation is integrable.

Conversely, if a highest weight module is integrable, then its highest weight vector v satisfies $k_\lambda.v = c_\lambda q^{(\lambda, \nu)} v$, where c_λ are complex numbers such that

- $c_0 = 1$,
- $c_\lambda c_\mu = c_{\lambda+\mu}$, for all weights λ and μ ,
- $c_{2\alpha_i} = 1$ for all i ,

and ν is dominant and integral.

As is the case for all Hopf algebras, the tensor product of two modules is another module. For an element x of $U_q(G)$, and for vectors v and w in the respective modules, $x.(v \otimes w) = \Delta(x).(v \otimes w)$, so that $k_\lambda.(v \otimes w) = k_\lambda.v \otimes k_\lambda.w$, and in the case of coproduct Δ_1 , $e_i.(v \otimes w) = k_i.v \otimes e_i.w + e_i.v \otimes w$ and $f_i.(v \otimes w) = v \otimes f_i.w + f_i.v \otimes k_i^{-1}.w$.

The integrable highest weight module described above is a tensor product of a one-dimensional module (on which $k_\lambda = c_\lambda$ for all λ , and $e_i = f_i = 0$ for all i) and a highest weight module generated by a nonzero vector v_0 , subject to $k_\lambda.v_0 = q^{(\lambda, \nu)} v_0$ for all weights λ , and $e_i.v_0 = 0$ for all i .

In the specific case where G is a finite-dimensional Lie algebra (as a special case of a Kac-Moody algebra), then the irreducible representations with dominant integral highest weights are also finite-dimensional.

In the case of a tensor product of highest weight modules, its decomposition into submodules is the same as for the tensor product of the corresponding modules of the Kac-Moody algebra (the highest weights are the same, as are their multiplicities).

Case 2: q is a root of unity

Quasitriangularity

Case 1: q is not a root of unity

Strictly, the quantum group $U_q(G)$ is not quasitriangular, but it can be thought of as being "nearly quasitriangular" in that there exists an infinite formal sum which plays the role of an R -matrix. This infinite formal sum is expressible in terms of generators e_i and f_i , and Cartan generators t_λ , where k_λ is formally identified with q^{t_λ} . The infinite formal sum is the product of two factors, $q^{\eta \sum_j t_{\lambda_j} \otimes t_{\mu_j}}$, and an infinite formal sum, where $\{\lambda_j\}$ is a basis for the dual space to the Cartan subalgebra, and $\{\mu_j\}$ is the dual basis, and η is a sign (+1 or -1).

The formal infinite sum which plays the part of the R -matrix has a well-defined action on the tensor product of two irreducible highest weight modules, and also on the tensor product of two lowest weight modules. Specifically, if v has weight α and w has weight β , then $q^{\eta \sum_j t_{\lambda_j} \otimes t_{\mu_j}}.(v \otimes w) = q^{\eta(\alpha, \beta)} v \otimes w$, and the fact that the modules are both highest weight modules or both lowest weight modules reduces the action of the other factor on $v \otimes w$ to a finite sum.

Specifically, if V is a highest weight module, then the formal infinite sum, R , has a well-defined, and invertible, action on $V \otimes V$, and this value of R (as an element of $\text{Hom}(V) \otimes \text{Hom}(V)$) satisfies the Yang-Baxter equation, and therefore allows us to determine a representation of the braid group, and to define quasi-invariants for knots, links and braids.

Case 2: q is a root of unity

Quantum groups at $q = 0$

Masaki Kashiwara has researched the limiting behaviour of quantum groups as $q \rightarrow 0$.

As a consequence of the defining relations for the quantum group $U_q(G)$, $U_q(G)$ can be regarded as a Hopf algebra over $\mathbb{Q}(q)$, the field of all rational functions of an indeterminate q over $\mathbb{Q}$.

For simple root α_i and non-negative integer n , define $e_i^{(n)} = e_i^n / [n]_{q_i}!$ and $f_i^{(n)} = f_i^n / [n]_{q_i}!$ (specifically, $e_i^{(0)} = f_i^{(0)} = 1$). In an integrable module M , and for weight λ , a vector $u \in M_\lambda$ (i.e. a vector u in M with weight λ) can be uniquely decomposed into the sums

$$\bullet \quad u = \sum_{n=0}^{\infty} f_i^{(n)} u_n = \sum_{n=0}^{\infty} e_i^{(n)} v_n,$$

where $u_n \in \ker(e_i) \cap M_{\lambda+n\alpha_i}$, $v_n \in \ker(f_i) \cap M_{\lambda-n\alpha_i}$, $u_n \neq 0$ only if $n + \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \geq 0$, and

$v_n \neq 0$ only if $n - \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \geq 0$. Linear mappings $\tilde{e}_i : M \rightarrow M$ and $\tilde{f}_i : M \rightarrow M$ can be

defined on M_λ by

$$\begin{aligned} \bullet \quad \tilde{e}_i u &= \sum_{n=1}^{\infty} f_i^{(n-1)} u_n = \sum_{n=0}^{\infty} e_i^{(n+1)} v_n, \\ \bullet \quad \tilde{f}_i u &= \sum_{n=0}^{\infty} f_i^{(n+1)} u_n = \sum_{n=1}^{\infty} e_i^{(n-1)} v_n. \end{aligned}$$

Let A be the integral domain of all rational functions in $\mathbb{Q}(q)$ which are regular at $q = 0$ (i.e. a rational function $f(q)$ is an element of A if and only if there exist polynomials $g(q)$ and $h(q)$ in the polynomial ring $\mathbb{Q}[q]$ such that $h(0) \neq 0$, and $f(q) = g(q)/h(q)$). A **crystal base** for M is an ordered pair (L, B) , such that

- L is a free A -submodule of M such that $M = \mathbb{Q}(q) \otimes_A L$;
- B is a $\mathbb{Q}$ -basis of the vector space L/qL over $\mathbb{Q}$,
- $L = \bigoplus_\lambda L_\lambda$ and $B = \sqcup_\lambda B_\lambda$, where $L_\lambda = L \cap M_\lambda$ and $B_\lambda = B \cap (L_\lambda/qL_\lambda)$,
- $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for all i ,
- $\tilde{e}_i B \subset B \cup \{0\}$ and $\tilde{f}_i B \subset B \cup \{0\}$ for all i ,
- for all $b \in B$ and $b' \in B$, and for all i , $\tilde{e}_i b = b'$ if and only if $\tilde{f}_i b' = b$.

To put this into a more informal setting, the actions of $e_i f_i$ and $f_i e_i$ are generally singular at $q = 0$ on an integrable module M . The linear mappings $\tilde{e}_i$ and $\tilde{f}_i$ on the module are introduced so that the actions of $\tilde{e}_i \tilde{f}_i$ and $\tilde{f}_i \tilde{e}_i$ are regular at $q = 0$ on the module. There exists a $\mathbb{Q}(q)$ -basis of weight vectors $\tilde{B}$ for M , with respect to which the actions of $\tilde{e}_i$ and $\tilde{f}_i$ are regular at $q = 0$ for all i . The module is then restricted to the free A -module generated by the basis, and the basis vectors, the A -submodule and the actions of $\tilde{e}_i$ and $\tilde{f}_i$ are evaluated at $q = 0$. Furthermore, the basis can be chosen such that at $q = 0$, for all i , $\tilde{e}_i$ and $\tilde{f}_i$ are represented by mutual transposes, and map basis vectors to basis vectors or 0.

A crystal base can be represented by a directed graph with labelled edges. Each vertex of the graph represents an element of the $\mathbb{Q}$ -basis B of L/qL , and a directed edge, labelled by i , and directed from vertex v_1 to vertex v_2 , represents that $b_2 = \tilde{f}_i b_1$ (and,

equivalently, that $b_1 = \tilde{e}_i b_2$), where b_1 is the basis element represented by v_1 , and b_2 is the basis element represented by v_2 . The graph completely determines the actions of $\tilde{e}_i$ and $\tilde{f}_i$ at $q = 0$. If an integrable module has a crystal base, then the module is irreducible if and only if the graph representing the crystal base is connected (a graph is called "connected" if the set of vertices cannot be partitioned into the union of nontrivial disjoint subsets V_1 and V_2 such that there are no edges joining any vertex in V_1 to any vertex in V_2).

For any integrable module with a crystal base, the weight spectrum for the crystal base is the same as the weight spectrum for the module, and therefore the weight spectrum for the crystal base is the same as the weight spectrum for the corresponding module of the appropriate Kac-Moody algebra. The multiplicities of the weights in the crystal base are also the same as their multiplicities in the corresponding module of the appropriate Kac-Moody algebra.

It is a theorem of Kashiwara that every integrable highest weight module has a crystal base. Similarly, every integrable lowest weight module has a crystal base.

Tensor products of crystal bases

Let M be an integrable module with crystal base (L, B) and M' be an integrable module with crystal base (L', B') . For crystal bases, the coproduct Δ , given by $\Delta(k_\lambda) = k_\lambda \otimes k_\lambda$, $\Delta(e_i) = e_i \otimes k_i^{-1} + 1 \otimes e_i$, $\Delta(f_i) = f_i \otimes 1 + k_i \otimes f_i$, is adopted. The integrable module $M \otimes_{\mathbb{Q}(q)} M'$ has crystal base $(L \otimes_A L', B \otimes B')$, where $B \otimes B' = \{b \otimes_{\mathbb{Q}} b' : b \in B, b' \in B'\}$. For a basis vector $b \in B$, define $\epsilon_i(b) = \max\{n \geq 0 : \tilde{e}_i^n b \neq 0\}$ and $\phi_i(b) = \max\{n \geq 0 : \tilde{f}_i^n b \neq 0\}$. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $b \otimes b'$ are given by

$$\begin{aligned} \bullet \quad \tilde{e}_i(b \otimes b') &= \begin{cases} \tilde{e}_i b \otimes b', & \text{if } \phi_i(b) \geq \epsilon_i(b'), \\ b \otimes \tilde{e}_i b', & \text{if } \phi_i(b) < \epsilon_i(b'), \end{cases} \\ \bullet \quad \tilde{f}_i(b \otimes b') &= \begin{cases} \tilde{f}_i b \otimes b', & \text{if } \phi_i(b) > \epsilon_i(b'), \\ b \otimes \tilde{f}_i b', & \text{if } \phi_i(b) \leq \epsilon_i(b'). \end{cases} \end{aligned}$$

The decomposition of the product two integrable highest weight modules into irreducible submodules is determined by the decomposition of the graph of the crystal base into its connected components (i.e. the highest weights of the submodules are determined, and the multiplicity of each highest weight is determined).

Compact matrix quantum groups

S.L. Woronowicz introduced compact matrix quantum groups. Compact matrix quantum groups are abstract structures on which the "continuous functions" on the structure are given by elements of a C*-algebra. The geometry of a compact matrix quantum group is a special case of a noncommutative geometry.

The continuous complex-valued functions on a compact Hausdorff topological space form a commutative C*-algebra. By the Gelfand theorem, a commutative C*-algebra is isomorphic to the C*-algebra of continuous complex-valued functions on a compact Hausdorff topological space, and the topological space is uniquely determined by the C*-algebra up to homeomorphism.

For a compact topological group, G , there exists a C*-algebra homomorphism $\Delta : C(G) \rightarrow C(G) \otimes C(G)$ (where $C(G) \otimes C(G)$ is the C*-algebra tensor product - the completion of the algebraic tensor product of $C(G)$ and $C(G)$), such that

$\Delta(f)(x, y) = f(xy)$ for all $f \in C(G)$, and for all $x, y \in G$ (where $(f \otimes g)(x, y) = f(x)g(y)$ for all $f, g \in C$ such that $\kappa(f)(x) = f(x^{-1})$ for all $f \in C(G)$ and all $x \in G$). Strictly, this does not make $C(G)$ a Hopf algebra. On the other hand, a finite-dimensional representation of G can be used to generate a $*$ -subalgebra of $C(G)$ which is also a Hopf $*$ -algebra. Specifically, if $g \mapsto (u_{ij}(g))_{i,j}$ is an n -dimensional representation of G , then $u_{ij} \in C(G)$ for all i, j , and $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ for all i, j . It follows that the $*$ -subalgebra generated by u_{ij} and $\kappa(u_{ij})$ for all i, j is a Hopf $*$ -algebra: the counit is determined by $\epsilon(u_{ij}) = \delta_{ij}$ for all i, j (where δ_{ij} is the Kronecker delta), the antipode is κ , and the unit is given by $1 = \sum_k u_{1k} \kappa(u_{k1}) = \sum_k \kappa(u_{1k}) u_{k1}$.

As a generalization, a compact matrix quantum group is defined as a pair (C, u) , where C is a C^* -algebra and $u = (u_{ij})_{i,j=1,\dots,n}$ is a matrix with entries in C such that

- The $*$ -subalgebra, C_0 , of C , which is generated by the matrix elements of u , is dense in C ;
- There exists a C^* -algebra homomorphism $\Delta : C \rightarrow C \otimes C$ (where $C \otimes C$ is the C^* -algebra tensor product - the completion of the algebraic tensor product of C and C) such that $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ for all i, j (Δ is called the comultiplication);
- There exists a linear antimultiplicative map $\kappa : C_0 \rightarrow C_0$ (the coinverse) such that $\kappa(\kappa(v)^*) = v$ for all $v \in C_0$ and $\sum_k \kappa(u_{ik}) u_{kj} = \sum_k u_{ik} \kappa(u_{kj}) = \delta_{ij} I$, where I is the identity element of C . Since κ is antimultiplicative, then $\kappa(vw) = \kappa(w) \kappa(v)$ for all $v, w \in C_0$.

As a consequence of continuity, the comultiplication on C is coassociative.

In general, C is not a bialgebra, and C_0 is a Hopf $*$ -algebra.

Informally, C can be regarded as the $*$ -algebra of continuous complex-valued functions over the compact matrix quantum group, and u can be regarded as a finite-dimensional representation of the compact matrix quantum group.

A representation of the compact matrix quantum group is given by a corepresentation of the Hopf $*$ -algebra (a corepresentation of a counital coassociative coalgebra A is a square matrix $v = (v_{ij})_{i,j=1,\dots,n}$ with entries in A (so $v \in M_n(A)$) such that $\Delta(v_{ij}) = \sum_{k=1}^n v_{ik} \otimes v_{kj}$ for all i, j and $\epsilon(v_{ij}) = \delta_{ij}$ for all i, j). Furthermore, a representation, v , is called unitary if the matrix for v is unitary (or equivalently, if $\kappa(v_{ij}) = v_{ji}^*$ for all i, j).

An example of a compact matrix quantum group is $SU_\mu(2)$, where the parameter μ is a positive real number. So $SU_\mu(2) = (C(SU_\mu(2)), u)$, where $C(SU_\mu(2))$ is the C^* -algebra generated by α and γ , subject to

$$\gamma\gamma^* = \gamma^*\gamma, \quad \alpha\gamma = \mu\gamma\alpha, \quad \alpha\gamma^* = \mu\gamma^*\alpha, \quad \alpha\alpha^* + \mu\gamma^*\gamma = \alpha^*\alpha + \mu^{-1}\gamma^*\gamma = I,$$

and $u = \begin{pmatrix} \alpha & \gamma \\ -\gamma^* & \alpha^* \end{pmatrix}$, so that the comultiplication is determined by $\Delta(\alpha) = \alpha \otimes \alpha - \gamma \otimes \gamma^*$, $\Delta(\gamma) = \alpha \otimes \gamma + \gamma \otimes \alpha^*$, and the coinverse is determined by $\kappa(\alpha) = \alpha^*$, $\kappa(\gamma) = -\mu^{-1}\gamma$, $\kappa(\gamma^*) = -\mu\gamma^*$, $\kappa(\alpha^*) = \alpha$. Note that u is a representation, but not a unitary representation. u is equivalent to the unitary representation $v = \begin{pmatrix} \alpha & \sqrt{\mu}\gamma \\ -\frac{1}{\sqrt{\mu}}\gamma^* & \alpha^* \end{pmatrix}$.

Equivalently, $SU_\mu(2) = (C(SU_\mu(2)), w)$, where $C(SU_\mu(2))$ is the C^* -algebra generated by α and β , subject to

$$\beta\beta^* = \beta^*\beta, \quad \alpha\beta = \mu\beta\alpha, \quad \alpha\beta^* = \mu\beta^*\alpha, \quad \alpha\alpha^* + \mu^2\beta^*\beta = \alpha^*\alpha + \beta^*\beta = I,$$

and $w = \begin{pmatrix} \alpha & \mu\beta \\ -\beta^* & \alpha^* \end{pmatrix}$, so that the comultiplication is determined by $\Delta(\alpha) = \alpha \otimes \alpha - \mu\beta \otimes \beta^*$, $\Delta(\beta) = \alpha \otimes \beta + \beta \otimes \alpha^*$, and the coinverse is determined by $\kappa(\alpha) = \alpha^*$, $\kappa(\beta) = -\mu^{-1}\beta$, $\kappa(\beta^*) = -\mu\beta^*$, $\kappa(\alpha^*) = \alpha$. Note that w is a unitary representation. The realizations can be identified by equating $\gamma = \sqrt{\mu}\beta$. When $\mu = 1$, then $SU_\mu(2)$ is equal to the concrete compact group $SU(2)$.

See also

- Lie bialgebra
- Poisson-Lie group
- Affine quantum group
- Quantum affine algebras

Notes

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Hopf algebra

In mathematics, a **Hopf algebra**, named after Heinz Hopf, is a structure that is simultaneously a (unital associative) algebra, a (counital coassociative) coalgebra, with these structures compatible making it a bialgebra, and moreover is equipped with an antiautomorphism satisfying a certain property.

Hopf algebras occur naturally in algebraic topology, where they originated and are related to the H-space concept, in group scheme theory, in group theory (via the concept of a group ring), and in numerous other places, making them probably the most familiar type of bialgebra. Hopf algebras are also studied in their own right, with much work on specific classes of examples on the one hand and classification problems on the other.

Formal definition

Formally, a Hopf algebra is a bialgebra H over a field K together with a K -linear map $S: H \rightarrow H$ (called the **antipode**) such that the following diagram commutes:

$$\begin{array}{ccccc}
 & H \otimes H & \xrightarrow{S \otimes \text{id}} & H \otimes H & \\
 \Delta \nearrow & & & & \searrow \nabla \\
 H & \xrightarrow{\varepsilon} & K & \xrightarrow{\eta} & H \\
 \Delta \searrow & & & & \nearrow \nabla \\
 & H \otimes H & \xrightarrow{\text{id} \otimes S} & H \otimes H &
 \end{array}$$

Here Δ is the comultiplication of the bialgebra, ∇ its multiplication, η its unit and ε its counit. In the sumless Sweedler notation, this property can also be expressed as

$$S(c_{(1)})c_{(2)} = c_{(1)}S(c_{(2)}) = \epsilon(c)1 \quad \text{for all } c \in H.$$

As for algebras, one can replace the underlying field K with a commutative ring R in the above definition.

The definition of Hopf algebra is self-dual (as reflected in the symmetry of the above diagram), so if one can define a dual of H (which is always possible if H is finite-dimensional), then it is automatically a Hopf algebra.

Properties of the antipode

S is sometimes required to have a K -linear inverse, which is automatic in the finite-dimensional case, or if H is commutative or cocommutative (or more generally quasitriangular).

In general, S is an antihomomorphism, so S^2 is a homomorphism, which is therefore an automorphism if S was invertible (as may be required).

If $S^2 = \text{Id}$, then the Hopf algebra is said to be **involutive** (and the underlying algebra with involution is a $*$ -algebra). If H is finite-dimensional semisimple over a field of characteristic zero, commutative, or cocommutative, then it is involutive.

If a bialgebra B admits an antipode S , then S is unique ("a bialgebra admits at most 1 Hopf algebra structure").

The antipode is an analog to the inversion map on a group that sends g to g^{-1} .^[1]

Examples

Group algebra. Suppose G is a group. The group algebra KG is a unital associative algebra over K . It turns into a Hopf algebra if we define

- $\Delta : KG \rightarrow KG \otimes KG$ by $\Delta(g) = g \otimes g$ for all g in G
- $\varepsilon : KG \rightarrow K$ by $\varepsilon(g) = 1$ for all g in G
- $S : KG \rightarrow KG$ by $S(g) = g^{-1}$ for all g in G .

Functions on a finite group. Suppose now that G is a *finite* group. Then the set K^G of all functions from G to K with pointwise addition and multiplication is a unital associative algebra over K , and $K^G \otimes K^G$ is naturally isomorphic to $K^{G \times G}$ (for G infinite, $K^G \otimes K^G$ is a proper subset of $K^{G \times G}$). The set K^G becomes a Hopf algebra if we define

- $\Delta : K^G \rightarrow K^{G \times G}$ by $\Delta(f)(x,y) = f(xy)$ for all f in K^G and all x,y in G
- $\varepsilon : K^G \rightarrow K$ by $\varepsilon(f) = f(e)$ for every f in K^G [here e is the identity element of G]
- $S : K^G \rightarrow K^G$ by $S(f)(x) = f(x^{-1})$ for all f in K^G and all x in G .

Note that functions on a finite group can be identified with the group ring, though these are more naturally thought of as dual – the group ring consists of *finite* sums of elements, and thus pairs with functions on the group by evaluating the function on the summed elements.

Regular functions on an algebraic group. Generalizing the previous example, we can use the same formulas to show that for a given algebraic group G over K , the set of all regular functions on G forms a Hopf algebra.

Universal enveloping algebra. Suppose g is a Lie algebra over the field K and U is its universal enveloping algebra. U becomes a Hopf algebra if we define

- $\Delta : U \rightarrow U \otimes U$ by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for every x in g (this rule is compatible with commutators and can therefore be uniquely extended to all of U).
- $\varepsilon : U \rightarrow K$ by $\varepsilon(x) = 0$ for all x in g (again, extended to U)
- $S : U \rightarrow U$ by $S(x) = -x$ for all x in g .

Cohomology of Lie groups

The cohomology algebra of a Lie group is a Hopf algebra: the multiplication is provided by the cup-product, and the comultiplication

$$H^*(G) \rightarrow H^*(G \times G) \cong H^*(G) \otimes H^*(G)$$

by the group multiplication $G \times G \rightarrow G$. This observation was actually a source of the notion of Hopf algebra. Using this structure, Hopf proved a structure theorem for the cohomology algebra of Lie groups.

Theorem (Hopf)^[2] Let A be a finite-dimensional, graded commutative, graded cocommutative Hopf algebra over a field of characteristic 0. Then A (as an algebra) is a free exterior algebra with generators of odd degree.

Quantum groups and non-commutative geometry

All examples above are either commutative (i.e. the multiplication is commutative) or co-commutative (i.e. $\Delta = T \circ \Delta$ where $T: H \otimes H \rightarrow H \otimes H$ is defined by $T(x \otimes y) = y \otimes x$). Other interesting Hopf algebras are certain "deformations" or "quantizations" of those from example 3 which are neither commutative nor co-commutative. These Hopf algebras are often called *quantum groups*, a term that is so far only loosely defined. They are important in noncommutative geometry, the idea being the following: a standard algebraic group is well described by its standard Hopf algebra of regular functions; we can then think of the deformed version of this Hopf algebra as describing a certain "non-standard" or "quantized" algebraic group (which is not an algebraic group at all). While there does not seem to be a direct way to define or manipulate these non-standard objects, one can still work with their Hopf algebras, and indeed one *identifies* them with their Hopf algebras. Hence the name "quantum group".

Related concepts

Graded Hopf algebras are often used in algebraic topology: they are the natural algebraic structure on the direct sum of all homology or cohomology groups of an H-space.

Locally compact quantum groups generalize Hopf algebras and carry a topology. The algebra of all continuous functions on a Lie group is a locally compact quantum group.

Quasi-Hopf algebras are also generalizations of Hopf algebras, where coassociativity only holds up to a twist.

Analogy with groups

Groups can be axiomatized by the same diagrams (equivalently, operations) as a Hopf algebra, where G is taken to be a set instead of a module. In this case:

- the field K is replaced by the 1-point set
- there is a natural counit (map to 1 point)
- there is a natural comultiplication (the diagonal map)
- the unit is the identity element of the group
- the multiplication is the multiplication in the group
- the antipode is the inverse

In this philosophy, a group can be thought of as a Hopf algebra over the "field with one element".^[3]

See also

- Quasitriangular Hopf algebra
- Algebra/set analogy
- Representation theory of Hopf algebras
- Ribbon Hopf algebra
- Superalgebra
- Supergroup
- Anyonic Lie algebra

Notes

- [1] Quantum groups lecture notes (http://www.mathematik.uni-muenchen.de/~pareigis/Vorlesungen/QuantGrp/ln2_1.pdf)
- [2] Hopf, 1941.
- [3] Group = Hopf algebra « Secret Blogging Seminar (<http://sbseminar.wordpress.com/2007/10/07/group-hopf-algebra/>), Group objects and Hopf algebras (<http://www.youtube.com/watch?v=p3kkm5dYH-w>), video of Simon Willerton.

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Quasitriangular Hopf algebra

In mathematics, a Hopf algebra, H , is **quasitriangular** if there exists an invertible element, R , of $H \otimes H$ such that

- $R \Delta(x) = (T \circ \Delta)(x) R$ for all $x \in H$, where Δ is the coproduct on H , and the linear map $T : H \otimes H \rightarrow H \otimes H$ is given by $T(x \otimes y) = y \otimes x$,
- $(\Delta \otimes 1)(R) = R_{13} R_{23}$,
- $(1 \otimes \Delta)(R) = R_{13} R_{12}$,

where $R_{12} = \phi_{12}(R)$, $R_{13} = \phi_{13}(R)$, and $R_{23} = \phi_{23}(R)$, where $\phi_{12} : H \otimes H \rightarrow H \otimes H \otimes H$, $\phi_{13} : H \otimes H \rightarrow H \otimes H \otimes H$, and $\phi_{23} : H \otimes H \rightarrow H \otimes H \otimes H$, are algebra morphisms determined by

$$\phi_{12}(a \otimes b) = a \otimes b \otimes 1,$$

$$\phi_{13}(a \otimes b) = a \otimes 1 \otimes b,$$

$$\phi_{23}(a \otimes b) = 1 \otimes a \otimes b.$$

R is called the R -matrix.

As a consequence of the properties of quasitriangularity, the R -matrix, R , is a solution of the Yang-Baxter equation (and so a module V of H can be used to determine quasi-invariants of braids, knots and links). Also as a consequence of the properties of quasitriangularity, $(\epsilon \otimes 1)R = (1 \otimes \epsilon)R = 1 \in H$; moreover $R^{-1} = (S \otimes 1)(R)$, $R = (1 \otimes S)(R^{-1})$, and $(S \otimes S)(R) = R$. One may further show that the antipode S must be a linear isomorphism, and thus S^2 is an automorphism. In fact, S^2 is given by conjugating by an invertible element: $S(x) = u x u^{-1}$ where $u = m(S \otimes 1)R^{21}$ (cf. Ribbon Hopf algebras).

It is possible to construct a quasitriangular Hopf algebra from a Hopf algebra and its dual, using the Drinfel'd quantum double construction.

Twisting

The property of being a quasi-triangular Hopf algebra is preserved by twisting via an invertible element $F = \sum_i f^i \otimes f_i \in \mathcal{A} \otimes \mathcal{A}$ such that $(\varepsilon \otimes id)F = (id \otimes \varepsilon)F = 1$ and satisfying the cocycle condition

$$(F \otimes 1) \circ (\Delta \otimes id)F = (1 \otimes F) \circ (id \otimes \Delta)F$$

Furthermore, $u = \sum_i f^i S(f_i)$ is invertible and the twisted antipode is given by $S'(a) = uS(a)u^{-1}$, with the twisted comultiplication, R-matrix and co-unit change according to those defined for the quasi-triangular Quasi-Hopf algebra. Such a twist is known as an admissible (or Drinfel'd) twist.

See also

- Quasi-triangular Quasi-Hopf algebra
- Ribbon Hopf algebra

Quantum double

Quantum double or **Drinfeld double** or **Drinfel'd double** or **Drienfeld quantum double** or **double** is a Hopf algebra construction and an example of a quasitriangular Hopf algebra introduced by Vladimir Drinfel'd. The Drinfel'd double is an important and complex construction that can be applied to a Hopf algebra to obtain a larger, quasi-triangular, Hopf algebra. However, the quantum double as a new quasi-triangular Hopf algebra is different from the standard quantum double that is q-deformations for Lie algebras or Lie superalgebras.^[1]

The duality in the "QUE-dual"^[2] of Drinfeld is very specific, and defines the quantum double in a particular context. But the quantum double can be defined more generally, for example in terms of the double cross-product of Majid. It is also of interest in the theory of monoidal categories. But, because of the problem of dualizing an infinite-dimensional Hopf algebra, it is supposed that the constructions are done for finite-dimensional Hopf algebras (group algebras of finite groups for example).^[3]

The quantum double (QD) can also be described as an example of a mathematical construction of ribbon Hopf algebra, a particular form of a Hopf algebra, which has a lot of mathematical structure and every piece of this structure corresponds to some physical property in gauge theory. A way to describe the symmetry in a theory featuring interactions of both fundamental charges and topological defects is to extend the residual gauge group to its quantum double.

Definition

A quantum double $D(A)$ definition can be seen in Definition 1.17 (page 57) of this link [4]

QD construction

Vladimir Gershonovich Drinfel'd in 1987 introduced the construction of the quantum double leading later to quantum supergroups and quantized enveloping superalgebras. Drinfel'd showed that any Hopf algebra H can be extended to a larger Hopf algebra $D(H)$ that is quasi-triangular. The kind of Hopf algebra that arises in the study of the Yang-Baxter equation are the quasi-triangular Hopf algebras. ^[5] One difficulty in the construction of $D(H)$ is the large increase in the space required to encode $D(H)$ as a matrix algebra. (See Hopf project in "See also" below)

The quantum double construction $D(H)$ can be used to derive the particle spectrum as well as the braid statistics for any two-dimensional model with a discrete and finite symmetry group H . This can be verified by considering a topological quantum error correcting code (topological quantum computer/toric code^[6]), which can be understood as a $\mathbb{Z}_2$ gauge theory^[7] on a lattice. The real advantage of the quantum double formalism lies in applying it to non-abelian models (when H is a non-abelian group). This is how quantum groups can be used to provide a unified language for describing anyon model. The language is closely related to topological quantum field theories. ^[8]

Exotic quantum double

The exotic quantum double (EQD) and its universal R-matrix for quantum Yang-Baxter (QYBE) equation are constructed in terms of Drinfeld's quantum double theory. But this new quasi-triangular Hopf algebra, is distinct from standard QDs that are the q -deformations for Lie algebras. By studying its representation theory, many-parameter representations of the exotic quantum double are obtained. The multi-parameter R-matrices for the quantum Yang-Baxter equation can result from the universal R-matrix of this exotic quantum double and these representations.

The QYBE has become a focus of the attention from both theoretical physicists and mathematicians. This is because the QYBE is a key to the complete integrability of many physical systems appearing in the quantum inverse scattering methods, the exactly-solvable models in statistical mechanics and low-dimensional quantum field theory.

In solving the QYBE in a general way and classifying its solutions (R-matrices) algebraically, a mathematical structure (the quasi-triangular Hopf algebra, loosely called quantum group) are found in connection with the QYBE. Among these developments, the Drinfeld's quantum double theory provides a general construction to systematically obtain solutions of the QYBE in terms of the QDs, which are usually the q -deformations of certain algebras, and their representations.

The standard QDs are only the quantum (universal enveloping) algebras and superalgebras and their parameterizations. They are the q -deformations of the universal algebras and possess a standard quantum double structure that both the subalgebras A and B are non-commutative and non-cocommutative. This symmetric structure reflects the duality of A and B . These standard quantum doubles approach the usual universal enveloping algebras in the classical limit as $q \rightarrow 1$. EQDs possess asymmetric dual structure that one of the subalgebras A and B is commutative but non-cocommutative and another cocommutative

but non-commutative.

See also

- Hopf project (Virginia Tech) – quantum double encoding difficulties^[9]
- Michio Jimbo
- Braided Hopf algebra

Further reading

- Foundations of Quantum Group Theory By Shahn Majid ^[10]

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External links

- Quantum Double - Planetmath Article Browser (<http://myyn.org/m/article/quantum-double/>)

Grassman algebra

1. REDIRECT Exterior algebra

Special unitary group

In mathematics, the **special unitary group** of degree n , denoted $SU(n)$, is the group of $n \times n$ unitary matrices with determinant 1. The group operation is that of matrix multiplication. The special unitary group is a subgroup of the unitary group $U(n)$, consisting of all $n \times n$ unitary matrices, which is itself a subgroup of the general linear group $GL(n, \mathbf{C})$.

The $SU(n)$ groups find wide application in the Standard Model of particle physics, especially $SU(2)$ in the electroweak interaction and $SU(3)$ in QCD.

The simplest case, $SU(1)$, is the trivial group, having only a single element. The group $SU(2)$ is isomorphic to the group of quaternions of absolute value 1, and is thus diffeomorphic to the 3-sphere. Since unit quaternions can be used to represent rotations in 3-dimensional space (up to sign), we have a surjective homomorphism from $SU(2)$ to the rotation group $SO(3)$ whose kernel is $\{+I, -I\}$.

Properties

The special unitary group $SU(n)$ is a real matrix Lie group of dimension $n^2 - 1$. Topologically, it is compact and simply connected. Algebraically, it is a simple Lie group (meaning its Lie algebra is simple; see below). The center of $SU(n)$ is isomorphic to the cyclic group $\mathbf{Z}_n$. Its outer automorphism group, for $n \geq 3$, is $\mathbf{Z}_2$, while the outer automorphism group of $SU(2)$ is the trivial group.

The $SU(n)$ algebra is generated by n^2 operators, which satisfy the commutator relationship (for $i, j, k, l = 1, 2, \dots, n$)

$$[\hat{O}_{ij}, \hat{O}_{kl}] = \delta_{jk} \hat{O}_{il} - \delta_{il} \hat{O}_{kj}$$

Additionally, the operator

$$\hat{N} = \sum_{i=1}^n \hat{O}_{ii}$$

satisfies

$$[\hat{N}, \hat{O}_{ij}] = 0$$

which implies that the number of *independent* generators of $SU(n)$ is $n^2 - 1$.^[1]

Generators

In general the infinitesimal generators of $SU(n)$, T , are represented as traceless Hermitian matrices. I.e:

- $\text{tr}(T_a) = 0$

and

- $T_a = T_a^\dagger$

Fundamental representation

In the defining or fundamental representation the generators are represented by $n \times n$ matrices where:

- $T_a T_b = \frac{1}{2n} \delta_{ab} I_n + \frac{1}{2} \sum_{c=1}^{n^2-1} (i f_{abc} + d_{abc}) T_c$

where the f are the *structure constants* and are antisymmetric in all indices, whilst the d are symmetric in all indices.

As a consequence:

- $[T_a, T_b]_+ = \frac{1}{n} \delta_{ab} + \sum_{c=1}^{n^2-1} d_{abc} T_c$
- $[T_a, T_b]_- = i \sum_{c=1}^{n^2-1} f_{abc} T_c$

We also have

- $\sum_{c,e=1}^{n^2-1} d_{ace} d_{bce} = \frac{n^2 - 4}{n} \delta_{ab}$

as a normalization convention.

Adjoint representation

In the adjoint representation the generators are represented by $(n^2 - 1) \times (n^2 - 1)$ matrices whose elements are defined by the structure constants:

- $(T_a)_{jk} = -i f_{ajk}$

$SU_2(\mathbb{C})$ and $\mathfrak{su}_2(\mathbb{C})$

A general matrix element of $SU_2(\mathbb{C})$ takes the form

$$U = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$. We can consider the following map $\varphi : \mathbb{C}^2 \rightarrow M(2, \mathbb{C})$, (where $M(2, \mathbb{C})$ denotes the set of 2 by 2 complex matrices), defined in the obvious way by

$$\varphi(\alpha, \beta) = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}.$$

By considering $\mathbb{C}^2$ diffeomorphic to $\mathbb{R}^4$ and $M(2, \mathbb{C})$ diffeomorphic to $\mathbb{R}^8$ we can see that φ is an injective real linear map and hence an embedding. Now considering the restriction of φ to the 3-sphere, denoted S^3 , we can see that this is an embedding of the 3-sphere onto a compact submanifold of $M(2, \mathbb{C})$. However it is also clear that $\varphi(S^3) = SU_2(\mathbb{C})$, which as

a manifold is diffeomorphic to $SU_2(\mathbb{C})$, making $SU_2(\mathbb{C})$ a compact, connected Lie group.

Now considering the Lie algebra $\mathfrak{su}_2(\mathbb{C})$, a general element takes the form

$$U' = \begin{pmatrix} ix & -\bar{\beta} \\ \beta & -ix \end{pmatrix}$$

where $x \in \mathbb{R}$ and $\beta \in \mathbb{C}$. It is easily verified that matrices of this form have trace zero and are antihermitian. The Lie algebra is then generated by the following matrices

$$u_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad u_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

which are easily seen to have the form of the general element specified above. These satisfy the relations $u_3 u_2 = -u_2 u_3 = u_1$ and $u_2 u_1 = -u_1 u_2 = u_3$. The commutator bracket is therefore specified by

$$[u_1, u_3] = 2u_2, \quad [u_2, u_1] = 2u_3, \quad [u_3, u_2] = 2u_1.$$

The above generators are related to the Pauli matrices by $u_1 = i\sigma_1$, $u_2 = -i\sigma_2$ and $u_3 = i\sigma_3$.

SU(3)

The generators of $SU(3)$, T , in the defining representation, are:

$$T_a = \frac{\lambda_a}{2}.$$

where λ , the Gell-Mann matrices, are the $SU(3)$ analog of the Pauli matrices for $SU(2)$:

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned}$$

Note that they are all traceless Hermitian matrices as required.

These obey the relations

$$\bullet \quad [T_a, T_b] = i \sum_{c=1}^8 f_{abc} T_c$$

where the f are the *structure constants*, as previously defined, and have values given by

$$\begin{aligned} f^{123} &= 1 \\ f^{147} &= -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2} \\ f^{458} &= f^{678} = \frac{\sqrt{3}}{2} \end{aligned}$$

The d take the values:

$$d^{118} = d^{228} = d^{338} = -d^{888} = \frac{1}{\sqrt{3}}$$

$$d^{448} = d^{558} = d^{668} = d^{778} = -\frac{1}{2\sqrt{3}}$$

$$d^{146} = d^{157} = -d^{247} = d^{256} = d^{344} = d^{355} = -d^{366} = -d^{377} = \frac{1}{2}$$

Lie algebra

The Lie algebra corresponding to $SU(n)$ is denoted by $\mathfrak{su}(n)$. Its standard mathematical representation consists of the traceless antihermitian $n \times n$ complex matrices, with the regular commutator as Lie bracket. A factor i is often inserted by particle physicists, so that all matrices become hermitian. This is simply a different, more convenient, representation of the same real Lie algebra. Note that $\mathfrak{su}(n)$ is a Lie algebra over $\mathbb{R}$.

For example, the following antihermitian matrices used in quantum mechanics form a basis for $\mathfrak{su}(2)$ over $\mathbb{R}$:

$$i\sigma_x = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$i\sigma_y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$i\sigma_z = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

(where i is the imaginary unit.)

This representation is often used in quantum mechanics (see *Pauli matrices* and *Gell-Mann matrices*), to represent the spin of fundamental particles such as electrons. They also serve as unit vectors for the description of our 3 spatial dimensions in quantum relativity.

Note that the product of any two different generators is another generator, and that the generators anticommute. Together with the identity matrix (times i),

$$iI_2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

these are also generators of the Lie algebra $\mathfrak{su}(2)$.

Here it depends of course on the problem whether one works finally, as in non-relativistic quantum mechanics, with 2-spinors; or, as in the relativistic Dirac theory, one needs an extension to 4-spinors; or in mathematics even to Clifford algebras.

Note: make clearer the fact that under matrix multiplication (which is anticommutative in this case), we generate the Clifford algebra Cl_3 , whereas you generate the Lie algebra $\mathfrak{su}(2)$ with commutator brackets instead.

Back to general $SU(n)$:

If we choose an (arbitrary) particular basis, then the subspace of traceless diagonal $n \times n$ matrices with imaginary entries forms an $n - 1$ dimensional Cartan subalgebra.

Complexify the Lie algebra, so that any traceless $n \times n$ matrix is now allowed. The weight eigenvectors are the Cartan subalgebra itself and the matrices with only one nonzero entry which is off diagonal. Even though the Cartan subalgebra has only $n - 1$ dimensional, to simplify calculations, it is often convenient to introduce an auxiliary element, the unit matrix which commutes with everything else (which should not be thought of as an element of the Lie algebra!) for the purpose of computing weights and that only. So, we have a basis where the i th basis vector is the matrix with 1 on the i th diagonal entry and zero

elsewhere. Weights would then be given by n coordinates and the sum over all n coordinates has to be zero (because the unit matrix is only auxiliary).

So, $SU(n)$ has a rank of $n - 1$ and its Dynkin diagram is given by A_{n-1} , a chain of $n - 1$ vertices.

Its root system consists of $n(n - 1)$ roots spanning a $n - 1$ Euclidean space. Here, we use n redundant coordinates instead of $n - 1$ to emphasize the symmetries of the root system (the n coordinates have to add up to zero). In other words, we are embedding this $n - 1$ dimensional vector space in an n -dimensional one. Then, the roots consists of all the $n(n - 1)$ permutations of $(1, -1, 0, \dots, 0)$. The construction given two paragraphs ago explains why. A choice of simple roots is

$$\begin{aligned} &(1, -1, 0, \dots, 0), \\ &(0, 1, -1, \dots, 0), \\ &\dots, \\ &(0, 0, 0, \dots, 1, -1). \end{aligned}$$

Its Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix}.$$

Its Weyl group or Coxeter group is the symmetric group S_n , the symmetry group of the $(n - 1)$ -simplex.

Generalized special unitary group

For a field F , the **generalized special unitary group over F** , $SU(p, q; F)$, is the group of all linear transformations of determinant 1 of a vector space of rank $n = p + q$ over F which leave invariant a nondegenerate, hermitian form of signature (p, q) . This group is often referred to as the **special unitary group of signature $p \ q$ over F** . The field F can be replaced by a commutative ring, in which case the vector space is replaced by a free module.

Specifically, fix a hermitian matrix A of signature $p \ q$ in $GL(n, R)$, then all

$$M \in SU(p, q, R)$$

satisfy

$$M^* A M = A$$

$$\det M = 1.$$

Often one will see the notation $SU_{p,q}$ without reference to a ring or field, in this case the ring or field being referred to is $\mathbf{C}$ and this gives one of the classical Lie groups. The standard choice for A when $F = \mathbf{C}$ is

$$A = \begin{bmatrix} 0 & 0 & i \\ 0 & I_{n-2} & 0 \\ -i & 0 & 0 \end{bmatrix}.$$

However there may be better choices for A for certain dimensions which exhibit more behaviour under restriction to subrings of $\mathbf{C}$.

Example

A very important example of this type of group is the Picard modular group $SU(2,1;\mathbf{Z}[i])$ which acts (projectively) on complex hyperbolic space of degree two, in the same way that $SL(2,\mathbf{Z})$ acts (projectively) on real hyperbolic space of dimension two. In 2003 Gábor Francsics and Peter Lax computed a fundamental domain for the action of this group on HC^2 , see [2]. Another example is $SU(1,1;\mathbf{C})$ which is isomorphic to $SL(2,\mathbf{R})$.

Important subgroups

In physics the special unitary group is used to represent bosonic symmetries. In theories of symmetry breaking it is important to be able to find the subgroups of the special unitary group. Subgroups of $SU(n)$ that are important in GUT physics are, for $p>1$, $n-p>1$:

$$SU(n) \supset SU(p) \times SU(n-p) \times U(1).$$

For completeness there are also the orthogonal and symplectic subgroups:

$$SU(n) \supset O(n)$$

$$SU(2n) \supset USp(2n).$$

Since the rank of $SU(n)$ is $n-1$ and of $U(1)$ is 1, a useful check is that the sum of the ranks of the subgroups is less than or equal to the rank of the original group. $SU(n)$ is a subgroup of various other lie groups:

$$SO(2n) \supset SU(n)$$

$$USp(2n) \supset SU(n)$$

$$Spin(4) = SU(2) \times SU(2) \text{ (see Spin group)}$$

$$E_6 \supset SU(6)$$

$$E_7 \supset SU(8)$$

$$G_2 \supset SU(3) \text{ (see Simple Lie groups for } E_6, E_7, \text{ and } G_2).$$

There are also the identities **$SU(4)=Spin(6)$** , **$SU(2)=Spin(3)=USp(2)$** and **$U(1)=Spin(2)=SO(2)$** .

One should finally mention that $SU(2)$ is the double covering group of $SO(3)$, a relation that plays an important role in the theory of rotations of 2-spinors in non-relativistic quantum mechanics.

See also

- Representation theory of $SU(2)$
- Projective special unitary group, $PSU(n)$

Notes

[1] R.R. Puri, *Mathematical Methods of Quantum Optics*, Springer, 2001.

[2] <http://www.esi.ac.at/Preprint-shadows/esi1273.html>

References

- Halzen, Francis; Martin, Alan (1984). *Quarks & Leptons: An Introductory Course in Modern Particle Physics*. John Wiley & Sons. ISBN 0-471-88741-2.
- Maximal Subgroups of Compact Lie Groups (<http://arxiv.org/abs/math/0605784v1>)

External links

- Physics 558 - Lecture 1, Winter 2003 (http://courses.washington.edu/phys55x/Physics558_lec1_03.htm)

Gauge theory

In physics, **gauge theory** is a field theory in which the Lagrangian is invariant under a certain continuous group of transformations.

The transformations (called *gauge transformations*) form a Lie group which is referred to as the *symmetry group* or the *gauge group* of the theory. Associated with a Lie group is a Lie Algebra of group generators. For each group generator there is a corresponding vector field called *gauge field*. Gauge fields are included in the Lagrangian to ensure its invariance under the group transformations (called *gauge invariance*). When such a theory is quantized, the quanta of the gauge fields are called *gauge bosons*.

If the symmetry group is non-commutative, the gauge theory is referred to as non-abelian or Yang-Mills theory.

Quantum electrodynamics is an abelian gauge theory with the symmetry group $U(1)$ and has one gauge field, the electromagnetic field, with the photon being the gauge boson.

The standard model is a non-abelian gauge theory with the symmetry group $U(1) \times SU(2) \times SU(3)$ and has a total of twelve gauge bosons: the photon, three weak bosons, Z^0 , W^+ and W^- ; and eight gluons.

Description

Global and local symmetries

In physics, the mathematical description of any physical situation usually contains excess degrees of freedom; the same physical situation is equally well described by many equivalent mathematical configurations. For instance, in Newtonian dynamics, if two configurations are related by a Galilean transformation—an inertial change of reference frame—they represent the same physical situation. These transformations form a group of "symmetries" of the theory, and a physical situation corresponds not to an individual mathematical configuration but to a class of configurations related to one another by this symmetry group. This idea can be generalized to include local as well as global symmetries, analogous to much more abstract "changes of coordinates" in a situation where there is no preferred "inertial" coordinate system that covers the entire physical system. A **gauge theory** is a mathematical model that has symmetries of this kind, together with a set of techniques for making physical predictions consistent with the symmetries of the model.

Example of global symmetry

When a quantity occurring in the mathematical configuration is not just a number but has some geometrical significance, such as a velocity or an axis of rotation, its representation as numbers arranged in a vector or matrix is also changed by a coordinate transformation. For instance, if one description of a pattern of fluid flow states that the fluid velocity in the neighborhood of $(x=1, y=0)$ is 1 m/s in the positive x direction, then a description of the

same situation in which the coordinate system has been rotated clockwise by 90 degrees will state that the fluid velocity in the neighborhood of $(x=0, y=1)$ is 1 m/s in the positive y direction. The coordinate transformation has affected both the coordinate system used to identify the *location* of the measurement and the basis in which its *value* is expressed. As long as this transformation is performed globally (affecting the coordinate basis in the same way at every point), the effect on values that represent the *rate of change* of some quantity along some path in space and time as it passes through point P is the same as the effect on values that are truly local to P .

Use of fiber bundles to describe local symmetries

In order to adequately describe physical situations in more complex theories, it is often necessary to introduce a "coordinate basis" for some of the objects of the theory that do not have this simple relationship to the coordinates used to label points in space and time. (In mathematical terms, the theory involves a fiber bundle in which the fiber at each point of the base space consists of possible coordinate bases for use when describing the values of objects at that point.) In order to spell out a mathematical configuration, one must choose a particular coordinate basis at each point (a *local section* of the fiber bundle) and express the values of the objects of the theory (usually "fields" in the physicist's sense) using this basis. Two such mathematical configurations are equivalent (describe the same physical situation) if they are related by a transformation of this abstract coordinate basis (a change of local section, or *gauge transformation*).

In most gauge theories, the set of possible transformations of the abstract gauge basis at an individual point in space and time is a finite-dimensional Lie group. The simplest such group is $U(1)$, which appears in the modern formulation of quantum electrodynamics (QED) via its use of complex numbers. QED is generally regarded as the first, and simplest, physical gauge theory. The set of possible gauge transformations of the entire configuration of a given gauge theory also forms a group, the *gauge group* of the theory. An element of the gauge group can be parameterized by a smoothly varying function from the points of spacetime to the (finite-dimensional) Lie group, whose value at each point represents the action of the gauge transformation on the fiber over that point.

A gauge transformation with constant parameter at every point in space and time is analogous to a rigid rotation of the geometric coordinate system; it represents a global symmetry of the gauge representation. As in the case of a rigid rotation, this gauge transformation affects expressions that represent the rate of change along a path of some gauge-dependent quantity in the same way as those that represent a truly local quantity. A gauge transformation whose parameter is *not* a constant function is referred to as a local symmetry; its effect on expressions that involve a derivative is qualitatively different from that on expressions that don't. (This is analogous to a non-inertial change of reference frame, which can produce a Coriolis effect.)

Gauge fields

The "gauge covariant" version of a gauge theory accounts for this effect by introducing a gauge field (in mathematical language, an Ehresmann connection) and formulating all rates of change in terms of the covariant derivative with respect to this connection. The gauge field becomes an essential part of the description of a mathematical configuration. A configuration in which the gauge field can be eliminated by a gauge transformation has the

property that its field strength (in mathematical language, its curvature) is zero everywhere; a gauge theory is *not* limited to these configurations. In other words, the distinguishing characteristic of a gauge theory is that the gauge field does not merely compensate for a poor choice of coordinate system; there is generally no gauge transformation that makes the gauge field vanish.

When analyzing the dynamics of a gauge theory, the gauge field must be treated as a dynamical variable, similarly to other objects in the description of a physical situation. In addition to its interaction with other objects via the covariant derivative, the gauge field typically contributes energy in the form of a "self-energy" term. One can obtain the equations for the gauge theory by:

- starting from a naïve ansatz without the gauge field (in which the derivatives appear in a "bare" form);
- listing those global symmetries of the theory that can be characterized by a continuous parameter (generally an abstract equivalent of a rotation angle);
- computing the correction terms that result from allowing the symmetry parameter to vary from place to place; and
- reinterpreting these correction terms as couplings to one or more gauge fields, and giving these fields appropriate self-energy terms and dynamical behavior.

This is the sense in which a gauge theory "extends" a global symmetry to a local symmetry, and closely resembles the historical development of the gauge theory of gravity known as **general relativity**.

Physical experiments

Gauge theories are used to model the results of physical experiments, essentially by:

- limiting the universe of possible configurations to those consistent with the information used to set up the experiment, and then
- computing the probability distribution of the possible outcomes that the experiment is designed to measure.

The mathematical descriptions of the "setup information" and the "possible measurement outcomes" (loosely speaking, the "boundary conditions" of the experiment) are generally not expressible without reference to a particular coordinate system, including a choice of gauge. (If nothing else, one assumes that the experiment has been adequately isolated from "external" influence, which is itself a gauge-dependent statement.) Mishandling gauge dependence in boundary conditions is a frequent source of anomalies in gauge theory calculations, and gauge theories can be broadly classified by their approaches to anomaly avoidance.

Continuum theories

The two gauge theories mentioned above (continuum electrodynamics and general relativity) are examples of continuum field theories. The techniques of calculation in a continuum theory implicitly assume that:

- given a completely fixed choice of gauge, the boundary conditions of an individual configuration can in principle be completely described;
 - given a completely fixed gauge and a complete set of boundary conditions, the principle of least action determines a unique mathematical configuration (and therefore a unique physical situation) consistent with these bounds;
-

- the likelihood of possible measurement outcomes can be determined by:
 - establishing a probability distribution over all physical situations determined by boundary conditions that are consistent with the setup information,
 - establishing a probability distribution of measurement outcomes for each possible physical situation, and
 - convolving these two probability distributions to get a distribution of possible measurement outcomes consistent with the setup information; and
- fixing the gauge introduces no anomalies in the calculation, due either to gauge dependence in describing partial information about boundary conditions or to incompleteness of the theory.

These assumptions are close enough to valid, across a wide range of energy scales and experimental conditions, to allow these theories to make accurate predictions about almost all of the phenomena encountered in daily life, from light, heat, and electricity to eclipses and spaceflight. They fail only at the smallest and largest scales (due to omissions in the theories themselves) and when the mathematical techniques themselves break down (most notably in the case of turbulence and other chaotic phenomena).

Quantum field theories

Other than these "classical" continuum field theories, the most widely known gauge theories are quantum field theories, including quantum electrodynamics and the Standard Model of elementary particle physics. The starting point of a quantum field theory is much like that of its continuum analog: a gauge-covariant action integral which characterizes "allowable" physical situations according to the principle of least action. However, continuum and quantum theories differ significantly in how they handle the excess degrees of freedom represented by gauge transformations. Continuum theories, and most pedagogical treatments of the simplest quantum field theories, use a gauge fixing prescription to reduce the orbit of mathematical configurations that represent a given physical situation to a smaller orbit related by a smaller gauge group (the global symmetry group, or perhaps even the trivial group).

More sophisticated quantum field theories, in particular those which involve a non-abelian gauge group, break the gauge symmetry within the techniques of perturbation theory by introducing additional fields (the Faddeev-Popov ghosts) and counterterms motivated by anomaly cancellation, in an approach known as BRST quantization. While these concerns are in one sense highly technical, they are also closely related to the nature of measurement, the limits on knowledge of a physical situation, and the interactions between incompletely specified experimental conditions and incompletely understood physical theory. The mathematical techniques that have been developed in order to make gauge theories tractable have found many other applications, from solid-state physics and crystallography to low-dimensional topology.

History

The earliest field theory having a gauge symmetry was Maxwell's formulation of electrodynamics in 1864. The importance of this symmetry remained unnoticed in the earliest formulations. Similarly unnoticed, Hilbert had derived the Einstein field equations by postulating the invariance of the action under a general coordinate transformation. Later Hermann Weyl, in an attempt to unify general relativity and electromagnetism, conjectured

(incorrectly, as it turned out) that *Eichinvarianz* or invariance under the change of scale (or "gauge") might also be a local symmetry of general relativity. After the development of quantum mechanics, Weyl, Vladimir Fock and Fritz London modified gauge by replacing the scale factor with a complex quantity and turned the scale transformation into a change of phase—a $U(1)$ gauge symmetry). This explained the electromagnetic field effect on the wave function of a charged quantum mechanical particle. This was the first widely recognised gauge theory, popularised by Pauli in the 1940s.^[1]

In 1954, attempting to resolve some of the great confusion in elementary particle physics, Chen Ning Yang and Robert Mills introduced **non-abelian gauge theories** as models to understand the strong interaction holding together nucleons in atomic nuclei. (Ronald Shaw, working under Abdus Salam, independently introduced the same notion in his doctoral thesis.) Generalizing the gauge invariance of electromagnetism, they attempted to construct a theory based on the action of the (non-abelian) $SU(2)$ symmetry group on the isospin doublet of protons and neutrons. This is similar to the action of the $U(1)$ group on the spinor fields of quantum electrodynamics. In particle physics the emphasis was on using **quantized gauge theories**.

This idea later found application in the quantum field theory of the weak force, and its unification with electromagnetism in the electroweak theory. Gauge theories became even more attractive when it was realized that non-abelian gauge theories reproduced a feature called asymptotic freedom. Asymptotic freedom was believed to be an important characteristic of strong interactions. This motivated searching for a strong force gauge theory. This theory, now known as quantum chromodynamics, is a gauge theory with the action of the $SU(3)$ group on the color triplet of quarks. The Standard Model unifies the description of electromagnetism, weak interactions and strong interactions in the language of gauge theory.

In the 1970s, Sir Michael Atiyah began studying the mathematics of solutions to the **classical Yang-Mills equations**. In 1983, Atiyah's student Simon Donaldson built on this work to show that the differentiable classification of smooth 4-manifolds is very different from their classification up to homeomorphism. Michael Freedman used Donaldson's work to exhibit exotic $\mathbf{R}^4$ s, that is, exotic differentiable structures on Euclidean 4-dimensional space. This led to an increasing interest in gauge theory for its own sake, independent of its successes in fundamental physics. In 1994, Edward Witten and Nathan Seiberg invented gauge-theoretic techniques based on supersymmetry which enabled the calculation of certain topological invariants. These contributions to mathematics from gauge theory have led to a renewed interest in this area.

See Pickering for more about the history of gauge and quantum field theories.^[2]

Explanation

Many powerful theories in physics are described by Lagrangians which are invariant under certain symmetry transformation groups. When they are invariant under a transformation identically performed at *every* point in the space in which the physical processes occur, they are said to have a global symmetry. The requirement of local symmetry is much more strict than the requirement of global symmetry. In fact, a global symmetry is just a local symmetry whose group's parameters are fixed in space-time. This can be viewed as a generalization of the equivalence principle of general relativity in which each point in spacetime is allowed a choice of local reference (coordinate) frame. As in that situation,

gauge "symmetries" reflect a redundancy in the description of a system. Historically, these ideas were first noticed in the context of classical electromagnetism and later in general relativity. However, the modern importance of gauge symmetries appeared in relativistic quantum mechanics of electrons (see discussions below). Today, gauge theories are useful in condensed matter, nuclear and high energy physics among other subfields.

Sometimes, the term 'gauge symmetry' is used in a more general sense to include any local symmetry, like for example, diffeomorphisms. This sense of the term will *not* be used in this article.

Yang-Mills theories are a particular example of gauge theories with non-abelian symmetry groups specified by the Yang-Mills action (Other gauge theories with a non-abelian gauge symmetry also exist, e.g., the Chern-Simons model).

There is a certain inaccuracy in the way the term symmetry is used in some physics literature, especially in more elementary books about elementary particles and field theory. In (quantum) physics, symmetry is a transformation between *physical* states that preserves the expectation values of all observables O (in particular the Hamiltonian). $S: |\varphi\rangle \rightarrow |\psi\rangle = S|\varphi\rangle$; $|\langle\psi|O|\psi\rangle|^2 = |\langle\varphi|O|\varphi\rangle|^2$. The usual formulation of physics theories uses fields, which sometimes are not physical quantities. Such are the gauge fields (fiber bundle connections for the mathematicians), which provide a redundant but convenient description of the physical degrees of freedom. The gauge (local) "symmetries" are a reflection of this redundancy. The physical quantities are certain equivalence classes of gauge fields. An analogy can be made with the construction of the real numbers. We can use sequences of rational numbers that have the same limit. Of course, each real number is represented by infinitely many such sequences. We can choose a particular well-defined sequence to represent the real number. This corresponds to the procedure of 'gauge fixing' in gauge theories. The fact that gauge fields are not physical degrees of freedom becomes very clear when we try to quantize them. Then we are forced to work in one way or another with the physical quantities by removing the redundancy (the gauge symmetry). Another important illustration of the problem with the gauge "symmetries" is when we have anomalies. By definition these are symmetries which exist in the classical system, but not in its quantum counterpart. Anomalies are quite usual and also an experimental fact — for example, the axial anomaly in the strong interactions (broken symmetries). However, because gauge symmetries are not symmetries, gauge anomalies are not something that just complicates a proposed quantum theory but something that kills it, i.e. there are no gauge "anomalies", because such theories don't exist. This is why having the exact relation between the number of flavours and quark colours in the Standard model is so important — otherwise there is a gauge anomaly and the theory does not exist. For the same reason, string theories are defined in 10 dimensions. Only then do the anomalies cancel.

Importance

The importance of gauge theories for physics stems from the tremendous success of the mathematical formalism in providing a unified framework to describe the quantum field theories of electromagnetism, the weak force and the strong force. This theory, known as the Standard Model, accurately describes experimental predictions regarding three of the four fundamental forces of nature, and is a gauge theory with the gauge group $SU(3) \times SU(2) \times U(1)$. Modern theories like string theory, as well as some formulations of general relativity, are, in one way or another, gauge theories.

A simple gauge symmetry example from electrodynamics

The definition of electrical ground in an electric circuit is an example of a gauge symmetry. When the electric potentials at all points in a circuit are raised by the same amount, the circuit will still operate identically, as the potential differences (voltages) in the circuit are unchanged. A common illustration of this fact is the sight of a small bird perched on a high voltage power line without electrocution, because the bird is insulated from the ground (as long as it doesn't complete the circuit by accidentally touching another wire or some grounded structure).

This is called a global gauge symmetry. The absolute value of the potential is immaterial; what matters to circuit operation is the potential *differences* across the components of the circuit. The definition of the ground point is arbitrary, but once that point is set, then that definition must be followed globally.

In contrast, if some symmetry could be defined arbitrarily from one position to the next, that would be a local gauge symmetry. In fact, the example above (of electromagnetism) is actually a local gauge symmetry where the particular transformation of the potential of the electric field is just the constant one (a global symmetry is a local one but not vice versa); there are an infinite number of transformations of the electromagnetic field's potential A that are not constant but can be made consistent as explained below. (Although the number of transformations is infinite, the number of classes of transformation is finite—just 1 class, the addition of the gradient of a scalar function.)

Classical gauge theory

This section requires some familiarity with classical or quantum field theory, and the use of Lagrangians.

Definitions in this section: gauge group, gauge field, interaction Lagrangian, gauge boson

An example: Scalar $O(n)$ gauge theory

The following illustrates how local gauge invariance can be "motivated" heuristically starting from global symmetry properties, and how it leads to an interaction between fields which were originally non-interacting.

Consider a set of n non-interacting scalar fields, with equal masses m . This system is described by an action which is the sum of the (usual) action for each scalar field φ_i

$$S = \int d^4x \sum_{i=1}^n \left[\frac{1}{2} \partial_\mu \varphi_i \partial^\mu \varphi_i - \frac{1}{2} m^2 \varphi_i^2 \right].$$

The Lagrangian (density) can be compactly written as

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \Phi)^T \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^T \Phi$$

by introducing a vector of fields

$$\Phi = (\varphi_1, \varphi_2, \dots, \varphi_n)^T.$$

The term ∂_μ is Einstein notation for the partial derivative of Φ in each of the four dimensions. It is now transparent that the Lagrangian is invariant under the transformation

$$\Phi \mapsto G\Phi$$

whenever G is a *constant* matrix belonging to the n -by- n orthogonal group $O(n)$. This is the *global* symmetry of this particular Lagrangian, and the symmetry group is often called the

gauge group; the mathematical term is **structure group**, especially in the theory of G-structures. Incidentally, Noether's theorem implies that invariance under this group of transformations leads to the conservation of the *current*

$$J_\mu^a = i\partial_\mu \Phi^T T^a \Phi$$

where the T^a matrices are generators of the $SO(n)$ group. There is one conserved current for every generator.

Now, demanding that this Lagrangian should have *local* $O(n)$ -invariance requires that the G matrices (which were earlier constant) should be allowed to become functions of the space-time coordinates x .

Unfortunately, the G matrices do not "pass through" the derivatives. When $G = G(x)$,

$$\partial_\mu (G\Phi)^T \partial^\mu (G\Phi) \neq \partial_\mu \Phi^T \partial^\mu \Phi.$$

This suggests defining a "derivative" D with the property

$$D_\mu (G(x)\Phi(x)) = G(x)D_\mu \Phi.$$

It can be checked that such a "derivative" (called a covariant derivative) is

$$D_\mu = \partial_\mu + gA_\mu(x)$$

where the **gauge field** $A(x)$ is defined to have the transformation law

$$A_\mu(x) \mapsto G(x)A_\mu(x)G^{-1}(x) + \frac{1}{g}[G(x)(\partial_\mu G^{-1}(x)) - (\partial_\mu G(x))G^{-1}(x)]$$

and g is the coupling constant - a quantity defining the strength of an interaction.

The gauge field is an element of the Lie algebra, and can therefore be expanded as

$$A_\mu(x) = \sum_a A_\mu^a(x) T^a$$

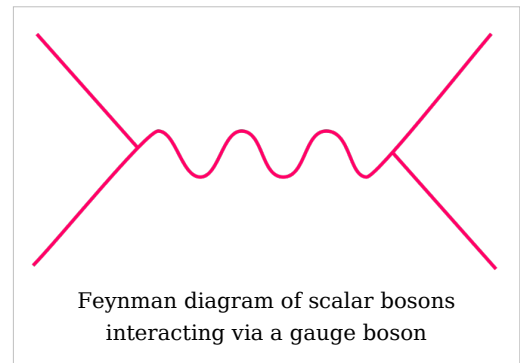
There are therefore as many gauge fields as there are generators of the Lie algebra.

Finally, we now have a *locally gauge invariant* Lagrangian

$$\mathcal{L}_{\text{loc}} = \frac{1}{2}(D_\mu \Phi)^T D^\mu \Phi - \frac{1}{2}m^2 \Phi^T \Phi.$$

Pauli calls *gauge transformation of the first type* to the one applied to fields as Φ , while the compensating transformation in A is said to be a *gauge transformation of the second type*.

The difference between this Lagrangian and the original *globally gauge-invariant* Lagrangian is seen to be the **interaction Lagrangian**



$$\mathcal{L}_{\text{int}} = \frac{g}{2} \Phi^T A_\mu^T \partial^\mu \Phi + \frac{g}{2} (\partial_\mu \Phi)^T A^\mu \Phi + \frac{g^2}{2} (A_\mu \Phi)^T A^\mu \Phi.$$

This term introduces interactions between the n scalar fields just as a consequence of the demand for local gauge invariance. In the quantized version of this classical field theory, the quanta of the gauge field $A(x)$ are called gauge bosons. The interpretation of the

interaction Lagrangian in quantum field theory is of scalar bosons interacting by the exchange of these gauge bosons.

The Yang-Mills Lagrangian for the gauge field

Our picture of classical gauge theory is almost complete except for the fact that to define the covariant derivatives D , one needs to know the value of the gauge field $A(x)$ at all space-time points. Instead of manually specifying the values of this field, it can be given as the solution to a field equation. Further requiring that the Lagrangian which generates this field equation is locally gauge invariant as well, one possible form for the gauge field Lagrangian is (conventionally) written as

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2} \text{Tr}(F^{\mu\nu} F_{\mu\nu})$$

with

$$F_{\mu\nu} = \frac{1}{ig} [D_\mu, D_\nu]$$

and the trace being taken over the vector space of the fields. This is called the **Yang-Mills action**. Other gauge invariant actions also exist (e.g. nonlinear electrodynamics, Born-Infeld action, Chern-Simons model, theta term etc.).

Note that in this Lagrangian there is not a field Φ whose transformation counterweights the one of A . Invariance of this term under gauge transformations is a particular case of *a priori* classical (that is geometrical) symmetry. This symmetry must be restricted in order to perform quantization, the procedure being denominated gauge fixing, but even after restriction, gauge transformations are possible (see Sakurai, *Advanced Quantum Mechanics*, sect 1-4).

The complete Lagrangian for the $O(n)$ gauge theory is now

$$\mathcal{L} = \mathcal{L}_{\text{loc}} + \mathcal{L}_{\text{gf}} = \mathcal{L}_{\text{global}} + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{gf}}$$

A simple example: Electrodynamics

As a simple application of the formalism developed in the previous sections, consider the case of electrodynamics, with only the electron field. The bare-bones action which generates the electron field's Dirac equation is

$$\mathcal{S} = \int \bar{\psi} (i\hbar c \gamma^\mu \partial_\mu - mc^2) \psi \, d^4x.$$

The global symmetry for this system is

$$\psi \mapsto e^{i\theta} \psi.$$

The gauge group here is $U(1)$, just the phase angle of the field, with a constant θ .

"Local"ising this symmetry implies the replacement of θ by $\theta(x)$.

An appropriate covariant derivative is then

$$D_\mu = \partial_\mu - i \frac{e}{\hbar} A_\mu.$$

Identifying the "charge" e with the usual electric charge (this is the origin of the usage of the term in gauge theories), and the gauge field $A(x)$ with the four-vector potential of electromagnetic field results in an interaction Lagrangian

$$\mathcal{L}_{\text{int}} = \frac{e}{\hbar} \bar{\psi}(x) \gamma^\mu \psi(x) A_\mu(x) = J^\mu(x) A_\mu(x).$$

where $J^\mu(x)$ is the usual four vector electric current density. The gauge principle is therefore seen to naturally introduce the so-called *minimal coupling* of the electromagnetic field to the electron field.

Adding a Lagrangian for the gauge field $A_\mu(x)$ in terms of the field strength tensor exactly as in electrodynamics, one obtains the Lagrangian which is used as the starting point in quantum electrodynamics.

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\hbar c \gamma^\mu D_\mu - mc^2)\psi - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}.$$

See also: Dirac equation, Maxwell's equations, Quantum electrodynamics

Mathematical formalism

Gauge theories are usually discussed in the language of differential geometry. Mathematically, a *gauge* is just a choice of a (local) section of some principal bundle. A **gauge transformation** is just a transformation between two such sections.

Although gauge theory is dominated by the study of connections (primarily because it's mainly studied by high-energy physicists), the idea of a connection is not central to gauge theory in general. In fact, a result in general gauge theory shows that affine representations (i.e. affine modules) of the gauge transformations can be classified as sections of a jet bundle satisfying certain properties. There are representations which transform covariantly pointwise (called by physicists gauge transformations of the first kind), representations which transform as a connection form (called by physicists gauge transformations of the second kind) (this is an affine representation) and other more general representations, such as the B field in BF theory. And of course, we can consider more general nonlinear representations (realizations), but that is extremely complicated. But still, nonlinear sigma models transform nonlinearly, so there are applications.

If we have a principal bundle P whose base space is space or spacetime and structure group is a Lie group, then the sections of P form a principal homogeneous space of the group of gauge transformations.

We can define a connection (gauge connection) on this principal bundle, yielding a covariant derivative ∇ in each associated vector bundle. If we choose a local frame (a local basis of sections) then we can represent this covariant derivative by the connection form A , a Lie algebra-valued 1-form which is called the **gauge potential** in physics and which is evidently not an intrinsic but a frame-dependent quantity. From this connection form we can construct the curvature form F , a Lie algebra-valued 2-form which is an intrinsic quantity, by

$$F = dA + A \wedge A$$

where d stands for the exterior derivative and $\wedge$ stands for the wedge product. (A is an element of the vector space spanned by the generators T^a , and so the components of A do not commute with one another. Hence the wedge product $A \wedge A$ does not vanish.)

Infinitesimal gauge transformations form a Lie algebra, which is characterized by a smooth Lie algebra valued scalar, ε . Under such an infinitesimal gauge transformation,

$$\delta_\varepsilon A = [\varepsilon, A] - d\varepsilon$$

where $[\cdot, \cdot]$ is the Lie bracket.

One nice thing is that if $\delta_\varepsilon X = \varepsilon X$, then $\delta_\varepsilon DX = \varepsilon DX$ where D is the covariant derivative

$$DX \stackrel{\text{def}}{=} dX + \mathbf{A}X.$$

Also, $\delta_\varepsilon \mathbf{F} = \varepsilon \mathbf{F}$, which means $\mathbf{F}$ transforms covariantly.

Not all gauge transformations can be generated by infinitesimal gauge transformations in general. An example is when the base manifold is a compact manifold without boundary such that the homotopy class of mappings from that manifold to the Lie group is nontrivial. See instanton for an example.

The *Yang-Mills action* is now given by

$$\frac{1}{4g^2} \int \text{Tr}[*F \wedge F]$$

where $*$ stands for the Hodge dual and the integral is defined as in differential geometry.

A quantity which is **gauge-invariant** i.e. invariant under gauge transformations is the Wilson loop, which is defined over any closed path, γ , as follows:

$$\chi^{(\rho)} \left(\mathcal{P} \left\{ e^{\int_\gamma A} \right\} \right)$$

where χ is the character of a complex representation ρ and $\mathcal{P}$ represents the path-ordered operator.

Quantization of gauge theories

Gauge theories may be quantized by specialization of methods which are applicable to any quantum field theory. However, because of the subtleties imposed by the gauge constraints (see section on Mathematical formalism, above) there are many technical problems to be solved which do not arise in other field theories. At the same time, the richer structure of gauge theories allow simplification of some computations: for example Ward identities connect different renormalization constants.

Methods and aims

The first gauge theory to be quantized was quantum electrodynamics (QED). The first methods developed for this involved gauge fixing and then applying canonical quantization. The Gupta-Bleuler method was also developed to handle this problem. Non-abelian gauge theories are now handled by a variety of means. Methods for quantization are covered in the article on quantization.

The main point to quantization is to be able to compute quantum amplitudes for various processes allowed by the theory. Technically, they reduce to the computations of certain correlation functions in the vacuum state. This involves a renormalization of the theory.

When the running coupling of the theory is small enough, then all required quantities may be computed in perturbation theory. Quantization schemes intended to simplify such computations (such as canonical quantization) may be called **perturbative quantization schemes**. At present some of these methods lead to the most precise experimental tests of gauge theories.

However, in most gauge theories, there are many interesting questions which are non-perturbative. Quantization schemes suited to these problems (such as lattice gauge theory) may be called **non-perturbative quantization schemes**. Precise computations in such schemes often require supercomputing, and are therefore less well-developed

currently than other schemes.

Anomalies

Some of the symmetries of the classical theory are then seen not to hold in the quantum theory — a phenomenon called an **anomaly**. Among the most well known are:

- The scale anomaly, which gives rise to a *running coupling constant*. In QED this gives rise to the phenomenon of the Landau pole. In Quantum Chromodynamics (QCD) this leads to asymptotic freedom.
- The chiral anomaly in either chiral or vector field theories with fermions. This has close connection with topology through the notion of instantons. In QCD this anomaly causes the decay of a pion to two photons.
- The gauge anomaly, which must cancel in any consistent physical theory. In the electroweak theory this cancellation requires an equal number of quarks and leptons.

Pure gauge

A pure gauge is the set of field configurations obtained by a gauge transform on the null field configuration. So it is a particular "gauge orbit" in the field configuration's space.

In the abelian case, where $A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu f(x)$, the pure gauge is the set of field configurations $A'_\mu(x) = \partial_\mu f(x)$ for all $f(x)$.

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See also

- Aharonov-Bohm effect
- Coulomb gauge
- Electroweak theory
- Gauge covariant derivative
- Gauge fixing
- Gauge gravitation theory
- Kaluza-Klein theory
- Lie algebra
- Lie group
- Lorenz gauge
- Quantum chromodynamics
- Quantum electrodynamics
- Quantum field theory
- Quantum gauge theory
- Standard Model
- Standard Model (mathematical formulation)
- Symmetry breaking
- Symmetry in physics
- Yang-Mills theory
- Yang-Mills existence and mass gap

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- [4] <http://www.arxiv.org/abs/physics/0204034>
- [5] <http://www.arxiv.org/abs/math-ph/9902027>

External links

- Yang-Mills equations on DispersiveWiki (http://tosio.math.toronto.edu/wiki/index.php/Yang-Mills_equations)
-

Yang-Mills theory

Yang-Mills theory is a gauge theory of quantum field theory based on the $SU(N)$ group. It was formulated by Yang and Mills in 1954^[1] in an effort to extend the original concept of gauge theory for an Abelian group, as was quantum electrodynamics, to the case of a nonabelian group with the intention to get an explanation for strong interactions. This initial idea was not a success as the quanta of the Yang-Mills field must be massless in order to maintain gauge invariance but such massless particles should have had long range effects that are not seen in experiments. So, the idea was put aside till the start of 1960 when the concept of breaking of symmetry in massless theories, initially due to Jeffrey Goldstone, Yoichiro Nambu and Giovanni Jona-Lasinio, with particles acquiring mass in this way, was put forward.

This prompted a significant restart of Yang-Mills theory studies that proved successful in the formulation of both electroweak unification and quantum chromodynamics (QCD). The electroweak interaction is described by $SU(2) \times U(1)$ group while QCD is a $SU(3)$ gauge theory. The electroweak theory was obtained combining $SU(2)$ with $U(1)$, being quantum electrodynamics described by $U(1)$, in order to produce the photon field. The Standard Model combines strong, weak and electromagnetic interaction through the symmetry group $SU(2) \times U(1) \times SU(3)$. Strong interactions are not currently unified with the other two but couplings have been shown to converge to a single value at higher energies in an experiment at LEP, assuming a higher order symmetry as supersymmetry holds.

Phenomenology at lower energies in quantum chromodynamics is not completely understood due to the difficulties of managing such a theory with a strong coupling. This is the reason confinement has not been theoretically proven, though it is a consistent experimental observation. Proof that QCD confines at low energy is a mathematical problem of great relevance, and an award has been proposed by the Clay Mathematics Institute for whoever is able to show that the Yang-Mills theory has a mass gap.

Mathematical overview

Yang-Mills theories are a special example of gauge theory with symmetry non-abelian group given by the Lagrangian

$$\mathcal{L}_{\text{gf}} = -\frac{1}{4} \text{Tr}(F_a^{\mu\nu} F_{\mu\nu}^a)$$

with the generators of the Lie algebra corresponding to the F -quantities (the curvature or field-strength form) satisfying

$$[T^a, T^b] = i f^{abc} T^c$$

and the covariant derivative defined as

$$D_\mu = \partial_\mu - ig T^a A_\mu^a$$

where A_μ^a is the vector potential, and g is the coupling constant. For $D=4$, g is a pure number and for a $SU(N)$ group one has

$$a, b, c = 1 \dots N^2 - 1.$$

The relation

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$$

can be derived by the commutator

$$[D_\mu, D_\nu] = -igT^a F_{\mu\nu}^a.$$

The field has the property of being self-interacting and equations of motion that one obtains are said semilinear, as nonlinearities are both with and without derivatives. This means that one can manage this theory only by perturbation theory, with small nonlinearities. The Yang-Mills equations of motion are

$$D^\mu F_{\mu\nu}^a = 0,$$

with Bianchi identities

$$D_\mu F_{\nu\kappa}^a + D_\kappa F_{\mu\nu}^a + D_\nu F_{\kappa\mu}^a = 0.$$

Note that the transition between "upper" ("contravariant") and "lower" ("covariant") vector or tensor components is trivial for a indexes (e.g. $f^{abc} = f_{abc}$), whereas for μ and ν it is nontrivial, corresponding e.g. to the usual Lorentz signature, $\eta_{\mu\nu} = \text{diag}(+ - - -)$.

Quantization of Yang-Mills theory

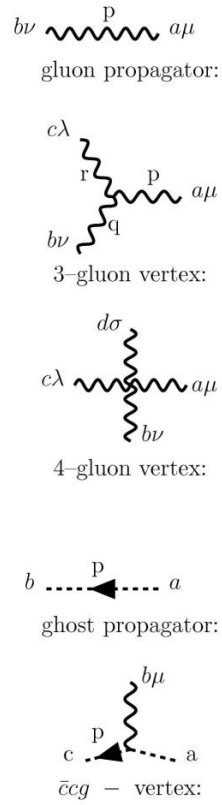
The most appropriate method to quantize the Yang-Mills theory is by functional methods, i.e. path integrals. One introduces a generating functional for n -point functions as

$$Z[j] = \int [dA] e^{-\frac{i}{4} \int d^4x \text{Tr}(F^{\mu\nu} F_{\mu\nu}) + i \int d^4x j_\mu^a(x) A^{a\mu}(x)}$$

but this integral has no meaning as is because the potential vector can be arbitrarily chosen due to the gauge freedom. This problem was already known for quantum electrodynamics but here becomes more severe due to non-abelian properties of the gauge group. A way out has been given by Ludvig Faddeev and Victor Popov with the introduction of a **ghost field** (see Faddeev-Popov ghost) that has the property of being unphysical as it agrees with Fermi-Dirac statistics but it is a complex scalar field, violating in this way the spin-statistics theorem. So, we can write the generating functional as

$$Z[j, \bar{\epsilon}, \epsilon] = \int [dA][d\bar{c}][dc] e^{-\frac{i}{4} \int d^4x \text{Tr}(F^{\mu\nu} F_{\mu\nu}) - i \int d^4x (\bar{c}^a \partial_\mu \partial^\mu c^a + g \bar{c}^a f^{abc} \partial_\mu A^{b\mu} c^c) - i \int d^4x \frac{1}{2\alpha} (\partial \cdot A)^2} \times \\ e^{i \int d^4x j_\mu^a(x) A^{a\mu}(x) + i \int d^4x [\bar{c}^a(x) \epsilon^a(x) + \bar{\epsilon}^a(x) c^a(x)]}$$

that is the expression commonly used to derive Feynman's rules (see Feynman diagram). Here we have c^a for the ghost field while α fixes the gauge's choice for the quantization. Feynman's rules obtained from this functional are the following



gluon propagator: $D_{\mu\nu}^{ab}(p) = \frac{-i\delta^{ab}}{p^2 + i0} \left[\eta_{\mu\nu} - \frac{(1-\xi)p_\mu p_\nu}{p^2 + i0} \right]$

3-gluon vertex: $\Gamma_{\mu\nu\lambda}^{abc}(p, q, r) = -gf^{abc}[(p-q)_\lambda \eta_{\mu\nu} + (q-r)_\mu \eta_{\nu\lambda} + (r-p)_\nu \eta_{\mu\lambda}]$

4-gluon vertex: $\Gamma_{\mu\nu\lambda\sigma}^{abcd} = -ig^2 f^{abe} f^{cde} (\eta_{\mu\lambda} \eta_{\nu\sigma} - \eta_{\mu\sigma} \eta_{\nu\lambda}) - ig^2 f^{ace} f^{bde} (\eta_{\mu\nu} \eta_{\sigma\lambda} - \eta_{\mu\sigma} \eta_{\nu\lambda}) - ig^2 f^{ade} f^{bce} (\eta_{\mu\nu} \eta_{\sigma\lambda} - \eta_{\mu\lambda} \eta_{\nu\sigma})$

ghost propagator: $C^{ab}(p) = \frac{i\delta^{ab}}{p^2 + i0}$

$\bar{c}cg$ - vertex: $\Gamma^{abc}(p) = gf^{abc} p_\mu$

These rules for Feynman diagrams are easily obtained when we realize that the generating functional given above can be rewritten as

$$Z[j, \bar{\epsilon}, \epsilon] = e^{-ig \int d^4x \frac{\delta}{i\delta \bar{c}^a(x)} f^{abc} \partial_\mu \frac{i\delta}{\delta j_\mu^b(x)} \frac{i\delta}{\delta c^c(x)}} e^{-ig \int d^4x f^{abc} \partial_\mu \frac{i\delta}{\delta j_\nu^a(x)} \frac{i\delta}{\delta j_\mu^b(x)} \frac{i\delta}{\delta j^{c\nu}(x)}} \times$$

$$e^{-i\frac{g^2}{4} \int d^4x f^{abc} f^{a\tau s} \frac{i\delta}{\delta j_\mu^b(x)} \frac{i\delta}{\delta j_\nu^c(x)} \frac{i\delta}{\delta j^{\tau\mu}(x)} \frac{i\delta}{\delta j^{s\nu}(x)}} Z_0[j, \bar{\epsilon}, \epsilon]$$

being

$$Z_0[j, \bar{\epsilon}, \epsilon] = e^{-\int d^4x d^4y \bar{\epsilon}^a(x) C^{ab}(x-y) \epsilon^b(y)} e^{\frac{1}{2} \int d^4x d^4y j_\mu^a(x) D^{ab\mu\nu}(x-y) j_\nu^b(y)}$$

the generating functional of the free theory. Expanding in g and computing the functional derivatives, we are able to obtain all the n -point functions with perturbation theory. Using LSZ reduction formula we get from the n -point functions the corresponding amplitudes for the given processes and cross sections and decay rates are promptly obtained. The theory is renormalizable and corrections are finite at any order of perturbation theory.

For quantum electrodynamics, being in this case abelian the gauge group (see abelian group), the ghost field decouples. This can be easily realized when we look at the coupling between the gauge field and the ghost field that is $\bar{c}^a f^{abc} \partial_\mu A^{b\mu} c^c$. For the Abelian case all the structure constants f^{abc} are zero and so there is no coupling. In the non-Abelian case, the ghost field appears as a useful way to rewrite the quantum field theory without physical consequences on the observables of the theory as cross sections or decay rates.

One of the most important results obtained for Yang-Mills theory is asymptotic freedom. This result can be obtained assuming the coupling constant g as small (so small nonlinearities), as indeed happens to high energies, and applying perturbation theory. The relevance of this result is due to the fact that a Yang-Mills theory describes strong interactions and asymptotic freedom permits to treat properly experimental results coming from deep inelastic scattering.

In order to obtain the behavior at high energies of the Yang-Mills theory, and so to prove asymptotic freedom, one does perturbation theory assuming a small coupling. This is verified a posteriori in the ultraviolet limit. In the opposite limit, infrared limit, the situation is quite the opposite being the coupling too large for perturbation theory to be reliable. Indeed, most of the difficulties that current research meets is just managing the theory at low energies that is the interesting one being inherent to the description of hadronic matter and, more generally, to all the observed bound states of gluons and quarks and their confinement (see hadrons). Then, the most used method to study the theory in this limit is to try to solve it on computers (see lattice gauge theory). In this case, large computational resources are needed to be sure the right limit of infinite volume (smaller lattice spacing) is hit. This is the limit the results have to be compared with. Smaller spacing and larger coupling are not independent each others and to accomplish both larger computational resources are demanded. As for today, the situation appears somewhat satisfactory for the hadronic spectrum and the computation of the gluon and ghost propagators but the glueball and hybrids spectra are yet a questioned matter also in view of the experimental observation of such exotic states. Indeed, σ resonance^{[2] [3]} is not seen in any of such lattice computations and contrasting interpretations have been put forward. This is currently a hot debated matter.

Open problems

The Yang-Mills theories were generally acknowledged in the physics community after Gerardus 't Hooft, in 1972, could prove their renormalizability. This applies even, if the gauge bosons described by this theory are massive, as in the electroweak theory. However, the mass is only an "acquired" one, namely, as suggested, by the famous Higgs mechanism.

Concerning the mathematics, it should be noted that presently, i.e. in 2009, the Yang-Mills theory is a very active field of research, yielding e.g. a classification of differentiable structures of four-dimensional manifolds by Simon Donaldson. Furthermore, the field of Yang-Mills theories was put by the Clay Mathematics Institute into its list of "Millennium Prize Problems". Here the prize-problem consists, especially, in a proof of the suggestion that the lowest excitations of a pure Yang-Mills theory (i.e. without matter fields) have a finite mass-gap with regard to the vacuum state. Another open problem, connected with this suggestion, is a proof of the confinement property in the presence of additional Fermion particles.

In physics the survey of Yang-Mills theories does usually not start from perturbation analysis or analytical methods, but more recently from systematic application of numerical methods to lattice gauge theories.

See also

- Yang-Mills existence and mass gap
- Aharonov-Bohm effect
- Bell's theorem
- Coulomb gauge
- Electroweak theory
- Field theoretical formulation of the standard model
- Gauge covariant derivative
- Kaluza-Klein theory
- Lorenz gauge
- Quantum chromodynamics
- Quantum gauge theory
- Symmetry in physics
- Weyl gauge

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External links

- Yang-Mills theory on DispersiveWiki (http://tosio.math.toronto.edu/wiki/index.php/Yang-Mills_equations)
 - The Clay Mathematics Institute (<http://www.claymath.org>)
 - The Millennium Prize Problems (<http://www.claymath.org/prizeproblems>)
-

Yangian

Yangian is an important structure in modern representation theory, a type of a quantum group with origins in physics. Yangians first appeared in the work of Ludvig Faddeev and his school concerning the quantum inverse scattering method in the late 1970s and early 1980s. Initially they were considered a convenient tool to generate the solutions of the quantum Yang-Baxter equation. The name *Yangian* was introduced by Vladimir Drinfeld in 1985 in honor of C.N. Yang.

Description

For any finite-dimensional semisimple Lie algebra a , Drinfeld defined an infinite-dimensional Hopf algebra $Y(a)$, called the *Yangian* of a . This Hopf algebra is a deformation of the universal enveloping algebra $U(a[z])$ of the Lie algebra of polynomial loops of a given by explicit generators and relations. The relations can be encoded by identities involving a rational R -matrix. Replacing it with a trigonometric R -matrix, one arrives at affine quantum groups, defined in the same paper of Drinfeld.

In the case of the general linear Lie algebra gl_N , the Yangian admits a simpler description in terms of a single *ternary* (or *RTT*) *relation* on the matrix generators due to Faddeev and coauthors. The Yangian $Y(gl_N)$ is defined to be the algebra generated by elements $t_{ij}^{(p)}$ with $1 \leq i, j \leq N$ and $p \geq 0$, subject to the relations

$$[t_{ij}^{(p+1)}, t_{kl}^{(q)}] - [t_{ij}^{(p)}, t_{kl}^{(q+1)}] = -(t_{kj}^{(p)} t_{il}^{(q)} - t_{kl}^{(q)} t_{ij}^{(p)}).$$

Defining $t_{ij}^{(-1)} = \delta_{ij}$, setting

$$T(z) = \sum_{p \geq -1} t_{ij}^{(p)} z^{-p+1}$$

and introducing the R -matrix $R(z) = I + z^{-1} P$ on $\mathbf{C}^N \otimes \mathbf{C}^N$, where P is the operator permuting the tensor factors, the above relations can be written more simply as the ternary relation:

$$R_{23}(z-w)T_{12}(z)T_{13}(w) = T_{13}(w)T_{12}(z)R_{23}(z-w).$$

The Yangian becomes a Hopf algebra with comultiplication Δ , counit ε and antipode s given by

$$(\Delta \otimes \text{id})T(z) = T_{12}(z)T_{13}(z), (\varepsilon \otimes \text{id})T(z) = I, (s \otimes \text{id})T(z) = T(z)^{-1}.$$

The **twisted Yangian** $Y(gl_{2N})$, introduced by G. I. Olshanky, is the sub-Hopf algebra generated by the coefficients of

$$S(z) = T(z)\sigma T(-z),$$

where σ is the involution of gl_{2N} given by

$$\sigma(E_{ij}) = (-1)^{i+j} E_{2N-j+1, 2N-i+1}.$$

Applications to classical representation theory

G.I. Olshansky and I. Cherednik discovered that the Yangian of gl_N is closely related with the branching properties of irreducible finite-dimensional representations of general linear algebras. In particular, the classical Gelfand–Tsetlin construction of a basis in the space of such a representation has a natural interpretation in the language of Yangians, studied by M. Nazarov and V. Tarasov. Olshansky, Nazarov and Molev later discovered a generalization of this theory to other classical Lie algebras, based on the twisted Yangian.

Representation theory of Yangians

Irreducible finite-dimensional representations of Yangians were parametrized by Drinfeld in a way similar to the highest weight theory in the representation theory of semisimple Lie algebras. The role of the highest weight is played by a finite set of *Drinfeld polynomials*. Drinfeld also discovered a generalization of the classical Schur–Weyl duality between representations of general linear and symmetric groups that involves the Yangian of sl_N and the degenerate affine Hecke algebra (graded Hecke algebra of type A, in George Lusztig's terminology).

Representations of Yangians have been extensively studied, but the theory is still under active development.

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Superalgebra

In mathematics and theoretical physics, a **superalgebra** is a $\mathbf{Z}_2$ -graded algebra. That is, it is an algebra over a commutative ring or field with a decomposition into "even" and "odd" pieces and a multiplication operator that respects the grading.

The prefix *super-* comes from the theory of supersymmetry in theoretical physics. Superalgebras and their representations, supermodules, provide an algebraic framework for formulating supersymmetry. The study of such objects is sometimes called super linear algebra. Superalgebras also play an important role in related field of supergeometry where they enter into the definitions of supermanifolds and superschemes.

Formal definition

Let K be a fixed commutative ring. In most applications, K is a field such as $\mathbf{R}$ or $\mathbf{C}$.

A **superalgebra** over K is a K -module A with a direct sum decomposition

$$A = A_0 \oplus A_1$$

together with a bilinear multiplication $A \times A \rightarrow A$ such that

$$A_i A_j \subseteq A_{i+j}$$

where the subscripts are read modulo 2.

A **superring**, or $\mathbf{Z}_2$ -graded ring, is a superalgebra over the ring of integers $\mathbf{Z}$.

The elements of A_i are said to be **homogeneous**. The **parity** of a homogeneous element x , denoted by $|x|$, is 0 or 1 according to whether it is in A_0 or A_1 . Elements of parity 0 are said to be **even** and those of parity 1 to be **odd**. If x and y are both homogeneous then so is the product xy and $|xy| = |x| + |y|$.

An **associative superalgebra** is one whose multiplication is associative and a **unital superalgebra** is one with a multiplicative identity element. The identity element in a unital superalgebra is necessarily even. Unless otherwise specified, all superalgebras in this article are assumed to be associative and unital.

A **commutative superalgebra** is one which satisfies a graded version of commutativity. Specifically, A is commutative if

$$yx = (-1)^{|x||y|}xy$$

for all homogeneous elements x and y of A .

Examples

- Any algebra over a commutative ring K may be regarded as a purely even superalgebra over K ; that is, by taking A_1 to be trivial.
- Any $\mathbf{Z}$ or $\mathbf{N}$ -graded algebra may be regarded as superalgebra by reading the grading modulo 2. This includes examples such as tensor algebras and polynomial rings over K .
- In particular, any exterior algebra over K is a superalgebra. The exterior algebra is the standard example of a supercommutative algebra.
- The symmetric polynomials and alternating polynomials together form a superalgebra, being the even and odd parts, respectively. Note that this is a different grading from the grading by degree.
- Clifford algebras are superalgebras. They are generally noncommutative.

- The set of all endomorphisms (both even and odd) of a super vector space forms a superalgebra under composition.
- The set of all square supermatrices with entries in K forms a superalgebra denoted by $M_{p|q}(K)$. This algebra may be identified with the algebra of endomorphisms of a free supermodule over K of rank $p|q$.
- Lie superalgebras are a graded analog of Lie algebras. Lie superalgebras are nonunital and nonassociative; however, one may construct the analog of a universal enveloping algebra of a Lie superalgebra which is a unital, associative superalgebra.

Further definitions and constructions

Even subalgebra

Let A be a superalgebra over a commutative ring K . The submodule A_0 , consisting of all even elements, is closed under multiplication and contains the identity of A and therefore forms a subalgebra of A , naturally called the **even subalgebra**. It forms an ordinary algebra over K .

The set of all odd elements A_1 is an A_0 -bimodule whose scalar multiplication is just multiplication in A . The product in A equips A_1 with a bilinear form

$$\mu : A_1 \otimes_{A_0} A_1 \rightarrow A_0$$

such that

$$\mu(x \otimes y) \cdot z = x \cdot \mu(y \otimes z)$$

for all x, y , and z in A_1 . This follows from the associativity of the product in A .

Grade involution

There is a canonical involutive automorphism on any superalgebra called the **grade involution**. It is given on homogeneous elements by

$$\hat{x} = (-1)^{|x|}x$$

and on arbitrary elements by

$$\hat{x} = x_0 - x_1$$

where x_i are the homogeneous parts of x . If A has no 2-torsion (in particular, if 2 is invertible) then the grade involution can be used to distinguish the even and odd parts of A :

$$A_i = \{x \in A : \hat{x} = (-1)^i x\}.$$

Supercommutativity

The **supercommutator** on A is the binary operator given by

$$[x, y] = xy - (-1)^{|x||y|}yx$$

on homogeneous elements. This can be extended to all of A by linearity. Elements x and y of A are said to **supercommute** if $[x, y] = 0$.

The **supercenter** of A is the set of all elements of A which supercommute with all elements of A :

$$Z(A) = \{a \in A : [a, x] = 0 \text{ for all } x \in A\}.$$

The supercenter of A is, in general, different than the center of A as an ungraded algebra. A commutative superalgebra is one whose supercenter is all of A .

Super tensor product

The graded tensor product of two superalgebras may be regarded as a superalgebra with a multiplication rule determined by:

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|}(a_1 a_2 \otimes b_1 b_2).$$

Generalizations and categorical definition

One can easily generalize the definition of superalgebras to include superalgebras over a commutative superring. The definition given above is then a specialization to the case where the base ring is purely even.

Let R be a commutative superring. A **superalgebra** over R is a R -supermodule A with a R -bilinear multiplication $A \times A \rightarrow A$ that respects the grading. Bilinearity here means that

$$r \cdot (xy) = (r \cdot x)y = (-1)^{|r||x|}x(r \cdot y)$$

for all homogeneous elements $r \in R$ and $x, y \in A$.

Equivalently, one may define a superalgebra over R as a superring A together with an superring homomorphism $R \rightarrow A$ whose image lies in the supercenter of A .

One may also define superalgebras categorically. The category of all R -supermodules forms a monoidal category under the super tensor product with R serving as the unit object. An associative, unital superalgebra over R can then be defined as a monoid in the category of R -supermodules. That is, a superalgebra is an R -supermodule A with two (even) morphisms

$$\mu : A \otimes A \rightarrow A$$

$$\eta : R \rightarrow A$$

for which the usual diagrams commute.

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Supergroup

Supergroup or **super group** may refer to:

- Supergroup (physics), a generalization of groups, used in the study of supersymmetry
- Supergroup (music), a term for music groups formed by artists who are already notable or respected in their fields
- Supergroup (City of Heroes), the term for player guilds in the *City of Heroes* MMORPG
- *SuperGroup*, a VH1 reality show
- Supergroup, a geological unit
- Super-group, a team of superheroes who work together
- Supergroup, a rarely used term in mathematics for the counterpart of a subgroup

Symmetry breaking

Symmetry breaking in physics describes a phenomenon where (infinitesimally) small fluctuations acting on a system crossing a critical point decide a system's fate, by determining which branch of a bifurcation is taken. For an outside observer unaware of the fluctuations (the "noise"), the choice will appear arbitrary. This process is called symmetry "breaking", because such transitions usually bring the system from a disorderly state into one of two more ordered, less probable states. Since disorder is more symmetric in the sense that small variations to it don't change its overall appearance, the symmetry gets "broken".

Symmetry breaking is supposed to play a major role in pattern formation.

In particular, we can distinguish between:

- An explicit symmetry breaking happens when the laws describing a system are themselves not invariant under the symmetry in question.
- Spontaneous symmetry breaking describes the case where the laws are invariant but it appears the system isn't because the background of the system, its vacuum, is noninvariant. Such a symmetry breaking is parametrized by an order parameter. A special case of this type of symmetry breaking is dynamical symmetry breaking.

In 1972, Nobel laureate P.W.Anderson used the idea of Symmetry breaking to show some of the drawbacks of Reductionism in his paper titled *More is different* in Science^[1].

See also

- Higgs mechanism
- QCD vacuum
- Goldstone boson

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"http://www.cmp.caltech.edu/~motrunch/Teaching/Phy135b_Winter07/MoreIsDifferent.pdf|More is Different".
Science **177** (4047): 393-396. doi: 10.1126/science.177.4047.393 (http://dx.doi.org/10.1126/science.177.4047.393). PMID 17796623. http://www.cmp.caltech.edu/~motrunch/Teaching/Phy135b_Winter07/MoreIsDifferent.pdf.
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Spontaneous symmetry breaking

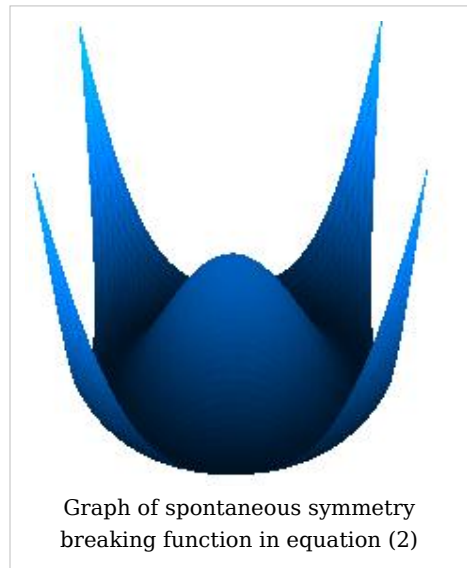
In physics, **spontaneous symmetry breaking** occurs when a system that is symmetric with respect to some symmetry group goes into a vacuum state that is not symmetric. When that happens, the system no longer appears to behave in a symmetric manner. It is a phenomenon that naturally occurs in many situations.

The symmetry group can be discrete, such as the space group of a crystal, or continuous (e.g., a Lie group), such as the rotational symmetry of space. However if the system contains only a single spatial dimension then only discrete symmetries may be broken in a vacuum state of the full quantum theory, although a classical solution may break a continuous symmetry.

A common example to help explain this phenomenon is a ball sitting on top of a hill. This ball is in a completely symmetric state. However, its state is unstable: the slightest perturbing force will cause the ball to roll down the hill in some particular direction. At that point, symmetry has been broken because the direction in which the ball rolled has a feature that distinguishes it from all other directions.

Mathematical example: the Mexican hat potential

In the simplest example, the spontaneously broken field is described by a scalar field theory. In physics, one way of seeing spontaneous symmetry breaking is through the use of Lagrangians. Lagrangians, which essentially dictate how a system will behave, can be split up into kinetic and potential terms



$$(1) \quad \mathcal{L} = \partial^\mu \phi \partial_\mu \phi - V(\phi).$$

It is in this potential term ($V(\phi)$) that the action of symmetry breaking occurs. An example of a potential is illustrated in the graph at the right.

$$(2) \quad V(\phi) = -10|\phi|^2 + |\phi|^4$$

This potential has many possible minima (vacuum states) given by

$$(3) \quad \phi = \sqrt{5}e^{i\theta}$$

for any real θ between 0 and 2π . The system also has an unstable vacuum state corresponding to $\phi = 0$. This state has a $U(1)$ symmetry. However, once the system falls into a specific stable vacuum state (corresponding to a choice of θ) this symmetry will be lost or spontaneously broken.

Higgs mechanism

In the Standard Model, spontaneous symmetry breaking is accomplished by using the Higgs boson and is responsible for the masses of the W and Z bosons. A slightly more technical presentation of this mechanism is given in the article on the Yukawa interaction, where it is shown how spontaneous symmetry breaking can be used to give mass to fermions.

Broader concept

More generally, we can have spontaneous symmetry breaking in nonvacuum situations and for systems not described by actions. The crucial concept here is the order parameter. If there is a field (often a background field) which acquires an expectation value (not necessarily a *vacuum* expectation value) which is not invariant under the symmetry in question, we say that the system is in the ordered phase and the symmetry is spontaneously broken. This is because other subsystems interact with the order parameter which forms a "frame of reference" to be measured against, so to speak.

If a vacuum state obeys the initial symmetry then the system is said to be in the **Wigner mode**, otherwise it is in the **Goldstone mode**.

Examples

- For ferromagnetic materials, the laws describing it are invariant under spatial rotations. Here, the order parameter is the magnetization, which measures the magnetic dipole density. Above the Curie temperature, the order parameter is zero, which is spatially invariant and there is no symmetry breaking. Below the Curie temperature, however, the magnetization acquires a constant (in the idealized situation where we have full equilibrium; otherwise, translational symmetry gets broken as well) nonzero value which points in a certain direction. The residual rotational symmetries which leaves the orientation of this vector invariant remain unbroken but the other rotations get spontaneously broken.
 - The laws describing a solid are invariant under the full Euclidean group, but the solid itself spontaneously breaks this group down to a space group. The displacement and the orientation are the order parameters.
 - The laws of physics are spatially invariant, but here on the surface of the Earth, we have a background gravitational field (which plays the role of the order parameter here) which points downwards, breaking the full rotational symmetry. This explains why up, down and the vertical directions are all "different" but all the horizontal directions are still isotropic.
 - General relativity has a Lorenz gauge symmetry, but in FRW cosmological models, the mean 4-velocity field defined by averaging over the velocities of the galaxies (the galaxies act like gas particles at cosmological scales) acts as an order parameter breaking this symmetry. Similar comments can be made about the cosmic microwave background.
 - Here on Earth, Galilean invariance (in the nonrelativistic approximation) is broken by the velocity field of the Earth/atmosphere, which acts as the order parameter here. This explains why people thought moving bodies tend towards rest before Galileo. We tend not to be aware of broken symmetries.
 - For the electroweak model, as explained earlier, the Higgs field acts as the order parameter breaking the electroweak gauge symmetry to the electromagnetic gauge symmetry. Like the ferromagnetic example, there is a phase transition at the electroweak
-

temperature. The same comment about us not tending to notice broken symmetries explains why it took so long for us to discover electroweak unification.

- For superconductors, there is a collective condensed matter field ψ which acts as the order parameter breaking the electromagnetic gauge symmetry.
- In general relativity, diffeomorphism covariance is broken by the nonzero order parameter, the metric tensor field.
- Take a flat plastic ruler which is identical on both sides and push both ends together. Before buckling, the system is symmetric under the reflection about the plane of the ruler. But after buckling, it either buckles upwards or downwards.
- Consider a uniform layer of fluid over an infinite horizontal plane. This system has all the symmetries of the Euclidean plane. But now heat the bottom surface uniformly so that it becomes much hotter than the upper surface. When the temperature gradient becomes large enough, convection cells will form, breaking the Euclidean symmetry.
- Consider a bead on a circular hoop that is rotated about a vertical diameter. As the rotational velocity is increased gradually from rest, the bead will initially stay at its initial equilibrium point at the bottom of the hoop (intuitively stable, lowest gravitational potential). At a certain critical rotational velocity, this point will become unstable and the bead will jump to one of two other newly created equilibria, equidistant from the center. Initially, the system is symmetric with respect to the diameter, yet after passing the critical velocity, the bead must choose between the two new equilibrium points, thus breaking symmetry. Note: This can easily be tried at home with an electric drill, a marble, and a pot cover, (or any other combination you can think of) and is the two-dimensional, mechanical analogue of the symmetry breaking that occurs in the Higgs Boson field.

Nobel Prize

On October 7, 2008, the Royal Swedish Academy of Sciences awarded the 2008 Nobel Prize in Physics to two Japanese citizens and a Japanese-born American for their work in subatomic physics. American Yoichiro Nambu, 87, of the University of Chicago, won half of the prize for the discovery of the mechanism of spontaneous broken symmetry. Japanese physicists Makoto Kobayashi and Toshihide Maskawa shared the other half of the prize for discovering the origin of the broken symmetry. ^[1] The trio shared the 10 million kronor (1.25 million USD) purse, as well as a diploma and an invitation to the prize ceremonies in Stockholm on December 10, 2008.

See also

- Autocatalytic reactions and order creation
- Catastrophe theory
- CP-violation
- Dynamical symmetry breaking
- Explicit symmetry breaking
- Gauge gravitation theory
- Goldstone boson
- Grand unified theory
- Mermin-Wagner theorem
- Second-order phase transition
- Symmetry breaking
- Vacuum fluctuation

Notes

[1] The Nobel Foundation. http://nobelprize.org/nobel_prizes/physics/laureates/2008/index.html|"The Nobel Prize in Physics 2008". *nobelprize.org*. http://nobelprize.org/nobel_prizes/physics/laureates/2008/index.html. Retrieved on January 15 2008.

External links

- Spontaneous symmetry breaking (<http://hyperphysics.phy-astr.gsu.edu/hbase/forces/unify.html#c2>)

Goldstone boson

In particle and condensed matter physics, **Goldstone bosons** (also known as **Nambu-Goldstone bosons**) are bosons that appear in models exhibiting spontaneous breakdown of continuous symmetries (cf. spontaneously broken symmetry). They were discovered by Nambu in the context of the BCS superconductivity mechanism,^[1] and subsequently elucidated by Goldstone,^[2] and systematically generalized in the context of quantum field theory.^[3]

These spinless bosons correspond to the spontaneously broken symmetry generators, and are characterized by the quantum numbers of these. They transform **nonlinearly** (shift) under the action of these generators, and can thus be excited out of the asymmetric vacuum by these generators. Thus, they can be thought of as the excitations of the field in the broken symmetry "directions" — and are massless if the spontaneously broken symmetry is *not also broken explicitly*. If, instead, the symmetry is not exact, i.e., if it is explicitly broken as well as spontaneously broken, then the Nambu-Goldstone bosons are not massless, though they typically remain relatively light; these are called **pseudo-Goldstone bosons** or **pseudo-Nambu-Goldstone bosons** (abbreviated *PNGBs*).

Goldstone's theorem

Goldstone's theorem examines a generic continuous symmetry which is spontaneously broken, i.e., its currents are conserved, but the ground state (vacuum) is not invariant under the action of the corresponding charges. Then, necessarily, new massless (or light, if the symmetry is not exact) scalar particles appear in the spectrum of possible excitations. There is one scalar particle — called a Goldstone boson — for each generator of the symmetry that is broken, i.e., that does not preserve the ground state.

There is an arguable loophole in the theorem. If one reads the theorem carefully, it only states that there exist non-vacuum states with arbitrarily small energies. Take for example a chiral $N = 1$ super QCD model with a nonzero squark VEV which is conformal in the IR. The chiral symmetry is a global symmetry which is (partially) spontaneously broken. Some of the "Goldstone bosons" associated with this SSB are charged under the unbroken gauge group and hence, these composite bosons have a continuous mass spectrum with arbitrarily small masses but yet there is no Goldstone boson with exactly zero mass. In other words, the Goldstone bosons are infraparticles.

In theories with gauge symmetry, the Goldstone bosons are "eaten" by the gauge bosons. The latter become massive and their new, longitudinal polarization is provided by the Goldstone boson.

A simple example

Consider a complex scalar field ϕ (phi), with the constraint that $\phi^* \phi = k^2$. One way to get a constraint of this sort is by including a potential

$$\lambda^2(\phi^* \phi - k^2)^2,$$

and taking the limit as $\lambda \rightarrow \infty$ (this is called the Abelian nonlinear σ -model). The constraint, and the action, below, are invariant under a phase transformation, $U(1)$, $\delta \phi = i \varepsilon \phi$.

The field can be redefined to give a real scalar field (i.e., a spin-zero particle) θ without any constraint by

$$\phi = k e^{i\theta}$$

where θ is the Goldstone boson (actually $k\theta$ is), and the $U(1)$ symmetry transformation effects a *shift* on θ , namely $\delta \theta = \varepsilon$, but does not preserve the ground state $|0\rangle$, (i.e. the above infinitesimal transformation does not annihilate it—the hallmark of invariance), as evident in the charge of the current below.

That is, corresponding Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu \phi^*) \partial_\mu \phi + m^2 \phi^* \phi = -\frac{1}{2}(-ik e^{-i\theta} \partial^\mu \theta)(ik e^{i\theta} \partial_\mu \theta) + m^2 k^2 = -\frac{k^2}{2}(\partial^\mu \theta)(\partial_\mu \theta) + m^2 k^2,$$

and the resulting conserved $U(1)$ current is $J_\mu = -k^2 \partial_\mu \theta$. The resulting charge Q shifts θ and the ground state. Thus, a ground state with $\langle \theta \rangle = 0$ will shift to a different ground state with $\langle \theta \rangle = -\varepsilon$.

Note that the constant term $m^2 k^2$ in the Lagrangian density has no physical significance, and the other term is simply the kinetic term for a massless scalar. In general the Goldstone boson is always massless, and parameterises the curve of possible vacuum states.

Goldstone's Argument

The principle of Goldstone's argument is that the charge operator for any symmetry current is time independent.

$$\frac{d}{dt} Q = \frac{d}{dt} \int_x J^0(x) = 0.$$

So acting with the charge operator on the vacuum either annihilates the vacuum, if that is symmetric; else, if not, it produces a zero-frequency state out of it, through its shift transformation feature illustrated. So, if the vacuum is not invariant under the symmetry, action of the charge operator produces a state which is different from the vacuum chosen, but which has zero frequency. This is a long wavelength oscillation of a field which is nearly stationary: there are states with zero frequency, so that the theory cannot have a mass gap.

This argument is clarified by taking the limit carefully. If an approximate charge operator is applied to the vacuum,

$$\frac{d}{dt} Q_A = \frac{d}{dt} \int_x e^{\frac{-x^2}{2A^2}} J^0(x) = - \int_x e^{\frac{-x^2}{2A^2}} \nabla \cdot J = \int_x \nabla(e^{\frac{-x^2}{2A^2}}) \cdot J,$$

a state with approximately zero time derivative is produced,

$$\left\| \frac{d}{dt} Q_A |0\rangle \right\| \approx \frac{1}{A} \|Q_A |0\rangle\|.$$

Assuming a mass gap m_0 , the frequency of any state which is orthogonal to the vacuum is at least m_0 .

$$||\frac{d}{dt}|\psi\rangle|| = ||H|\psi\rangle|| \geq m_0 |||\psi\rangle||$$

Letting A become large leads to a contradiction.

This argument fails completely when the symmetry is gauged, because then the symmetry generator is only performing a gauge transformation. A gauge transformed state is the same exact state, so that acting with a symmetry generator does not get one out of the vacuum.

Nonrelativistic theories

A version of Goldstone's theorem also applies to nonrelativistic theories (and also relativistic theories with spontaneously broken Lorentz symmetry). It basically states that for each spontaneously broken global symmetry, there corresponds a quasiparticle with no energy gap (the nonrelativistic version of the mass gap). It's important to note that the energy here is really $H - \mu N - \vec{\alpha} \cdot \vec{P}$ and not H . However, two different spontaneously broken generators may now give rise to the same Nambu-Goldstone boson. For example, in a superfluid, both the $U(1)$ particle number symmetry and Galilean symmetry are spontaneously broken. However, the phonon is the Goldstone boson for both.

In general, the phonon is effectively the Nambu-Goldstone boson for spontaneously broken Galilean/Lorentz symmetry.

Goldstone fermions

Spontaneously broken global fermionic symmetries, which occur in some supersymmetric models, lead to Goldstone fermions, or *goldstinos*.^{[4] [5]} These have spin $\frac{1}{2}$, instead of 0, and carry all quantum numbers of the respective supersymmetry generators broken spontaneously. Bosonic superpartners of these goldstinos under supersymmetries, called *sgoldstinos*, might sometimes also appear. Strictly speaking, however, spontaneous symmetry, and supersymmetry, breaking smashes up ("reduces") multiplet and supermultiplet structures, into the characteristic nonlinear realizations of broken supersymmetry, so that different particles need not be connected by supersymmetry transformations anymore; even though, perturbatively in the symmetry-breaking parameter, they may be usefully allowed to be perceived to be.

Goldstone bosons in nature

- In fluids, the phonon is longitudinal and it is the Goldstone boson of the spontaneously broken Galilean symmetry. In solids, the situation is more complicated; the Goldstone bosons are the longitudinal and transverse phonons and they happen to be the Goldstone bosons of spontaneously broken Galilean, translational and rotational symmetry with no simple one-to-one correspondence between the Goldstone modes and the broken symmetries.
- In magnets, the original rotational symmetry (present in the absence of an external magnetic field) is spontaneously broken such that the magnetization points into a specific direction. The Goldstone bosons then are the *magnons*, i.e. spin waves in which the local magnetization direction oscillates.
- The *pions* are the pseudo-Goldstone bosons that result from the spontaneous breakdown of the chiral-flavor symmetries of QCD effected by quark condensation due to the strong

interaction. These symmetries are further explicitly broken by the masses of the quarks, so that the pions are not massless, but their mass is *significantly smaller* than typical hadron masses.

- The longitudinal polarization components of the W and Z bosons correspond to the Goldstone bosons of the spontaneously broken part of the electroweak symmetry $SU(2) \times U(1)$. Because this symmetry is gauged, the three would-be Goldstone bosons are "eaten" by the three gauge bosons corresponding to the three broken generators; this gives these three gauge bosons a mass, and the associated necessary third polarization degree of freedom. This is described in the Standard Model through the Higgs mechanism.

See also

- Pseudo-Goldstone boson
- Majoron
- Higgs mechanism
- Higgs boson
- Mermin-Wagner theorem

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- [1] Nambu, Y (1960). "Quasiparticles and Gauge Invariance in the Theory of Superconductivity". *Physical Review* **117**: 648 – 663. doi: 10.1103/PhysRev.117.648 (<http://dx.doi.org/10.1103/PhysRev.117.648>).
 - [2] Goldstone, J (1961). "Field Theories with Superconductor Solutions". *Nuovo Cimento* **19**: 154 – 164. doi: 10.1007/BF02812722 (<http://dx.doi.org/10.1007/BF02812722>).
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Higgs boson

<i>Higgs boson</i>	
Composition:	Elementary particle
Family:	Boson
Status:	Hypothetical
Theorized:	F. Englert, R. Brout, P. Higgs, G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble 1964
Spin:	0

The **Higgs boson** is a massive scalar elementary particle predicted to exist by the Standard Model in particle physics. At present there are no known fundamental scalar particles in nature.

The Higgs boson is the only Standard Model particle that has not yet been observed. Experimental detection of the Higgs boson would help explain the origin of mass in the universe. More specifically, the Higgs boson would explain the difference between the massless photon, which mediates electromagnetism, and the massive W and Z bosons, which mediate the weak force. If the Higgs boson exists, it is an integral and pervasive component of the material world.

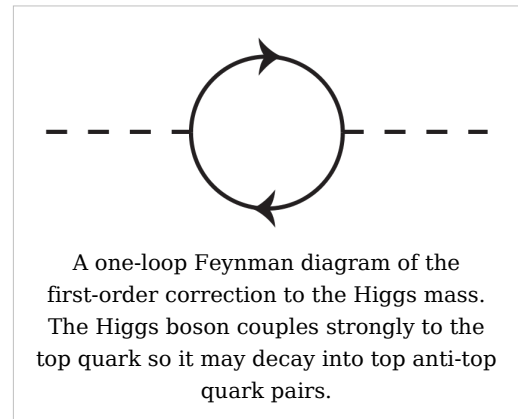
The Large Hadron Collider (LHC) at CERN in Geneva, which came online on September 10, 2008 is scheduled to become fully operational by late 2009, and is expected to provide experimental evidence either confirming or refuting the Higgs boson's existence. An accident in September 2008 has the LHC temporarily out of commission; ongoing experiments at Fermilab continue previous attempts at detection (although hindered by the lower energy of the Fermilab Tevatron accelerator). It has been reported that Fermilab physicists suggest the odds of Tevatron detecting the Higgs boson are between 50% and 96%, depending on its precise mass.^[1]

Origin

The Higgs mechanism, which gives mass to vector bosons, was theorized in 1964 by François Englert and Robert Brout ("boson scalaire");^[2] in October of the same year by Peter Higgs,^[3] working from the ideas of Philip Anderson; and independently by Gerald Guralnik, C. R. Hagen, and Tom Kibble,^[4] who worked out the results by the spring of 1963.^[5] The three papers written on this discovery by Guralnik, Hagen, Kibble, Higgs, Brout, and Englert were each recognized as milestone papers during *Physical Review Letters* 50th anniversary celebration.^[6] Steven Weinberg and Abdus Salam were the first to apply the Higgs mechanism to the electroweak symmetry breaking. The electroweak theory predicts a neutral particle whose mass is not far from that of the W and Z bosons.

Theoretical overview

The Higgs boson particle is one quantum component of the theoretical Higgs field. In empty space, the Higgs field has an amplitude different from zero, i.e., a non-zero vacuum expectation value. The existence of this non-zero vacuum expectation plays a fundamental role: it gives mass to every elementary particle which should have mass, including the Higgs boson itself. In particular, the acquisition of a non-zero vacuum expectation value spontaneously breaks electroweak gauge symmetry, which scientists often refer to as the Higgs mechanism. This is the simplest mechanism capable of giving mass to the gauge bosons while remaining compatible with gauge theories. In essence, this field is analogous to a pool of molasses that "sticks" to the otherwise massless fundamental particles which travel through the field, converting them into particles with mass which form, for example, the components of atoms.

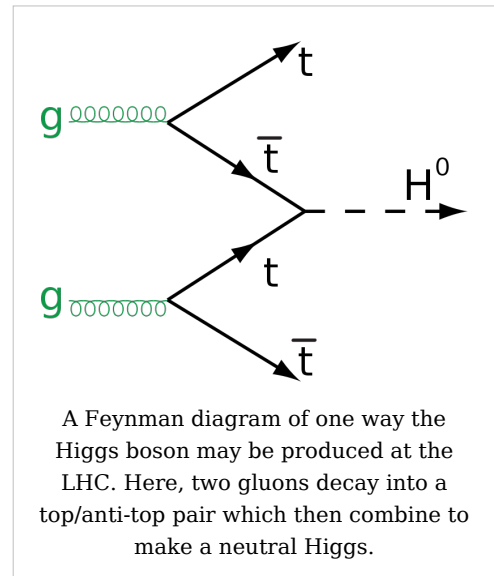


In the Standard Model, the Higgs field consists of two neutral and two charged component fields. Both of the charged components and one of the neutral fields are Goldstone bosons, which act as the longitudinal third-polarization components of the massive W^+ , W^- , and Z bosons. The quantum of the remaining neutral component corresponds to the massive Higgs boson. Since the Higgs field is a scalar field, the Higgs boson has no spin, hence no intrinsic angular momentum. The Higgs boson is also its own antiparticle and is CP-even.

The Standard Model does not predict the mass of the Higgs boson. If that mass is between 115 and 180 GeV/c^2 , then the Standard Model can be valid at energy scales all the way up to the Planck scale (10^{16} TeV). Many theorists expect new physics beyond the Standard Model to emerge at the TeV-scale, based on unsatisfactory properties of the Standard Model. The highest possible mass scale allowed for the Higgs boson (or some other electroweak symmetry breaking mechanism) is 1.4 TeV; beyond this point, the Standard Model becomes inconsistent without such a mechanism because unitarity is violated in certain scattering processes. Many models of supersymmetry predict that the lightest Higgs boson (of several) will have a mass only slightly above the current experimental limits, at around 120 GeV or less.

Experimental search

As of June 2009[7], the Higgs boson has yet to be observed experimentally, despite large efforts invested in accelerator experiments at CERN and Fermilab. The data gathered at the LEP collider at CERN allowed an experimental lower bound to be set for the mass of the Standard Model Higgs boson of $114.4 \text{ GeV}/c^2$ at 95% confidence level. The same experiment has produced a small number of events that could be interpreted as resulting from Higgs bosons with mass just above said cutoff - around 115 GeV - but the number of events was insufficient to draw definite conclusions.^[8] The LEP was shut down in 2000 due to construction of its successor - the Large Hadron Collider (LHC). The LHC, due to begin proper experimentation in 2009 after initial calibration, is expected to be able to confirm or reject the existence of the Higgs boson. Full operational mode has been delayed until late September 2009, because of problems discovered with a number of magnets during the calibration and startup phase.



At the Fermilab Tevatron, there are ongoing experiments searching for the Higgs boson. As of March 2009[7], combined data from CDF and D0 experiments at the Tevatron were sufficient to exclude the Higgs boson in the range between $160 \text{ GeV}/c^2$ and $170 \text{ GeV}/c^2$ at the 95% confidence level.^[9] Continued data collection is aimed at raising this lower bound.

It may be possible to estimate the mass of the Higgs Boson indirectly. In the Standard Model, the Higgs has a number of indirect effects; most notably, Higgs loops result in tiny corrections to W and Z masses. Precision measurements of electroweak parameters, such as the Fermi constant and masses of W/Z bosons, can be used to constrain the mass of the Higgs. As of 2006, measurements of electroweak observables allowed the exclusion of a Standard Model Higgs boson having a mass greater than $285 \text{ GeV}/c^2$ at 95% CL, and estimated its mass to be $129+74-49 \text{ GeV}/c^2$ (approximately 138 proton masses).^[10] As of early 2009, Standard Model Higgs is excluded by electroweak measurements above 185 GeV at 95% CL. However, it should be noted that these indirect constraints make assumptions about post-SM physics (or, more specifically, lack thereof). One may still discover a Higgs above 185 GeV if it's accompanied by other particles between Standard Model and GUT scale.

Some have argued that there exists potential evidence of the Higgs Boson,^{[11] [12]} but to date no such evidence has convinced the physics community.

Alternatives to the Higgs mechanism for electroweak symmetry breaking

In the years since the Higgs boson was proposed, several alternatives to the Higgs mechanism have been proposed. All of the alternative mechanisms use strongly interacting dynamics to produce a vacuum expectation value that breaks electroweak symmetry. A partial list of these alternative mechanisms are:

- Technicolor^[13] is a class of models that attempts to mimic the dynamics of the strong force as a way of breaking electroweak symmetry.
- Extra dimensional Higgsless models where the role of the Higgs field is played by the fifth component of the gauge field.^[14]
- Abbott-Farhi models of composite W and Z vector bosons.^[15]
- Top quark condensate

In popular culture

The Higgs boson is sometimes referred to as "the God particle," after the title of Leon Lederman's book for lay readers.^[16] The term mistakenly implies that the Higgs boson would complete our understanding of physics. In fact, while the discovery of the Higgs boson would be a groundbreaking stage in the story of electroweak unification, it would leave remaining the question of unification with Quantum Chromodynamics (QCD), gravity, and the ultimate origins and early evolution of the universe. Being an atheist, Peter Higgs dislikes the epithet "God particle".^[17] The term is rarely used by particle physicists when discussing the Higgs Boson; its prevalence is primarily due to its usage in popular media.

See also

- List of particles
- Yukawa interaction
- ZZ diboson
- Quantum triviality

Notes

- [1] <http://news.bbc.co.uk/2/hi/science/nature/7893689.stm>|"Race for 'God particle' heats up". <http://news.bbc.co.uk/2/hi/science/nature/7893689.stm>.
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External links

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 - Physics World, Introducing the little Higgs (<http://physicsworld.com/cws/article/print/11353>)
 - A quasi-political Explanation of the Higgs Boson (<http://www.hep.ucl.ac.uk/~djm/higgsa.html>)
 - *The Atom Smashers*, a blog about the making of a documentary about the search for the Higgs boson (<http://www.theatomsmashers.blogspot.com/>)
 - In CERN Courier, Steven Weinberg reflects on spontaneous symmetry breaking (<http://cerncourier.com/cws/article/cern/32522>)
 - Steven Weinberg Praises Teams for Higgs Boson Theory (http://www.pas.rochester.edu/urpas/news/Hagen_030708)
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 - Provides a new and simple explanation of the mass and gravity enigma (<http://www.higgs-boson.org/>)
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Higgs mechanism

In the standard model of particle physics, the *Higgs mechanism* is a theoretical framework which explains how the masses of the W and Z bosons arise as a result of electroweak symmetry breaking.^[1]

More generally, the *Higgs mechanism* is the way that the gauge bosons in any gauge theory, like the standard model, get a nonzero mass. It requires an extra field, a **Higgs field**, which interacts with the gauge fields, and which has a nonzero value in its lowest energy state, a vacuum expectation value. This means that all of space is filled with the background Higgs field, the so-called **Higgs condensate**. Interaction with this background field changes the low-energy spectrum of the gauge fields and the gauge bosons become massive.

The Higgs field has a non-trivial self-interaction, like the Mexican hat potential, which leads to spontaneous symmetry breaking: in the lowest energy state the symmetry of the potential (which includes the gauge symmetry) is broken by the condensate. Analysis of small fluctuations of the fields near the minimum reveals that the gauge bosons and other particles become massive.

In particle language, the constant Higgs field is a superfluid of charged particles, and a charged superfluid is a superconductor. Inside a superconductor, the gauge electric and magnetic fields both become short-ranged, or massive.

The Higgs field in the standard model is a SU(2) doublet, a complex spinor with four real components, which is charged under the standard model U(1). After symmetry breaking, three of the four degrees of freedom in the Higgs field mix with the W and Z bosons, while the one remaining degree of freedom becomes the Higgs boson - a new scalar particle. Although the evidence for the *Higgs mechanism* is overwhelming, accelerators have yet to produce the Higgs boson and evaluate its physical properties, so it is not even known if the Higgs is an elementary or a composite particle. It is hoped that the Large Hadron Collider at CERN will find the Higgs, and allow physicists to determine its properties.

The Higgs mechanism in the standard model successfully predicts the mass of the $W^\pm$, and Z weak gauge bosons, which are naturally massless. If the Higgs mechanism wasn't there, these particles would get much smaller masses from Higgslike QCD quark condensates instead. A part of the Higgs field is expected to show up as a new particle, called the Higgs boson.^[2]

In the standard model of particle physics, the same Higgs mechanism which breaks the standard model gauge group to electromagnetism is also responsible for giving all the leptons and quarks their masses. The fermions in the standard model are chiral and different chiralities have different charges. The chiral fermions can come together in pairs to make massive fermions by absorbing Higgs bosons from the condensate.

History and naming

The mechanism is also called the **Brout-Englert-Higgs mechanism**, or **Higgs-Brout-Englert-Guralnik-Hagen-Kibble mechanism**, or **Anderson-Higgs mechanism**.

It was proposed in 1964 by Robert Brout and Francois Englert,^[3] independently by Peter Higgs^[4] and by Gerald Guralnik, C. R. Hagen, and Tom Kibble.^[5] It was inspired by the BCS theory of superconductivity, vacuum structure work by Yoichiro Nambu, the preceding Ginzburg–Landau theory, and the suggestion by Philip Anderson that superconductivity could be important for relativistic physics. It was anticipated by earlier work of Ernst Stückelberg on massive quantum electrodynamics. It was named the *Higgs mechanism* by Gerardus 't Hooft in 1971. The three papers written on this discovery by Guralnik, Hagen, Kibble, Higgs, Brout, and Englert were each recognized as milestone papers by Physical Review Letters 50th anniversary celebration.^[6]

General discussion

The problem with spontaneous symmetry breaking models in particle physics is that, according to Goldstone's theorem, they come with massless scalar particles. If a symmetry is broken by a condensate, acting with a symmetry generator on the condensate gives a second state with the same energy. So certain oscillations do not have any energy, and in quantum field theory the particles associated with these oscillations have zero mass.

The only observed particles which could be interpreted as Goldstone bosons were the pions. Since the symmetry is approximate, the pions are not exactly massless. Yoichiro Nambu, writing before Jeffrey Goldstone, suggested that the pions were the bosons associated with chiral symmetry breaking. This explained their pseudoscalar nature, the reason they couple to nucleons through derivative couplings, and the Goldberger–Treiman relation. Aside from the pions, no other Goldstone particle was observed.

A similar problem arises in Yang–Mills theory, also known as nonabelian gauge theory. These theories predict massless spin 1 gauge bosons, which (apart from the photon) are also not observed. It was Higgs' insight that when a gauge theory is combined with a spontaneous symmetry-breaking model the (unobserved) massless bosons acquire a mass, which are observed, solving the problem.

Higgs' original article presenting the model was rejected by Physical Review Letters when first submitted, apparently because it did not predict any new detectable effects. So he added a sentence at the end, mentioning that it implies the existence of one or more new, massive scalar bosons, which do not form complete representations of the symmetry. These are the Higgs bosons.

The Higgs mechanism was incorporated into modern particle physics by Steven Weinberg and is an essential part of the Standard Model.

In the standard model, at temperatures high enough so that the symmetry is unbroken, all elementary particles except the scalar Higgs boson are massless. At a critical temperature, the Higgs field spontaneously slides from the point of maximum energy in a randomly chosen direction. Once the symmetry is broken, the gauge boson particles, such as the W bosons and Z boson, acquire masses. The mass can be interpreted the result of the interactions of the particles with the "Higgs ocean".

Fermions, such as the leptons and quarks in the Standard Model, acquire mass as a result of their interaction with the Higgs field, but not in the same way as the gauge bosons.

Superconductivity

The Higgs mechanism can be considered as the superconductivity in the vacuum. It occurs when all of space is filled with a sea of particles which are charged, or in field language, when a charged field has a nonzero vacuum expectation value. Interaction with the quantum fluid filling the space prevents certain forces from propagating over long distances.

A superconductor expels all magnetic fields from its interior, a phenomenon known as the Meissner effect. This was mysterious for a long time, because it implies that electromagnetic forces somehow become short-range inside the superconductor. Contrast this with the behavior of an ordinary metal. In a metal, the conductivity shields electric fields by rearranging charges on the surface until the total field cancels in the interior. But magnetic fields can penetrate to any distance, and if a magnetic monopole (an isolated magnetic pole) is surrounded by a metal the field can escape without collimating into a string. In a superconductor, however, electric charges move with no dissipation, and this allows for permanent surface currents, not just surface charges. When magnetic fields are introduced at the boundary of a superconductor, they produce surface currents which exactly neutralize them. The Meissner effect is due to currents in a thin surface layer, whose thickness, the London penetration depth, can be calculated from a simple model.

This simple model, due to Lev Landau and Vitaly Ginzburg, treats superconductivity as a charged Bose-Einstein condensate. Suppose that a superconductor contains bosons with charge q . The wavefunction of the bosons can be described by introducing a quantum field, ψ , which obeys the Schrödinger equation as a field equation (in units where $\hbar$, the Planck quantum divided by 2π , is replaced by 1):

$$i\frac{\partial}{\partial t}\psi = \frac{(\nabla - iqA)^2}{2m}\psi$$

The operator $\psi(x)$ annihilates a boson at the point x , while its adjoint $\psi^\dagger$ creates a new boson at the same point. The wavefunction of the Bose-Einstein condensate is then the expectation value Ψ of $\psi(x)$, which is a classical function that obeys the same equation. The interpretation of the expectation value is that it is the phase that one should give to a newly created boson so that it will coherently superpose with all the other bosons already in the condensate.

When there is a charged condensate, the electromagnetic interactions are screened. To see this, consider the effect of a gauge transformation on the field. A gauge transformation rotates the phase of the condensate by an amount which changes from point to point, and shifts the vector potential by a gradient.

$$\begin{aligned}\psi &\rightarrow e^{iq\phi(x)}\psi \\ A &\rightarrow A + \nabla\phi\end{aligned}$$

When there is no condensate, this transformation only changes the definition of the phase of ψ at every point. But when there is a condensate, the phase of the condensate defines a preferred choice of phase.

The condensate wavefunction can be written as

$$\psi(x) = \rho(x) e^{i\theta(x)},$$

where ρ is real amplitude, which determines the local density of the condensate. If the condensate were neutral, the flow would be along the gradients of θ , the direction in which the phase of the Schrödinger field changes. If the phase θ changes slowly, the flow is slow and has very little energy. But now θ can be made equal to zero just by making a gauge transformation to rotate the phase of the field.

The energy of slow changes of phase can be calculated from the Schrödinger kinetic energy,

$$H = \frac{1}{2m} |(qA + \nabla)\psi|^2,$$

and taking the density of the condensate ρ to be constant,

$$H \approx \frac{\rho^2}{2m} (qA + \nabla\theta)^2.$$

Fixing the choice of gauge so that the condensate has the same phase everywhere, the electromagnetic field energy has an extra term,

$$\frac{q^2 \rho^2}{2m} A^2.$$

When this term is present, electromagnetic interactions become short-ranged. Every field mode, no matter how long the wavelength, oscillates with a nonzero frequency. The lowest frequency can be read off from the energy of a long wavelength A mode,

$$E \approx \frac{\dot{A}^2}{2} + \frac{q^2 \rho^2}{2m} A^2.$$

This is a harmonic oscillator with frequency $\sqrt{q^2 \rho^2 / m}$. The quantity $|\psi|^2 (= \rho^2)$ is the density of the condensate of superconducting particles.

In an actual superconductor, the charged particles are electrons, which are fermions not bosons. So in order to have superconductivity, the electrons need to somehow bind into Cooper pairs. The charge of the condensate q is therefore twice the electron charge e . The pairing in a normal superconductor is due to lattice vibrations, and is in fact very weak; this means that the pairs are very loosely bound. The description of a Bose-Einstein condensate of loosely bound pairs is actually more difficult than the description of a condensate of elementary particles, and was only worked out in 1957 by Bardeen, Cooper and Schrieffer in the famous BCS theory.

Abelian Higgs model

In a relativistic gauge theory, the vector bosons are naively massless, like the photon, leading to long-range forces. This is fine for electromagnetism, where the force is actually long-range, but it means that the description of short-range weak forces by a gauge theory requires a modification.

Gauge invariance means that certain transformations of the gauge field do not change the energy at all. If an arbitrary gradient is added to A, the energy of the field is exactly the same. This makes it difficult to add a mass term, because a mass term tends to push the field toward the value zero. But the zero value of the vector potential is not a gauge invariant idea. What is zero in one gauge is nonzero in another.

So in order to give mass to a gauge theory, the gauge invariance must be broken by a condensate. The condensate will then define a preferred phase, and the phase of the condensate will define the zero value of the field in a gauge invariant way. The gauge

invariant definition is that a gauge field is zero when the phase change along any path from parallel transport is equal to the phase difference in the condensate wavefunction.

The condensate value is described by a quantum field with an expectation value, just as in the Landau-Ginzburg model. To make sure that the condensate value of the field does not pick out a preferred direction in space-time, it must be a scalar field. In order for the phase of the condensate to define a gauge, the field must be charged.

In order for a scalar field Φ to be charged, it must be complex. Equivalently, it should contain two fields with a symmetry which rotates them into each other, the real and imaginary parts. The vector potential changes the phase of the quanta produced by the field when they move from point to point. In terms of fields, it defines how much to rotate the real and imaginary parts of the fields into each other when comparing field values at nearby points.

The only renormalizable model where a complex scalar field Φ acquires a nonzero value is the Mexican-hat model, where the field energy has a minimum away from zero.

$$S(\phi) = \int \frac{1}{2} |\partial\phi|^2 - \lambda \cdot (|\phi|^2 - \Phi^2)^2$$

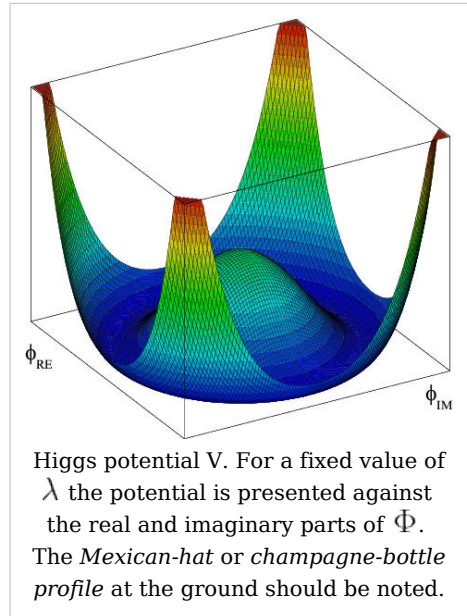
This defines the following Hamiltonian:

$$H(\phi) = \frac{1}{2} |\dot{\phi}|^2 + |\nabla\phi|^2 + V(|\phi|)$$

The first term is the kinetic energy of the field. The second term is the extra potential energy when the field varies from point to point. The third term is the potential energy when the field has any given magnitude.

This potential energy $V(z, \Phi) = \lambda \cdot (|z|^2 - \Phi^2)^2$ has a graph which looks like a Mexican hat, which gives the model its name. In particular, the minimum energy value is not at $z=0$, but on the circle of points where the magnitude of z is Φ .

When the field $\Phi(x)$ is not coupled to electromagnetism, the Mexican-hat potential has flat directions. Starting in any one of the circle of vacua and changing the phase of the field from point to point costs very little energy. Mathematically, if



$$\phi(x) = \Phi e^{i\theta(x)},$$

with a constant prefactor, then the action for the field $\theta(x)$, i.e., the "phase" of the Higgs field $\Phi(x)$, has only derivative terms. This is not a surprise. Adding a constant to $\theta(x)$ is a symmetry of the original theory, so different values of $\theta(x)$ cannot have different energies.

This is an example of Goldstone's theorem: spontaneously broken continuous symmetries lead to massless particles.

The Abelian Higgs model is the Mexican-hat model coupled to electromagnetism:

$$S(\phi, A) = \int \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + |(\partial - iqA)\phi|^2 + \lambda \cdot (|\phi|^2 - \Phi^2)^2.$$

The classical vacuum is again at the minimum of the potential, where the magnitude of the complex field ϕ is equal to Φ . But now the phase of the field is arbitrary, because gauge transformations change it. This means that the field $\theta(x)$ can be set to zero by a gauge transformation, and does not represent any degrees of freedom at all.

Furthermore, choosing a gauge where the phase of the condensate is fixed, the potential energy for fluctuations of the vector field is nonzero, just as it is in the Landau–Ginzburg model. So in the abelian Higgs model, the gauge field acquires a mass. To calculate the magnitude of the mass, consider a constant value of the vector potential A in the x direction in the gauge where the condensate has constant phase. This is the same as a sinusoidally varying condensate in the gauge where the vector potential is zero. In the gauge where A is zero, the potential energy density in the condensate is the scalar gradient energy:

$$E = \frac{1}{2} |\partial(\Phi e^{iqAx})|^2 = \frac{1}{2} q^2 \Phi^2 A^2$$

And this energy is the same as a mass term $m^2 A^2/2$ where $m = q\Phi$.

Nonabelian Higgs mechanism

The Nonabelian Higgs model has the following action:

$$S(\phi, \mathbf{A}) = \int \frac{1}{4g^2} \text{tr}(F^{\mu\nu} F_{\mu\nu}) + |D\phi|^2 + V(|\phi|),$$

where now the nonabelian field $\mathbf{A}$ is contained in D and in the tensor components $F^{\mu,\nu}$ and $F_{\mu,\nu}$ (the relation between $\mathbf{A}$ and those components is well-known from the Yang–Mills theory).

It is exactly analogous to the Abelian Higgs model. Now the field ϕ is in a representation of the gauge group, and the gauge covariant derivative is defined by the rate of change of the field minus the rate of change from parallel transport using the gauge field A as a connection.

$$D\phi = \partial\phi - iA^k t_k \phi$$

Again, the expectation value of Φ defines a preferred gauge where the condensate is constant, and fixing this gauge, fluctuations in the gauge field A come with a nonzero energy cost.

Depending on the representation of the scalar field, not every gauge field acquires a mass. A simple example is in the renormalizable version of an early electroweak model due to Julian Schwinger. In this model, the gauge group is $SO(3)$ (or $SU(2)$ —there are no spinor representations in the model), and the gauge invariance is broken down to $U(1)$ or $SO(2)$ at long distances. To make a consistent renormalizable version using the Higgs mechanism, introduce a scalar field ϕ^a which transforms as a vector (a triplet) of $SO(3)$. If this field has a vacuum expectation value, it points in some direction in field space. Without loss of generality, one can choose the z -axis in field space to be the direction that ϕ is pointing, and then the vacuum expectation value of ϕ is $(0, 0, A)$, where A is a constant with dimensions of mass ($\hbar=1$).

Rotations around the z axis form a U(1) subgroup of SO(3) which preserves the vacuum expectation value of ϕ , and this is the unbroken gauge group. Rotations around the x and y axis do not preserve the vacuum, and the components of the SO(3) gauge field which generate these rotations become massive vector mesons. There are two massive W mesons in the Schwinger model, with a mass set by the mass scale A, and one massless U(1) gauge boson, similar to the photon.

The Schwinger model predicts magnetic monopoles at the electroweak unification scale, and does not predict the Z meson. It doesn't break electroweak symmetry properly as in nature. But historically, a model similar to this (but not using the Higgs mechanism) was the first in which the weak force and the electromagnetic force were unified.

Standard model Higgs mechanism

The gauge group of the electroweak part of the standard model is $SU(2) \times U(1)$. The Higgs mechanism is by a scalar field which is a weak SU(2) doublet with weak hypercharge -1, it has four real components or two complex components, and it transforms as a spinor under SU(2) and gets multiplied by a phase under U(1) rotations. Note that this is not the same as two complex spinors which mix under U(1), which would have eight real components, rather this is the spinor representation of the group U(2)--- multiplying by a phase mixes the real and imaginary part of the complex spinor into each other.

The group SU(2) is all unitary matrices, all the orthonormal changes of coordinates in a complex two dimensional vector space. Rotating the coordinates so that the first basis vector in the direction of H makes the vacuum expectation value of H the spinor $(A, 0)$. The generators for rotations about the x,y,z axes are by half the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$, so that a rotation of angle θ about the z axis takes the vacuum to:

$$(Ae^{i\theta/2}, 0)$$

While the X and Y generators mix up the top and bottom components, the Z rotations only multiply by a phase. This phase can be undone by a U(1) rotation of angle $\theta/2$, which multiplies by the opposite phase, since the Higgs has charge -1. Under both an SU(2) z-rotation and a U(1) rotation by an amount $\theta/2$, the vacuum is invariant. This combination of generators:

$$Q = W_z + Y/2$$

defines the unbroken gauge group, where W_z is the generator of rotations around the z-axis in the SU(2) and Y is the generator of the U(1). This combination of generators--- perform a z rotation in the SU(2) and simultaneously perform a U(1) rotation by half the angle--- preserves the vacuum, and defines the unbroken gauge group in the standard model. The part of the gauge field in this direction stays massless, and this gauge field is the actual photon.

The phase that a field acquires under this combination of generators is its electric charge, and this is the formula for the electric charge in the standard model. In this convention, all the Y charges in the standard model are multiples of $1/3$. To make all the Y-charges in the standard model integers, you can rescale the Y part of the formula by tripling all the Y-charges if you like, and rewrite the charge formula as $Q = W_z + Y/6$, but the normalization with Y/2 is the universal standard.

Affine Higgs Mechanism

Ernst Stueckelberg discovered a version of the Higgs mechanism by analyzing the theory of quantum electrodynamics with a massive photon. Stueckelberg's model is a limit of the regular Mexican hat Abelian Higgs model, where the vacuum expectation value H goes to infinity and the charge of the Higgs field goes to zero in such a way that their product stays fixed. The mass of the Higgs boson is proportional to H , so the Higgs boson becomes infinitely massive and disappears. The vector meson mass is equal to the product eH , and stays finite.

The interpretation is that when a $U(1)$ gauge field does not require quantized charges, it is possible to keep only the angular part of the Higgs oscillations, and discard the radial part. The angular part of the Higgs field θ has the following gauge transformation law:

$$\theta \rightarrow \theta + e\alpha$$

$$A \rightarrow A + \alpha$$

The gauge covariant derivative for the angle (which is actually gauge invariant) is:

$$D\theta = \partial\theta - eA$$

In order to keep θ fluctuations finite and nonzero in this limit, θ should be rescaled by H , so that its kinetic term in the action stays normalized. The action for the theta field is read off from the Mexican hat action by substituting $\phi = He^{i\theta/H}$.

$$S = \int \frac{1}{4}F^2 + \frac{1}{2}(D\theta)^2 = \int \frac{1}{4}F^2 + \frac{1}{2}(\partial\theta - HeA)^2 = \int \frac{1}{4}F^2 + \frac{1}{2}(\partial\theta - mA)^2$$

since eH is the gauge boson mass. By making a gauge transformation to set $\theta=0$, the gauge freedom in the action is eliminated, and the action becomes that of a massive vector field:

$$S = \int \frac{1}{4}F^2 + \frac{m^2}{2}A^2$$

To have arbitrarily small charges requires that the $U(1)$ is not the circle of unit complex numbers under multiplication, but the real numbers $\mathbb{R}$ under addition, which is only different in the global topology. Such a $U(1)$ group is *non-compact*. The field θ transforms as an affine representation of the gauge group. Among the allowed gauge groups, only non-compact $U(1)$ admits affine representations, and the $U(1)$ of electromagnetism is experimentally known to be compact, since charge quantization holds to extremely high accuracy.

The Higgs condensate in this model has infinitesimal charge, so interactions with the Higgs boson do not violate charge conservation. The theory of quantum electrodynamics with a massive photon is still a renormalizable theory, one in which electric charge is still conserved, but magnetic monopoles are not allowed. For nonabelian gauge theory, there is no affine limit, and the Higgs oscillations cannot be too much more massive than the vectors.

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See also

- Tachyon condensation
- Top quark condensate
- Goldstone boson
- Symmetry breaking
- QCD vacuum

External links

- A Generalized Higgs Model (<http://www.arXiv.org/abs/hep-th/9904080>)
 - Global Conservation Laws and Massless Particles (http://prola.aps.org/abstract/PRL/v13/i20/p585_1)
 - In CERN Courier, Steven Weinberg reflects on spontaneous symmetry breaking (<http://cerncourier.com/cws/article/cern/32522>)
 - Steven Weinberg Praises Teams for Higgs Boson Theory (http://www.pas.rochester.edu/urpas/news/Hagen_030708)
 - Physical Review Letters – 50th Anniversary Milestone Papers (<http://prl.aps.org/50years/milestones#1964>)
 - Imperial College London on PRL 50th Anniversary Milestone Papers (http://www3.imperial.ac.uk/newsandeventspggrp/imperialcollege/newssummary/news_13-6-2008-12-42-20?newsid=38514)
 - Physics World, Introducing the little Higgs (<http://physicsworld.com/cws/article/print/11353>)
 - Englert-Brout-Higgs-Guralnik-Hagen-Kibble Mechanism on Scholarpedia (http://www.scholarpedia.org/article/Englert-Brout-Higgs-Guralnik-Hagen-Kibble_mechanism)
 - History of Englert-Brout-Higgs-Guralnik-Hagen-Kibble Mechanism on Scholarpedia ([http://www.scholarpedia.org/article/Englert-Brout-Higgs-Guralnik-Hagen-Kibble_mechanism_\(history\)](http://www.scholarpedia.org/article/Englert-Brout-Higgs-Guralnik-Hagen-Kibble_mechanism_(history)))
 - God Particle Overview (<http://www.godparticle.com/>)
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Topological order

In physics, **topological order** ^[1] is a new kind of order (a new kind of organization of particles) in a quantum state that is beyond the Landau symmetry-breaking description. It cannot be described by local order parameters and long range correlations. However, topological orders can be described by a new set of quantum numbers, such as ground state degeneracy, quasiparticle fractional statistics, edge states, topological entropy, etc. Roughly speaking, topological order is a pattern of long-range quantum entanglement in quantum states. States with different topological orders can change into each other only through a phase transition.

Background

Although all matter is formed by atoms, matter can have very different properties and appear in very different forms, such as solid, liquid, superfluid, magnet, etc. According to condensed matter physics and the principle of emergence, the different properties of materials originate from the different ways in which the atoms are organized in the materials. Those different organizations of the atoms (or other particles) are formally called the orders in the materials.

Atoms can organize in many ways which lead to many different orders and many different types of materials. With so many different orders, we need a general understanding of the orders. Landau symmetry-breaking theory provides such a general understanding. It points out that different orders really correspond to different symmetries in the organizations of the constituent atoms. As a material changes from one order to another order (i.e., as the material undergoes a phase transition), what happens is that the symmetry of the organization of the atoms changes.

For example, atoms have a random distribution in a liquid, so a liquid remains the same as we displace it by an arbitrary distance. We say that a liquid has a *continuous translation symmetry*. After a phase transition, a liquid can turn into a crystal. In a crystal, atoms organize into a regular array (a lattice). A lattice remains unchanged only when we displace it by a particular distance (integer times of lattice constant), so a crystal has only *discrete translation symmetry*. The phase transition between a liquid and a crystal is a transition that reduces the continuous translation symmetry of the liquid to the discrete symmetry of the crystal. Such change in symmetry is called *symmetry breaking*. The essence of the difference between liquids and crystals is therefore that the organizations of atoms have different symmetries in the two phases.

Landau symmetry-breaking theory is a very successful theory. For a long time, physicists believed that Landau symmetry-breaking theory describes all possible orders in materials, and all possible (continuous) phase transitions.

The discovery and characterization of topological order

However, over the last thirty-two years, it has become gradually apparent that Landau symmetry-breaking theory may not describe all possible orders. In 1980s, physicists discovered fraction quantum Hall (FQH) states ^{[2] [3]} and introduced chiral spin state. ^{[4] [5]} chiral spin state is introduced in an attempt to explain high temperature superconductivity^[6]. At first, physicists still wanted to use Landau symmetry-breaking

theory to describe the chiral spin state. They identified the chiral spin state as a state that breaks the time reversal and parity symmetries, but not the spin rotation symmetry. However, it was quickly realized that there are many different chiral spin states that have exactly the same symmetry, so symmetry alone was not enough to characterize different chiral spin states. This means that the chiral spin states contain a new kind of order that is beyond the usual symmetry description^[7]. The proposed, new kind of order was named "topological order"^[8]. (The name "topological order" is motivated by the low energy effective theory of the chiral spin states which is a topological quantum field theory (TQFT)^{[9] [10] [11]}). New quantum numbers, such as ground state degeneracy^[12] and the non-Abelian Berry's phase of degenerate ground states^[13], were introduced to characterize the different topological orders in chiral spin states. Recently, it was shown that topological orders can also be characterized by topological entropy^{[14] [15]}.

But experiments soon indicated that chiral spin states do not describe high-temperature superconductors, and the theory of topological order became a theory with no experimental realization. However, the similarity between chiral spin states and quantum Hall states allows one to use the theory of topological order to describe different quantum Hall states.^[16] Just like chiral spin states, different quantum Hall states all have the same symmetry and are beyond the Landau symmetry-breaking description. One finds that the different orders in different quantum Hall states can indeed be described by topological orders, so the topological order does have experimental realizations.

Mechanism of topological order

A large class of topological orders is realized through a mechanism called string-net condensation^[17]. This class of topological orders can be classified by utilizing tensor category (or monoidal category) theory. One finds that string-net condensation can generate infinitely many different types of topological orders, which may indicate that there are many different new types of materials remaining to be discovered.

The collective motions of condensed strings give rise to excitations above the string-net condensed states. Those excitations turn out to be gauge bosons. The ends of strings are defects which correspond to another type of excitations. Those excitations are the gauge charges and can carry Fermi or fractional statistics.^[18]

The condensations of other extended objects such as 'membranes', [Hamma et al, 2005] "brane-nets", [Bombin, M.A. Martin-Delgado, 2006] and fractals also lead to topologically ordered phases^[19] and "quantum glassiness"^[20]

Mathematical foundation of topological order

We know that group theory is the mathematical foundation of symmetry breaking orders. What is the mathematical foundation of topological order? The string-net condensation suggests that tensor category (or monoidal category) theory may be the mathematical foundation of topological order. Quantum operator algebra is a very important mathematical tool in studying topological orders. Some also suggest that topological order is mathematically described by *extended* quantum symmetry^[21].

Applications

The materials described by Landau symmetry-breaking theory have had a substantial impact on technology. For example, Ferromagnetic materials that break spin rotation symmetry can be used as the media of digital information storage. A hard drive made of ferromagnetic materials can store gigabytes of information. Liquid crystals that break the rotational symmetry of molecules find wide application in display technology; nowadays one can hardly find a household without a liquid crystal display somewhere in it. Crystals that break translation symmetry lead to well defined electronic bands which in turn allow us to make semiconducting devices such as transistors. Topologically ordered states are found in a new class of materials that are even richer than those exhibiting symmetry breaking states. This suggests their potential for exciting, novel applications.

One theorized application would be to use topologically ordered states as media for quantum computing in a technique known as topological quantum computing. A topologically ordered state is a state with complicated non-local quantum entanglement. The non-locality means that the quantum entanglement in a topologically ordered state is distributed among many different particles. As a result, the pattern of quantum entanglements cannot be destroyed by local perturbations. This significantly reduces the effect of decoherence. This suggests that if we use different quantum entanglements in a topologically ordered state to encode quantum information, the information may last much longer. ^[22] The quantum information encoded by the topological quantum entanglements can also be manipulated by dragging the topological defects around each other. This process may provide a physical apparatus for performing quantum computations. ^[23] Therefore, topologically ordered states may provide natural media for both quantum memory and quantum computation. Such realizations of quantum memory and quantum computation may potentially be made fault tolerant. ^[24]

Potential impact

Why is topological order important? Landau symmetry-breaking theory is a cornerstone of condensed matter physics. It is used to define the territory of condensed matter research. The existence of topological order appears to indicate that nature is much richer than Landau symmetry-breaking theory has so far indicated. The exciting time of condensed matter physics is still ahead of us. Some suggest that topological order (or more precisely, string-net condensation) and the local bosonic (spin) models have the potential to provide a unified origin ^[25] for photons, electrons and other elementary particles in our universe. ^[26]

See also

- Quantum phase transitions
 - Quantum critical point
 - Quantum topology
 - Topological defect
 - Topological entropy in physics
 - Topological quantum field theory
 - Topological quantum number
 - Topological string theory
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Topological quantum field theory

A **topological quantum field theory** (or **topological field theory** or **TQFT**) is a quantum field theory which computes topological invariants.

Although TQFTs were invented by physicists, they are also of mathematical interest, being related to, among other things, knot theory and the theory of four-manifolds in algebraic topology, and to the theory of moduli spaces in algebraic geometry. Donaldson, Jones, Witten, and Kontsevich have all won Fields Medals for work related to topological field theory.

In condensed matter physics, topological quantum field theories are the low energy effective theories of topologically ordered states, such as fractional quantum Hall states, string-net condensed states, and other strongly correlated quantum liquid states.

Overview

In a topological field theory, the correlation functions do not depend on the metric on spacetime. This means that the theory is not sensitive to changes in the shape of spacetime; if the spacetime warps or contracts, the correlation functions do not change. Consequently, they are topological invariants.

Topological field theories are not very interesting on the flat Minkowski spacetime used in particle physics. Minkowski space can be contracted to a point, so a TQFT on Minkowski space computes only trivial topological invariants. Consequently, TQFTs are usually studied on curved spacetimes, such as, for example, Riemann surfaces. Most of the known topological field theories are defined on spacetimes of dimension less than five. It seems that a few higher dimensional theories exist, but they are not very well understood.

Quantum gravity is believed to be background-independent (in some suitable sense), and TQFTs provide examples of background independent quantum field theories. This has prompted ongoing theoretical investigation of this class of models.

(Caveat: It is often said that TQFTs have only finitely many degrees of freedom. This is not a fundamental property. It happens to be true in most of the examples that physicists and mathematicians study, but it is not necessary. A topological sigma model with target infinite-dimensional projective space, if such a thing could be defined, would have countably infinitely many degrees of freedom.)

Specific models

The known topological field theories fall into two general classes: Schwarz-type TQFTs and Witten-type TQFTs. Witten TQFTs are also sometimes referred to as cohomological field theories.

Schwarz-type TQFTs

In Schwarz-type TQFTs, the correlation functions computed by the path integral are topological invariants because the path integral measure and the quantum field observables are explicitly independent of the metric. For instance, in the BF model, the spacetime is a two-dimensional manifold M , the observables are constructed from a two-form F , an auxiliary scalar B , and their derivatives. The action (which determines the path integral) is

$$S = \int_M BF$$

The spacetime metric does not appear anywhere in this theory, so the theory is explicitly topologically invariant. Another, more famous example is Chern-Simons theory, which can be used to compute knot invariants.

Witten-type TQFTs

In Witten-type topological field theories, the topological invariance is more subtle. For example the Lagrangian for the WZW model does depend explicitly on the metric, but one shows by calculation that the expectation value of the partition function and a special class of correlation functions are in fact diffeomorphism invariant.

Mathematical formulations

Atiyah-Segal axioms

Atiyah suggested a set of axioms for topological quantum field theory which was inspired by Segal's proposed axioms for conformal field theory, (*Atiyah 1988*). These axioms have been relatively useful for mathematical treatments of Schwarz-type QFTs, although it isn't clear that they capture the whole structure of Witten-type QFTs. The basic idea is that a TQFT is a functor from a certain category of cobordisms to the category of vector spaces.

There are in fact two different sets of axioms which could reasonably be called the Atiyah axioms. These axioms differ basically in whether or not they study a TQFT defined on a single fixed n -dimensional Riemannian / Lorentzian spacetime M or a TQFT defined on all n -dimensional spacetimes at once.

[ed. *What follows is still in rough draft form and should be regarded suspiciously.*]

The case of a fixed spacetime

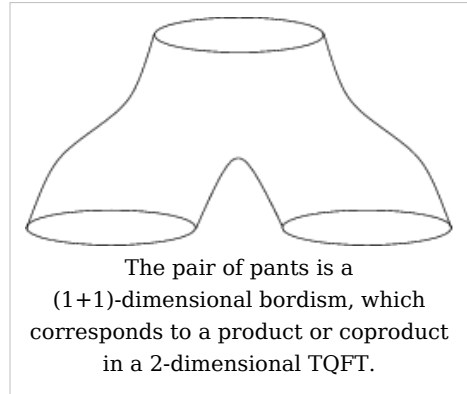
Let $Bord_M$ be the category whose morphisms are n -dimensional submanifolds of M and whose objects are connected components of the boundaries of such submanifolds. Regard two morphisms as equivalent if they are homotopic via submanifolds of M , and so form the quotient category $hBord_M$: The objects in $hBord_M$ are the objects of $Bord_M$, and the morphisms of $hBord_M$ are homotopy equivalence classes of morphisms in $Bord_M$. A TQFT on M is a symmetric monoidal functor from $hBord_M$ to the category of vector spaces.

Note that cobordisms can, if their boundaries match up, be sewn together to form a new bordism. This is the composition law for morphisms in the cobordism category. Since functors are required to preserve composition, this says that the linear map corresponding to a sewn together morphism is just the composition of the linear map for each piece.

There is an equivalence of categories between the category of 2-dimensional topological quantum field theories and the category of commutative Frobenius algebras.

All n -dimensional spacetimes at once

To consider all spacetimes at once, it is necessary to replace $hBord_M$ by a larger category. So let $Bord_n$ be the category of bordisms, i.e. the category whose morphisms are n -dimensional manifolds with boundary, and whose objects are the connected components of the boundaries of n -dimensional manifolds. (Note that any $(n-1)$ -dimensional manifold may appear as an object in $Bord_n$.) As above, regard two morphisms in $Bord_n$ as equivalent if they are homotopic, and form the quotient category $hBord_n$. $Bord_n$ is a monoidal category under the operation which takes two bordisms to the bordism made from their disjoint union. A TQFT on n -dimensional manifolds is then a functor from $hBord_n$ to the category of vector spaces, which takes disjoint unions of bordisms to the tensor product $\otimes$ [ed. *unfinished*]



For example, for $(1+1)$ -dimensional bordisms (2-dimensional bordisms between 1-dimensional manifolds), the map associated with a pair of pants gives a product or coproduct, depending on how the boundary components are grouped – which is commutative or cocommutative, while the map associated with a disk gives a counit (trace) or unit (scalars), depending on grouping of boundary, and thus $(1+1)$ -dimension TQFTs correspond to Frobenius algebras.

Generalizations

For some applications, it is convenient to demand extra topological structure on the morphisms, such as a choice of orientation.

See also

- Quantum topology
- Topological defect
- Topological entropy in physics
- Topological order
- Topological quantum number
- Topological string theory

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Quantum Geometry

Quantum geometry

In theoretical physics, **quantum geometry** is the set of new mathematical concepts generalizing the concepts of geometry whose understanding is necessary to describe the physical phenomena at very short distance scales (comparable to Planck length). At these distances, quantum mechanics has a profound effect on physics.

Each theory of quantum gravity uses the term *quantum geometry* in a slightly different fashion. String theory, a leading candidate for a quantum theory of gravity, uses the term quantum geometry to describe exotic phenomena such as T-duality and other geometric dualities, mirror symmetry, topology-changing transitions, minimal possible distance scale, and other effects that challenge our usual geometrical intuition. More technically, quantum geometry refers to the shape of the spacetime manifold as seen by D-branes which includes the quantum corrections to the metric tensor, such as the worldsheet instantons. For example, the quantum volume of a cycle is computed from the mass of a brane wrapped on this cycle.

In an alternative approach to quantum gravity called loop quantum gravity (LQG), the phrase **quantum geometry** usually refers to the formalism within LQG where the observables that capture the information about the geometry are now well defined operators on a Hilbert space. In particular, certain physical observables, such as the area, have a discrete spectrum. It has also been shown that the loop quantum geometry is non-commutative.

It is possible (but considered unlikely) that this strictly quantized understanding of geometry will be consistent with the quantum picture of geometry arising from string theory.

Another approach, which tries to reconstruct the geometry of space-time from "first principles" is Discrete Lorentzian quantum gravity.

See also

- Noncommutative geometry

External links

- Space and Time: From Antiquity to Einstein and Beyond ^[1]
- Quantum Geometry and its Applications ^[2]

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 [2] <http://cgpg.gravity.psu.edu/people/Ashtekar/articles/qgfinal.pdf>
-

Noncommutative geometry

Noncommutative geometry, or **NCG**, is a branch of mathematics concerned with the possible spatial interpretations of algebraic structures for which the commutative law fails, that is, for which xy does not always equal yx . For example; 3 steps of 4 units and 4 steps of 3 units length might be different in noncommutative spaces. Although one could technically construct geometries by simply removing this condition (commutativity), the results are typically trivial or uninteresting. The most common usage of the term, therefore, refers to what is properly called differential noncommutative geometry, a subject which was developed extensively by French mathematician Alain Connes. The challenge of NCG theory is to get around the lack of commutative multiplication, which is a requirement of previous geometric theories of algebraic structures.

History

Some of the theory developed by Alain Connes to handle noncommutative geometry at a technical level has roots in older attempts, in particular in ergodic theory. The proposal of George Mackey to create a *virtual subgroup* theory, with respect to which ergodic group actions would become homogeneous spaces of an extended kind, has by now been subsumed.

Motivation

In mathematics, there is a close relationship between *spaces*, which are geometric in nature, and the numerical functions on them. In general, such functions will form a commutative ring. For instance, one may take the ring $C(X)$ of continuous complex-valued functions on a topological space X . In many important cases (*e.g.*, if X is a compact Hausdorff space), we can recover X from $C(X)$, and therefore it makes some sense to say that X has *commutative geometry*.

For other cases and applications, including mathematical physics^[1] and functional analysis, non-commutative rings arise as the natural candidates for a ring of functions on some non-commutative "space". "Non-commutative spaces", however defined, cannot be too similar to ordinary topological spaces, as these are known to correspond to commutative rings in many important cases. For this reason, the field is also called non-commutative topology — some of the motivating questions of the theory are concerned with extending known topological invariants to these new spaces.

The problem is that on the microscopic level our traditional (non-quantum mechanical) notion of distance is no longer sufficient. For example, it is not possible to determine both the length and height of an object at the same time. A mathematician describes this situation by saying that the space coordinates of length and height do not commute with each other. This is good motivation to develop new mathematical tools, such as non-commutative geometry.

Non-commutative C*-algebras, von Neumann algebras

Non-commutative C*-algebras are often now called non-commutative spaces. This is by analogy with the Gelfand representation, which shows that commutative C*-algebras are dual to locally compact Hausdorff spaces. In general, one can associate to any C*-algebra S a topological space $\hat{S}$; see spectrum of a C*-algebra.

For the duality between σ -finite measure spaces and commutative von Neumann algebras, noncommutative von Neumann algebras are called *non-commutative measure spaces*.

Non-commutative differentiable manifolds

A smooth Riemannian manifold M is a topological space with a lot of extra structure. From its algebra of continuous functions $C(M)$ we only recover M topologically. The algebraic invariant that recovers the Riemannian structure is a spectral triple. It is constructed from a smooth vector bundle E over M , e.g. the exterior algebra bundle. The Hilbert space $L^2(M, E)$ of square integrable sections of E carries a representation of $C(M)$ by multiplication operators, and we consider an unbounded operator D in $L^2(M, E)$ with compact resolvent (e.g. the signature operator), such that the commutators $[D, f]$ are bounded whenever f is smooth. A recent deep theorem states that M as a Riemannian manifold can be recovered from this data.

This suggests that one might define a noncommutative Riemannian manifold as a spectral triple (A, H, D) , consisting of a representation of a C*-algebra A on a Hilbert space H , together with an unbounded operator D on H , with compact resolvent, such that $[D, a]$ is bounded for all a in some dense subalgebra of A . Research in spectral triples is very active, and many examples of noncommutative manifolds have been constructed.

The theory of characteristic classes of smooth manifolds has been extended to spectral triples, employing the tools of operator K-theory and cyclic cohomology. Several generalizations of now classical index theorems allow for effective extraction of numerical invariants from spectral triples.

Non-commutative affine schemes

In analogy to the duality between affine schemes and commutative rings, we can also have noncommutative affine schemes. For example, there exist an analog of the celebrated Serre duality for noncommutative projective schemes. [2]

Examples of non-commutative spaces

- In Weyl quantization, the symplectic phase space of classical mechanics is deformed into a non-commutative phase space generated by the position and momentum operators.
- The standard model of particle physics is another example of a noncommutative geometry, cf noncommutative standard model.
- The noncommutative torus, deformation of the function algebra of the ordinary torus, can be given the structure of a spectral triple. This class of examples has been studied intensively and still functions as a test case for more complicated situations.
- Noncommutative algebras arising from foliations.
- Examples related to dynamical systems arising from number theory, such as the Gauss shift on continued fractions, give rise to noncommutative algebras that appear to have

interesting noncommutative geometries.

- Quantum tori in quantum field theory.

See also

- Commutativity
- Moyal product
- Fuzzy sphere

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External links

Elements of Non-commutative geometry by Gracia-Bondia, Varilly and Figueroa

- Introduction to Quantum Geometry (<http://www.matem.unam.mx/~micho/papers/qgeom.pdf>) by Micho Đurđevich
- Noncommutative Geometry (<http://www.alainconnes.org/docs/book94bigpdf.pdf>) by Alain Connes
- Lectures on Noncommutative Geometry (<http://arxiv.org/abs/math/0506603>) by Victor Ginzburg
- Very Basic Noncommutative Geometry (<http://arxiv.org/abs/math/0408416>) by Masoud Khalkhali
- A walk in the noncommutative garden (<http://arxiv.org/abs/math/0601054>) by Alain Connes and Matilde Marcolli
- Lectures on Arithmetic Noncommutative Geometry (<http://arxiv.org/abs/math.qa/0409520>) by Matilde Marcolli
- An Introduction to Noncommutative Spaces and their Geometry (<http://arxiv.org/abs/hep-th/9701078>) by Giovanni Landi
- Noncommutative Geometry for Pedestrians (<http://arxiv.org/abs/gr-qc/9906059>) by J. Madore
- An informal introduction to the ideas and concepts of noncommutative geometry (<http://arxiv.org/abs/math-ph/0612012>) by Thierry Masson (an easier introduction that is still rather technical)
- Noncommutative geometry on arxiv.org (<http://xstructure.inr.ac.ru/x-bin/subthemes3.py?level=2&index1=-173391&skip=0>)
- Noncommutative Geometry, Quantum Fields and Motives (<http://www.alainconnes.org/docs/bookwebfinal.pdf>) by Alain Connes and Matilde Marcolli
- Noncommutative Geometry Blog (<http://noncommutativegeometry.blogspot.com/>)
- Noncommutative Geometry Resources (<http://www.noncommutativegeometry.net/>)

Noncommutative quantum field theory

In physics, **noncommutative quantum field theory** (or quantum field theory on noncommutative space-time) is a branch of quantum field theory born from noncommutative geometry and Index theory in which the spatial coordinates^[1] do not commute. One (commonly studied) version of such theories has the "canonical" commutation relation:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$

which means that (with any given set of axes), it is impossible to accurately measure the position of a particle with respect to more than one axis. In fact, this leads to an uncertainty relation for the coordinates analogous to the Heisenberg uncertainty principle.

Various lower limits have been claimed for the noncommutative scale, (i.e. how accurately positions can be measured) but there is currently no experimental evidence in favour of such theory or grounds for ruling them out.

One of the novel features of noncommutative field theories is the UV/IR mixing^[2] phenomenon in which the physics at high energies affects the physics at low energies which does not occur in quantum field theories in which the coordinates commute.

Other features include violation of Lorentz invariance due to the preferred direction of noncommutativity. Relativistic invariance can however be retained in the sense of twisted Poincaré invariance of the theory^[3]. Causality condition is possibly modified from that of the commutative theories.

History and motivation

The idea of extension of noncommutativity to the coordinates was first suggested by Heisenberg as a possible solution for removing the infinite quantities of field theories before the renormalization procedure was developed and had gained acceptance. The first paper on the subject was published in 1947 by Hartland Snyder. The success of renormalization theory however drained interest from the subject for some time. In 1980s noncommutative geometry was studied and developed by mathematicians, most notably Alain Connes. The notion of differential structure was generalized to a noncommutative setting. This led to an operator algebraic description of noncommutative space-times and a Yang-Mills theory on a noncommutative torus was developed.

The recent interest by the particle physics community was driven by a by Nathan Seiberg and Edward Witten^[4]. They argued in the context of string theory that the coordinate functions of the endpoints of open strings constrained to a D-brane in the presence of a constant Neveu-Schwartz B-field -- equivalent to a constant magnetic field on the brane -- would satisfy the noncommutative algebra presented above. The implication is that a quantum field theory on noncommutative space-time can be interpreted as a low energy limit of the theory of open strings.

Another paper^[5] possible motivation for the noncommutativity of space-time was presented by Sergio Doplicher, Klaus Fredenhagen and John Roberts. Their arguments is as follows: According to general relativity, when the energy density grows sufficiently large, a black hole is formed. On the other hand according to the Heisenberg uncertainty principle, a

measurement of a space-time separation causes an uncertainty in momentum inversely proportional to the separation. Thus energy of scale corresponding to the uncertainty in momentum is localized in the system within a region corresponding to the uncertainty in position. When the separation is small enough, the Schwarzschild radius of the system is reached and a black hole is formed, preventing any information to escape the region. Thus a lower limit is introduced for the measurement of length. A sufficient condition for preventing the gravitational collapse can be expressed as a form of uncertainty relation for the coordinates. This relation in turn can be derived from a commutation relation for the coordinates.

See also

- Weyl quantization
- Moyal product
- Noncommutative geometry
- Quantum tori

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- [1] It is possible to have a noncommuting time coordinate but this causes many problems, such as the violation of unitarity of the S-matrix, and most research is restricted to so-called "space-space" noncommutativity. There have been attempts to avoid these problems by redefining the perturbation theory. String theoretical derivation of noncommutative coordinates however excludes time-space noncommutativity.
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See statistics for Noncommutative quantum field theory (<http://xstructure.inr.ac.ru/x-bin/theme3.py?level=2&index1=-173391>) on arxiv.org

Noncommutative standard model

The **non-commutative Standard Model** is an extension of the Standard Model to include a modified form of general relativity, developed by Alain Connes using his theory of noncommutative geometry. This unification implies a few constraints on the parameters of the Standard Model. One of the constraints determines the mass of the Higgs boson to be around 170 GeV, comfortably within the range of the Large Hadron Collider. However, a Higgs mass of 170 GeV has been excluded to a confidence level of 95% using data from Tevatron experiments.^[1]

Background

Today it is thought that there are four elementary forces: the gravitational force, the electromagnetic force, the weak force, and the strong force. Gravity has an elegant and experimentally precise theory: Einstein's general relativity. It is based on Riemannian geometry and interprets the gravitational force as curvature of space-time. It has a Lagrangian formulation with only two real parameters, the gravitational constant and the cosmological constant.

The other three forces also have a Lagrangian theory, called the Standard Model. Its underlying idea is that they are mediated by the exchange of spin-one particles, the so-called gauge bosons. The one responsible for electromagnetism is the photon. The weak force has the W and Z bosons, and the strong force is thought to result from gluon exchange. The gauge Lagrangian is much more complicated than the gravitational one: at present, it involves some 30 real parameters. This number may still increase. What is more, the gauge Lagrangian must also contain a spin-zero particle, the 'Higgs boson'. So far this particle has not been observed and if it does not show up at the Large Hadron Collider in Geneva, the consistency of the Standard Model is questionable.

Alain Connes has generalized Bernhard Riemann's geometry to noncommutative geometry. It describes spaces with curvature and uncertainty. Historically, the first example of such a geometry is quantum mechanics, which introduces Heisenberg's uncertainty relation by turning the classical observables of position and momentum into noncommuting operators. Still, noncommutative geometry is close enough to Riemannian geometry that Connes was able to redo general relativity with it. In doing so he obtained the gauge Lagrangian as a companion of the gravitational one, a truly geometric unification of all four forces. Connes has thus provided a totally geometric formulation of the Standard Model where all the parameters are geometric invariants of a noncommutative space. Hence, electron parameters like mass are now analogous to π !

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External links

- <http://www.alainconnes.org/>

Fuzzy sphere

Fuzzy sphere is one of the simplest and most canonical examples of non-commutative geometry. Ordinarily, the functions defined on a sphere form a commuting algebra. A fuzzy sphere differs from an ordinary sphere because the algebra of functions on it is not commutative. It is generated by spherical harmonics whose spin l is at most equal to some j . The terms in the product of two spherical harmonics that involve spherical harmonics with spin exceeding j are simply omitted in the product. This truncation replaces an infinite-dimensional commutative algebra by a j^2 -dimensional non-commutative algebra.

The simplest way to see this sphere is to realize this truncated algebra of functions as a matrix algebra on some finite dimensional vector space. Take the three j -dimensional matrices J_a , $a = 1, 2, 3$ that form a basis for the j dimensional irreducible representation of the Lie algebra $SU(2)$. They satisfy the relations $[J_a, J_b] = i\epsilon_{abc}J_c$, where ϵ_{abc} is the totally anti-commuting tensor with $\epsilon_{123} = 1$, and generate via the matrix product the algebra M_j of j dimensional matrices. The value of the $SU(2)$ Casimir operator in this representation is

$$J_1^2 + J_2^2 + J_3^2 = \frac{1}{3}(j^2 - 1)I$$

where I is the j -dimensional identity matrix. Thus, if we define the 'coordinates' $x_a = kr^{-1}J_a$ where r is the radius of the sphere and k is a parameter, related to r and j by $3r^4 = k^2(j^2 - 1)$, then the above equation concerning the Casimir operator can be rewritten as

$$x_1^2 + x_2^2 + x_3^2 = r^2,$$

which is the usual relation for the coordinates on a sphere of radius r embedded in three dimensional space.

One can define an integral on this space, by

$$\int_{S^2} f d\Omega := 2\pi k \text{Tr}(F)$$

where F is the matrix corresponding to the function f . For example, the integral of unity, which gives the surface of the sphere in the commutative case is here equal to

$$2\pi k \text{Tr}(I) = 2\pi k j = 4\pi r^2 \frac{j}{\sqrt{j^2 - 1}}$$

which converges to the value of the surface of the sphere if one takes j to infinity.

See also

- Fuzzy torus

Notes

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Quantum torus

A **quantum torus** (plural: *quantum tori*) is a "space" whose noncommutative algebra of functions is obtained by deformation of the commutative algebra of functions on a torus $(\mathbf{R}/\mathbf{Z})^d$. It is a noncommutative analogue of the algebra of Laurent polynomials. Quantum tori appear in extended affine Lie algebra and quantum physics. In noncommutative geometry or quantum physics, a noncommutative torus is a special type of quantum tori.

In the 1980s and 1990s, Fields medallist Alain Connes introduced a range of new geometric objects designed to put quantum physics on a firm mathematical foundation. One of the most important examples was the quantum torus, an abstract, quantum version of the traditional doughnut-shaped torus. While a classical torus is easy to visualise, it is impossible to picture a quantum torus in the same way. This is because quantum geometry replaces the traditional notions of shape, area, curvature and so on with a more abstract concept of a mathematical space. Boris Zilber's work proved that if Schanuel's conjecture is true then the quantum torus is what is known as a stable structure. Stable structures have been studied intensively for years and are well known which gives better insight or intuition into quantum torus abstraction. Regardless, the quantum torus is fundamental to ongoing efforts to model the quantum universe.^[1]

The Schwartz kernel of the multiplication operation on a quantum torus is the distributional boundary value of a classical multivariate theta function. The kernel satisfies a Schrodinger equation in which the role of time is played by the deformation parameter $\hbar$ and the role of the hamiltonian by a Poisson structure associated with the deformation. At least in some special cases, the kernel can be written as a sum of products of single variable theta functions. Thus, the evolution in Planck's constant of a torus is like the time evolution of a free particle, with the initial state (the Schwartz kernel of the operator of pointwise

multiplication) being a delta function, like the initial state of a particle with certain position and totally uncertain momentum.

Kernel of multiplication on a quantum torus

Multiplication (Star product, Weyl-Moyal) on quantum tori are particularly useful in taking care of convergence problems in predefined noncommutative tori having some deformation parameter (deformation quantization). In particular, these kernels satisfy a Schrödinger equation, in which the role of time is played by the deformation parameter, and the role of the hamiltonian is played by the Poisson structure associated with the deformation. Thus, the evolution in Planck's constant of a torus is like the time evolution of a free particle, with the initial state being a delta function, like the initial state of a particle with certain position and totally uncertain momentum.

Applications

Examples of applications include: operator algebras (K-theory and KK-theory of C^* -algebra and their classification); topology (successful resolution of the Novikov conjecture and the Baum-Connes conjecture for large classes of groups); global analysis and geometry (various new formulations of index theory problems beyond their classical realms, on singular spaces like the space of leaves of a foliation); algebra (algebraic K-theory computations through topological Hochschild and cyclic homology, the idempotent conjecture for group algebras, the theory of quantum groups and Hopf algebra and their homological invariants); number theory (Connes' new approach to the Riemann hypothesis, relations between Hecke algebra and quantum statistical mechanics via noncommutative geometry, emerging relations with Arakelov theory, hidden quantum symmetries in the space of modular forms recently discovered by Connes and Moscovici); physics, in particular high energy physics (solid state physics of quasicrystals, new formulations of the standard model, relations with string theory, gauge theory, and M-theory).

The quantum Gromov-Hausdorff distance defines a framework to investigate claims found in the physics literature that classical or quantum spaces can be approximated by matrix algebra. These approximations are related to M-theory and may serve as tools to construct quantum field theory over quantum spaces. Quantum tori can be approximated with matrix algebras generated by unitaries in finite dimension.

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Homological mirror symmetry

Homological mirror symmetry is a mathematical conjecture made by Maxim Kontsevich. It seeks a systematic mathematical explanation for a phenomenon called mirror symmetry first observed by physicists studying string theory.

History

In an address to the 1994 International Congress of Mathematicians in Zurich, Kontsevich speculated that mirror symmetry for a pair of Calabi-Yau manifolds X and Y could be explained as an equivalence of a triangulated category constructed from the algebraic geometry of X and another triangulated category constructed from the symplectic geometry of Y .

Edward Witten originally described the topological twisting of the $N=(2,2)$ supersymmetric field theory into what he called the A and B model topological string theories. These models concern maps from Riemann surfaces into a fixed target - usually a Calabi-Yau manifold. Most of the mathematical predictions of mirror symmetry are embedded in the physical equivalence of the A-model on Y with the B-model on its mirror X . When the Riemann surfaces have empty boundary, they represent the worldsheets of closed strings. To cover the case of open strings, one must introduce boundary conditions to preserve the supersymmetry. In the A-model, these boundary conditions come in the form of Lagrangian submanifolds of Y with some additional structure (often called a brane structure). In the B-model, the boundary conditions come in the form of holomorphic (or algebraic) submanifolds of X with holomorphic (or algebraic) vector bundles on them. These are the objects one uses to build the relevant categories. They are often called A and B branes respectively. Morphisms in the categories are given by the massless spectrum of open strings stretching between two branes.

The closed string A and B models only capture the so-called topological sector - a small portion of the full string theory. Similarly, the branes in these models are only topological approximations to the full dynamical objects that are D-branes. Even so, the mathematics resulting from this small piece of string theory has been both deep and difficult.

Examples

Only in a few examples have mathematicians been able to verify the conjecture. In his seminal address, Kontsevich commented that the conjecture could be proved in the case of elliptic curves using theta functions. Following this route, Alexander Polishchuk and Eric Zaslow provided a proof of a version of the conjecture for elliptic curves. Kenji Fukaya was able to establish elements of the conjecture for abelian varieties. Later, Kontsevich and Yan Soibelman provided a proof of the majority of the conjecture for nonsingular torus bundles over affine manifolds using ideas from the SYZ conjecture. Recently, Paul Seidel proved the conjecture in the case of the quartic surface.

See also

- Topological quantum field theory
- Category theory
- Floer homology
- Fukaya category
- Derived category

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-

Algebra

Algebra

Algebra is a branch of mathematics concerning the study of structure, relation, and quantity. Together with geometry, analysis, combinatorics, and number theory, algebra is one of the main branches of mathematics. Elementary algebra is often part of the curriculum in secondary education and provides an introduction to the basic ideas of algebra, including effects of adding and multiplying numbers, the concept of variables, definition of polynomials, along with factorization and determining their roots.

Algebra is much broader than elementary algebra and can be generalized. In addition to working directly with numbers, algebra covers working with symbols, variables, and set elements. Addition and multiplication are viewed as general operations, and their precise definitions lead to structures such as groups, rings and fields.

History

While the word "algebra" comes from Arabic word (al-jabr, ربحل), its origins can be traced to the ancient Babylonians,^[1] who developed an advanced arithmetical system with which they were able to do calculations in an algebraic fashion. With the use of this system they were able to apply formulas and calculate solutions for unknown values for a class of problems typically solved today by using linear equations, quadratic equations, and indeterminate linear equations. By contrast, most Egyptians of this era, and most Indian, Greek and Chinese mathematicians in the first millennium BC, usually solved such equations by geometric methods, such as those described in the *Rhind Mathematical Papyrus*, *Sulba Sutras*, Euclid's *Elements*, and *The Nine Chapters on the Mathematical Art*. The geometric work of the Greeks, typified in the *Elements*, provided the framework for generalizing formulae beyond the solution of particular problems into more general systems of stating and solving equations, though this would not be realized until the medieval Muslim mathematicians.



A page from Al-Khwārizmī's *al-Kitāb al-muḥtaṣar fī ḥisāb al-ğabr wa-l-muqābala*

The Hellenistic mathematicians Hero of Alexandria and Diophantus^[2] as well as Indian mathematicians such as Brahmagupta continued the traditions of Egypt and Babylon, though Diophantus' *Arithmetica* and Brahmagupta's *Brahmasphuṭasiddhanta* are on a higher level.^[3] Later, Arab and Muslim mathematicians developed algebraic methods to a much higher degree of sophistication. Although Diophantus and the Babylonians used

mostly special ad hoc methods to solve equations, Al-Khowarazmi was the first to solve equations using general methods. He solved the linear indeterminate equations, quadratic equations, second order indeterminate equations and equations with multiple variable.

The word "algebra" is named after the Arabic word "*al-jabr* , ربحل" from the title of the book *al-Kitāb al-muḥtaṣar fī ḥisāb al-ğabr wa-l-muqābala* , ربحل باسح يف رصتخملا باتكلال , meaning *The book of Summary Concerning Calculating by Transposition and Reduction*, a book written by the Islamic Persian mathematician, Muhammad ibn Mūsā al-Khwārizmī (considered the "father of algebra"), in 820. The word *Al-Jabr* means "*reunion*". The Hellenistic mathematician Diophantus has traditionally been known as the "father of algebra" but in more recent times there is much debate over whether al-Khwarizmi, who founded the discipline of *al-jabr*, deserves that title instead.^[4] Those who support Diophantus point to the fact that the algebra found in *Al-Jabr* is slightly more elementary than the algebra found in *Arithmetica* and that *Arithmetica* is syncopated while *Al-Jabr* is fully rhetorical.^[5] Those who support Al-Khwarizmi point to the fact that he introduced the methods of "reduction" and "balancing" (the transposition of subtracted terms to the other side of an equation, that is, the cancellation of like terms on opposite sides of the equation) which the term *al-jabr* originally referred to,^[6] and that he gave an exhaustive explanation of solving quadratic equations,^[7] supported by geometric proofs, while treating algebra as an independent discipline in its own right.^[8] His algebra was also no longer concerned "with a series of problems to be resolved, but an exposition which starts with primitive terms in which the combinations must give all possible prototypes for equations, which henceforward explicitly constitute the true object of study." He also studied an equation for its own sake and "in a generic manner, insofar as it does not simply emerge in the course of solving a problem, but is specifically called on to define an infinite class of problems."^[9]

The Persian mathematician Omar Khayyam developed algebraic geometry and found the general geometric solution of the cubic equation. Another Persian mathematician, Sharaf al-Dīn al-Tūsī, found algebraic and numerical solutions to various cases of cubic equations.^[10] He also developed the concept of a function.^[11] The Indian mathematicians Mahavira and Bhaskara II, the Persian mathematician Al-Karaji,^[12] and the Chinese mathematician Zhu Shijie, solved various cases of cubic, quartic, quintic and higher-order polynomial equations using numerical methods.

Another key event in the further development of algebra was the general algebraic solution of the cubic and quartic equations, developed in the mid-16th century. The idea of a determinant was developed by Japanese mathematician Kowa Seki in the 17th century, followed by Gottfried Leibniz ten years later, for the purpose of solving systems of simultaneous linear equations using matrices. Gabriel Cramer also did some work on matrices and determinants in the 18th century. Abstract algebra was developed in the 19th century, initially focusing on what is now called Galois theory, and on constructibility issues.

Classification

Algebra may be divided roughly into the following categories:

- **Elementary algebra**, in which the properties of operations on the real number system are recorded using symbols as "place holders" to denote constants and variables, and the rules governing mathematical expressions and equations involving these symbols are studied (note that this usually includes the subject matter of courses called *intermediate algebra* and *college algebra*), also called second year and third year algebra;
- **Abstract algebra**, sometimes also called *modern algebra*, in which algebraic structures such as groups, rings and fields are axiomatically defined and investigated.
- **Linear algebra**, in which the specific properties of vector spaces are studied (including matrices);
- **Universal algebra**, in which properties common to all algebraic structures are studied.
- **Algebraic number theory**, in which the properties of numbers are studied through algebraic systems. Number theory inspired much of the original abstraction in algebra.
- **Algebraic geometry** in its algebraic aspect.
- **Algebraic combinatorics**, in which abstract algebraic methods are used to study combinatorial questions.

In some directions of advanced study, axiomatic algebraic systems such as groups, rings, fields, and algebras over a field are investigated in the presence of a geometric structure (a metric or a topology) which is compatible with the algebraic structure. The list includes a number of areas of functional analysis:

- Normed linear spaces
- Banach spaces
- Hilbert spaces
- Banach algebras
- Normed algebras
- Topological algebras
- Topological groups

Elementary algebra

Elementary algebra is the most basic form of algebra. It is taught to students who are presumed to have no knowledge of mathematics beyond the basic principles of arithmetic. In arithmetic, only numbers and their arithmetical operations (such as $+$, $-$, $\times$, $\div$) occur. In algebra, numbers are often denoted by symbols (such as a , x , or y). This is useful because:

- It allows the general formulation of arithmetical laws (such as $a + b = b + a$ for all a and b), and thus is the first step to a systematic exploration of the properties of the real number system.
 - It allows the reference to "unknown" numbers, the formulation of equations and the study of how to solve these (for instance, "Find a number x such that $3x + 1 = 10$ ").
 - It allows the formulation of functional relationships (such as "If you sell x tickets, then your profit will be $3x - 10$ dollars, or $f(x) = 3x - 10$, where f is the function, and x is the number to which the function is applied.").
-

Polynomials

A **polynomial** is an expression that is constructed from one or more variables and constants, using only the operations of addition, subtraction, and multiplication (where repeated multiplication of the same variable is standardly denoted as exponentiation with a constant non-negative whole number exponent). For example, $x^2 + 2x - 3$ is a polynomial in the single variable x .

An important class of problems in algebra is factorization of polynomials, that is, expressing a given polynomial as a product of other polynomials. The example polynomial above can be factored as $(x - 1)(x + 3)$. A related class of problems is finding algebraic expressions for the roots of a polynomial in a single variable.

Abstract algebra

Abstract algebra extends the familiar concepts found in elementary algebra and arithmetic of numbers to more general concepts.

Sets: Rather than just considering the different types of numbers, abstract algebra deals with the more general concept of *sets*: a collection of all objects (called elements) selected by property, specific for the set. All collections of the familiar types of numbers are sets. Other examples of sets include the set of all two-by-two matrices, the set of all second-degree polynomials ($ax^2 + bx + c$), the set of all two dimensional vectors in the plane, and the various finite groups such as the cyclic groups which are the group of integers modulo n . Set theory is a branch of logic and not technically a branch of algebra.

Binary operations: The notion of addition (+) is abstracted to give a *binary operation*, $\square$ say. The notion of binary operation is meaningless without the set on which the operation is defined. For two elements a and b in a set S , $a \square b$ is another element in the set; this condition is called closure. Addition (+), subtraction (-), multiplication ($\times$), and division ($\div$) can be binary operations when defined on different sets, as is addition and multiplication of matrices, vectors, and polynomials.

Identity elements: The numbers zero and one are abstracted to give the notion of an *identity element* for an operation. Zero is the identity element for addition and one is the identity element for multiplication. For a general binary operator $\square$ the identity element e must satisfy $a \square e = a$ and $e \square a = a$. This holds for addition as $a + 0 = a$ and $0 + a = a$ and multiplication $a \times 1 = a$ and $1 \times a = a$. Not all set and operator combinations have an identity element; for example, the positive natural numbers (1, 2, 3, ...) have no identity element for addition.

Inverse elements: The negative numbers give rise to the concept of *inverse elements*. For addition, the inverse of a is $-a$, and for multiplication the inverse is $1/a$. A general inverse element a^{-1} must satisfy the property that $a \square a^{-1} = e$ and $a^{-1} \square a = e$.

Associativity: Addition of integers has a property called associativity. That is, the grouping of the numbers to be added does not affect the sum. For example: $(2 + 3) + 4 = 2 + (3 + 4)$. In general, this becomes $(a \square b) \square c = a \square (b \square c)$. This property is shared by most binary operations, but not subtraction or division or octonion multiplication.

Commutativity: Addition of integers also has a property called commutativity. That is, the order of the numbers to be added does not affect the sum. For example: $2+3=3+2$. In general, this becomes $a \square b = b \square a$. Only some binary operations have this property. It holds for the integers with addition and multiplication, but it does not hold for matrix

multiplication or quaternion multiplication .

Groups - structures of a set with a single binary operation

Combining the above concepts gives one of the most important structures in mathematics: a **group**. A group is a combination of a set S and a single binary operation $\square$, defined in any way you choose, but with the following properties:

- An identity element e exists, such that for every member a of S , $e \square a$ and $a \square e$ are both identical to a .
- Every element has an inverse: for every member a of S , there exists a member a^{-1} such that $a \square a^{-1}$ and $a^{-1} \square a$ are both identical to the identity element.
- The operation is associative: if a , b and c are members of S , then $(a \square b) \square c$ is identical to $a \square (b \square c)$.

If a group is also commutative—that is, for any two members a and b of S , $a \square b$ is identical to $b \square a$ —then the group is said to be Abelian.

For example, the set of integers under the operation of addition is a group. In this group, the identity element is 0 and the inverse of any element a is its negation, $-a$. The associativity requirement is met, because for any integers a , b and c , $(a + b) + c = a + (b + c)$

The nonzero rational numbers form a group under multiplication. Here, the identity element is 1, since $1 \times a = a \times 1 = a$ for any rational number a . The inverse of a is $1/a$, since $a \times 1/a = 1$.

The integers under the multiplication operation, however, do not form a group. This is because, in general, the multiplicative inverse of an integer is not an integer. For example, 4 is an integer, but its multiplicative inverse is $1/4$, which is not an integer.

The theory of groups is studied in group theory. A major result in this theory is the classification of finite simple groups, mostly published between about 1955 and 1983, which is thought to classify all of the finite simple groups into roughly 30 basic types.

Examples										
Set:	Natural numbers $\mathbf{N}$		Integers $\mathbf{Z}$		Rational numbers $\mathbf{Q}$ (also real $\mathbf{R}$ and complex $\mathbf{C}$ numbers)				Integers modulo 3: $\mathbf{Z}_3 = \{0, 1, 2\}$	
Operation	+	$\times$ (w/o zero)	+	$\times$ (w/o zero)	+	–	$\times$ (w/o zero)	$\div$ (w/o zero)	+	$\times$ (w/o zero)
Closed	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Identity	0	1	0	1	0	N/A	1	N/A	0	1
Inverse	N/A	N/A	$-a$	N/A	$-a$	N/A	$1/a$	N/A	0, 2, 1, respectively	N/A, 1, respectively
Associative	Yes	Yes	Yes	Yes	Yes	No	Yes	No	Yes	Yes
Commutative	Yes	Yes	Yes	Yes	Yes	No	Yes	No	Yes	Yes
Structure	monoid	monoid	Abelian group	monoid	Abelian group	quasigroup	Abelian group	quasigroup	Abelian group	Abelian group ($\mathbf{Z}_2$)

Semigroups, quasigroups, and monoids are structures similar to groups, but more general. They comprise a set and a closed binary operation, but do not necessarily satisfy the other

conditions. A semigroup has an *associative* binary operation, but might not have an identity element. A monoid is a semigroup which does have an identity but might not have an inverse for every element. A quasigroup satisfies a requirement that any element can be turned into any other by a unique pre- or post-operation; however the binary operation might not be associative.

All groups are monoids, and all monoids are semigroups.

Rings and fields—structures of a set with two particular binary operations, (+) and (×)

Groups just have one binary operation. To fully explain the behaviour of the different types of numbers, structures with two operators need to be studied. The most important of these are rings, and fields.

Distributivity generalised the *distributive law* for numbers, and specifies the order in which the operators should be applied, (called the precedence). For the integers $(a + b) \times c = a \times c + b \times c$ and $c \times (a + b) = c \times a + c \times b$, and $\times$ is said to be *distributive* over $+$.

A **ring** has two binary operations (+) and (×), with $\times$ distributive over $+$. Under the first operator (+) it forms an *Abelian group*. Under the second operator (×) it is associative, but it does not need to have identity, or inverse, so division is not allowed. The additive (+) identity element is written as 0 and the additive inverse of a is written as $-a$.

The integers are an example of a ring. The integers have additional properties which make it an **integral domain**.

A **field** is a *ring* with the additional property that all the elements excluding 0 form an *Abelian group* under $\times$. The multiplicative (×) identity is written as 1 and the multiplicative inverse of a is written as a^{-1} .

The rational numbers, the real numbers and the complex numbers are all examples of fields.

Objects called algebras

The word **algebra** is also used for various algebraic structures:

- Algebra over a field or more generally Algebra over a ring
- Algebra over a set
- Boolean algebra
- F-algebra and F-coalgebra in category theory
- Relational algebra
- Sigma-algebra
- T-Algebras of monads.

See also

- Fundamental theorem of algebra
- List of basic algebra topics
- List of mathematics articles
- Order of operations

Notes

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- [2] Diophantus, Father of Algebra (<http://library.thinkquest.org/25672/diophan.htm>)
- [3] History of Algebra (<http://www.algebra.com/algebra/about/history/>)
- [4] Carl B. Boyer, *A History of Mathematics, Second Edition* (Wiley, 1991), pages 178, 181
- [5] Carl B. Boyer, *A History of Mathematics, Second Edition* (Wiley, 1991), page 228
- [6] (Boyer 1991, "The Arabic Hegemony" p. 229) "It is not certain just what the terms *al-jabr* and *muqabalah* mean, but the usual interpretation is similar to that implied in the translation above. The word *al-jabr* presumably meant something like "restoration" or "completion" and seems to refer to the transposition of subtracted terms to the other side of an equation; the word *muqabalah* is said to refer to "reduction" or "balancing" - that is, the cancellation of like terms on opposite sides of the equation."
- [7] (Boyer 1991, "The Arabic Hegemony" p. 230) "The six cases of equations given above exhaust all possibilities for linear and quadratic equations having positive root. So systematic and exhaustive was al-Khwarizmi's exposition that his readers must have had little difficulty in mastering the solutions."
- [8] Gandz and Saloman (1936), *The sources of al-Khwarizmi's algebra*, Osiris i, p. 263-277: "In a sense, Khwarizmi is more entitled to be called "the father of algebra" than Diophantus because Khwarizmi is the first to teach algebra in an elementary form and for its own sake, Diophantus is primarily concerned with the theory of numbers".
- [9] Rashed, R.; Armstrong, Angela (1994), *The Development of Arabic Mathematics*, Springer, pp. 11-2, ISBN 0792325656, OCLC 29181926 (<http://worldcat.org/oclc/29181926>)
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- [11] Victor J. Katz, Bill Barton (October 2007), "Stages in the History of Algebra with Implications for Teaching", *Educational Studies in Mathematics* (Springer Netherlands) **66** (2): 185-201 [192], doi: 10.1007/s10649-006-9023-7 (<http://dx.doi.org/10.1007/s10649-006-9023-7>)
- [12] (Boyer 1991, "The Arabic Hegemony" p. 239) "Abu'l Wefa was a capable algebraist as well as a trigonometer. [...] His successor al-Karkhi evidently used this translation to become an Arabic disciple of Diophantus - but without Diophantine analysis! [...] In particular, to al-Karkhi is attributed the first numerical solution of equations of the form $ax^{2n} + bx^n = c$ (only equations with positive roots were considered),"

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- L. Euler: *Elements of Algebra* (<http://web.mat.bham.ac.uk/C.J.Sangwin/euler/>), ISBN 978-1-89961-873-6

External links

- 4000 Years of Algebra (<http://www.gresham.ac.uk/event.asp?PageId=45&EventId=620>), lecture by Robin Wilson, at Gresham College, October 17, 2007 (available for MP3 and MP4 download, as well as a text file).
- Algebra (<http://plato.stanford.edu/entries/algebra>) entry in the *Stanford Encyclopedia of Philosophy* by Vaughan Pratt

Abstract Algebra

1. REDIRECT Abstract algebra

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Algebraic logic

In mathematical logic, **algebraic logic** is the study of logic presented in an algebraic style.

Algebras as models of logics

Algebraic logic treats algebraic structures, often bounded lattices, as models (interpretations) of certain logics, making logic a branch of order theory.

In algebraic logic:

- Variables are tacitly universally quantified over some universe of discourse. There are no existentially quantified variables or open formulas;
- Terms are built up from variables using primitive and defined operations. There are no connectives;
- Formulas, built from terms in the usual way, can be equated if they are logically equivalent. To express a tautology, equate a formula with a truth value;
- The rules of proof are the substitution of equals for equals, and uniform replacement. Modus ponens remains valid, but is seldom employed.

In the table below, the left column contains one or more logical or mathematical systems, and the algebraic structure which are its models are shown on the right in the same row. Some of these structures are either Boolean algebras or proper extensions thereof. Modal and other nonclassical logics are typically modeled by what are called "Boolean algebras with operators."

Algebraic formalisms going beyond first-order logic in at least some respects include:

- Combinatory logic, having the expressive power of set theory;
-

- Relation algebra, arguably the paradigmatic algebraic logic, can express Peano arithmetic and most axiomatic set theories, including the canonical ZFC.

logical system	its models
Classical sentential logic	Lindenbaum-Tarski algebra Two-element Boolean algebra
Intuitionistic propositional logic	Heyting algebra
Łukasiewicz logic	MV-algebra
Modal logic K	Modal algebra
Lewis's S4	Interior algebra
Lewis's S5; Monadic predicate logic	Monadic Boolean algebra
First-order logic	Cylindric algebra Polyadic algebra Predicate functor logic
Set theory	Combinatory logic Relation algebra

History

On the history of algebraic logic before World War II, see Brady (2000) and Grattan-Guinness (2000) and their ample references. On the postwar history, see Maddux (1991) and Quine (1976).

Algebraic logic has at least two meanings:

- The study of Boolean algebra, begun by George Boole, and of relation algebra, begun by Augustus DeMorgan, extended by Charles Sanders Peirce, and taking definitive form in the work of Ernst Schröder;
- Abstract algebraic logic, a branch of contemporary mathematical logic.

Perhaps surprisingly, algebraic logic is the oldest approach to formal logic, arguably beginning with a number of memoranda Leibniz wrote in the 1680s, some of which were published in the 19th century and translated into English by Clarence Lewis in 1918. But nearly all of Leibniz's known work on algebraic logic was published only in 1903, after Louis Couturat discovered it in Leibniz's Nachlass. Parkinson (1966) and Loemker (1969) translated selections from Couturat's volume into English.

Brady (2000) discusses the rich historical connections between algebraic logic and model theory. The founders of model theory, Ernst Schroder and Leopold Loewenheim, were logicians in the algebraic tradition. Alfred Tarski, the founder of set theoretic model theory as a major branch of contemporary mathematical logic, also:

- Co-discovered Lindenbaum-Tarski algebra;
- Invented cylindric algebra;
- Wrote the 1940 paper that revived relation algebra, and that can be seen as the starting point of abstract algebraic logic.

Modern mathematical logic began in 1847, with two pamphlets whose respective authors were Augustus DeMorgan and George Boole. They, and later C.S. Peirce, Hugh MacColl, Frege, Peano, Bertrand Russell, and A. N. Whitehead all shared Leibniz's dream of combining symbolic logic, mathematics, and philosophy. Relation algebra is arguably the culmination of Leibniz's approach to logic. With the exception of some writings by Leopold Loewenheim and Thoralf Skolem, algebraic logic went into eclipse soon after the 1910-13 publication of *Principia Mathematica*, not to revive until Tarski's 1940 reexposition of

relation algebra.

Leibniz had no influence on the rise of algebraic logic because his logical writings were little studied before the Parkinson and Loemker translations. Our present understanding of Leibniz the logician stems mainly from the work of Wolfgang Lenzen, summarized in Lenzen (2004).^[1] To see how present-day work in logic and metaphysics can draw inspiration from, and shed light on, Leibniz's thought, see Zalta (2000).^[2]

See also

- Abstract algebraic logic
- Algebraic structure
- Boolean algebra (logic)
- Boolean algebra (structure)
- Cylindric algebra
- Lindenbaum-Tarski algebra
- Mathematical logic
- Model theory
- Monadic Boolean algebra
- Predicate functor logic
- Relation algebra
- Universal algebra

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External links

- Stanford Encyclopedia of Philosophy: "Propositional Consequence Relations and Algebraic Logic ^[6]" -- by Ramon Jansana.

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- [3] http://www.elsevier.com/wps/find/bookdescription.cws_home/621535/description
- [4] <http://plato.stanford.edu/entries/algebra-logic-tradition/>
- [5] <http://mally.stanford.edu/leibniz.pdf>
- [6] <http://plato.stanford.edu/entries/consequence-algebraic/>

Galois group

In mathematics, a **Galois group** is a group associated with a certain type of field extension. The study of field extensions (and polynomials which give rise to them) via Galois groups is called Galois theory after Évariste Galois who first invented them.

For a more elementary discussion of Galois groups in terms of permutation groups, see the article on Galois theory.

Definition

Suppose that E is an extension of the field F (written as E/F and read E over F). Consider the set of all automorphisms of E/F (that is, isomorphisms α from E to itself such that $\alpha(x) = x$ for every x in F). This set of automorphisms with the operation of function composition forms a group, sometimes denoted by $\text{Aut}(E/F)$.

If E/F is a Galois extension, then $\text{Aut}(E/F)$ is called the **Galois group of (the extension) E over F** , and is usually denoted by $\text{Gal}(E/F)$.

Examples

In the following examples F is a field, and $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Q}$ are the fields of complex, real, and rational numbers, respectively. The notation $F(a)$ indicates the field extension obtained by adjoining an element a to the field F .

- $\text{Gal}(F/F)$ is the trivial group that has a single element, namely the identity automorphism.
- $\text{Gal}(\mathbf{C}/\mathbf{R})$ has two elements, the identity automorphism and the complex conjugation automorphism.
- $\text{Aut}(\mathbf{R}/\mathbf{Q})$ is trivial. Indeed it can be shown that any $\mathbf{Q}$ -automorphism must preserve the ordering of the real numbers and hence must be the identity.
- $\text{Aut}(\mathbf{C}/\mathbf{Q})$ is an infinite group.
- $\text{Gal}(\mathbf{Q}(\sqrt{2})/\mathbf{Q})$ has two elements, the identity automorphism and the automorphism which exchanges $\sqrt{2}$ and $-\sqrt{2}$.
- Consider the field $K = \mathbf{Q}(\sqrt[3]{2})$. The group $\text{Aut}(K/\mathbf{Q})$ contains only the identity automorphism. This is because K is not a normal extension, since the other two cube roots of 2 (both complex) are missing from the extension — in other words K is not a splitting field.

- Consider now $L = \mathbf{Q}(\sqrt[3]{2}, \omega)$, where ω is a primitive third root of unity. The group $\text{Gal}(L/\mathbf{Q})$ is isomorphic to S_3 , the dihedral group of order 6, and L is in fact the splitting field of $x^3 - 2$ over $\mathbf{Q}$.

Facts

The significance of an extension being Galois is that it obeys the fundamental theorem of Galois theory: the subgroups of the Galois group correspond to the intermediate fields of the field extension.

If E/F is a Galois extension, then $\text{Gal}(E/F)$ can be given a topology, called the Krull topology, that makes it into a profinite group.

See also

- Absolute Galois group

External links

- Galois Groups ^[1] at MathPages

References

[1] <http://www.mathpages.com/home/kmath290/kmath290.htm>

Galois theory

In mathematics, more specifically in abstract algebra, **Galois theory**, named after Évariste Galois, provides a connection between field theory and group theory. Using Galois theory, certain problems in field theory can be reduced to group theory, which is in some sense simpler and better understood.

Originally Galois used permutation groups to describe how the various roots of a given polynomial equation are related to each other. The modern approach to Galois theory, developed by Richard Dedekind, Leopold Kronecker and Emil Artin, among others, involves studying automorphisms of field extensions.

Further abstraction of Galois theory is achieved by the theory of Galois connections.

Application to classical problems

The birth of Galois theory was originally motivated by the following question, whose answer is known as the Abel-Ruffini theorem.

"Why is there no formula for the roots of a fifth (or higher) degree polynomial equation in terms of the coefficients of the polynomial, using only the usual algebraic operations (addition, subtraction, multiplication, division) and application of radicals (square roots, cube roots, etc)?"

Galois theory not only provides a beautiful answer to this question, it also explains in detail why it is possible to solve equations of degree four or lower in the above manner, and why their solutions take the form that they do. Further, it gives a conceptually clear, and often practical, means of telling when some particular equation of higher degree can be solved in

that manner.

Galois theory also gives a clear insight into questions concerning problems in compass and straightedge construction. It gives an elegant characterisation of the ratios of lengths that can be constructed with this method. Using this, it becomes relatively easy to answer such classical problems of geometry as

"Which regular polygons are constructible polygons?"

"Why is it not possible to trisect every angle?"

History

Galois theory originated in the study of symmetric functions – the coefficients of a polynomial are (up to sign) the elementary symmetric polynomials in the roots. For instance, $(x - a)(x - b) = x^2 - (a + b)x + ab$, where $a + b$ and ab are the elementary polynomials of degree 1 and 2 in 2 variables.

This was first formalized by the 16th century French mathematician François Viète, in Viète's formulas, for the case of positive real roots. In the opinion of the 18th century British mathematician Charles Hutton,^[1] the expression of coefficients of a polynomial in terms of the roots (not only for positive roots) was first understood by the 17th century French mathematician Albert Girard; Hutton writes:

...[Girard was] the first person who understood the general doctrine of the formation of the coefficients of the powers from the sum of the roots and their products. He was the first who discovered the rules for summing the powers of the roots of any equation.

In this vein, the discriminant is a symmetric function in the coefficients (and hence also in the roots) which reflects properties of the roots – it is zero if and only if the polynomial has a multiple root, and for quadratic and cubic polynomials it is positive if and only if all roots are real and distinct, and negative if and only if there is a pair of distinct complex conjugate roots. See Discriminant: nature of the roots for details.

The cubic was first partly solved by the 15th/16th century Italian mathematician Scipione del Ferro, who did not however publish his results; this method only solved one of three classes, as the others involved taking square roots of negative numbers, and complex numbers were not known at the time. This solution was then rediscovered independently in 1535 by Niccolò Fontana Tartaglia, who shared it with Gerolamo Cardano, asking him to not publish it. Cardano then extended this to the other two cases, using square roots of negatives as intermediate steps; see details at Cardano's method. After the discovery of Ferro's work, he felt that Tartaglia's method was no longer secret, and thus he published his complete solution in his 1545 *Ars Magna*. His student Lodovico Ferrari solved the quartic polynomial, which solution Cardano included in *Ars Magna*.

A further step was the 1770 paper *Réflexions sur la résolution algébrique des équations* by the French-Italian mathematician Joseph Louis Lagrange, in his method of Lagrange resolvents, where he analyzed Cardano and Ferrari's solution of cubics and quartics by considering them in terms of *permutations* of the roots, which yielded an auxiliary polynomial of lower degree, providing a unified understanding of the solutions and laying the groundwork for group theory and Galois theory. Crucially, however, he did not consider *composition* of permutations. Lagrange's method did not extend to quintic equations or higher, because the resolvent had higher degree.

The quintic was almost proven to have no general solutions by radicals by Paolo Ruffini in 1799, whose key insight was to use permutation *groups*, not just a single permutation. His solution contained a gap, which Cauchy considered minor, though this was not patched until the work of Norwegian mathematician Niels Henrik Abel, who published a proof in 1824, thus establishing the Abel-Ruffini theorem.

While Ruffini and Abel established that the *general* quintic could not be solved, some *particular* quintics can be solved, such as $(x - 1)^5$, and the precise criterion by which a *given* quintic or higher polynomial could be determined to be solvable or not was given by Évariste Galois, who showed that whether a polynomial was solvable or not was equivalent to whether or not the permutation group of its roots – in modern terms, its Galois group – had a certain structure – in modern terms, whether or not it was a solvable group. This group was always solvable for polynomials of degree four or less, but not always so for polynomials of degree five and greater, which explains why there is no general solution in higher degree.

The permutation group approach to Galois theory

If we are given a polynomial, it may happen that some of the roots of the polynomial are connected by various algebraic equations. For example, it may turn out that for two of the roots, say A and B , the equation $A^2 + 5B^3 = 7$ holds. The central idea of Galois theory is to consider those permutations (or rearrangements) of the roots having the property that *any* algebraic equation satisfied by the roots is *still satisfied* after the roots have been permuted. An important proviso is that we restrict ourselves to algebraic equations whose coefficients are rational numbers. (One might instead specify a certain field in which the coefficients should lie, but for the simple examples below, we will restrict ourselves to the field of rational numbers.)

These permutations together form a permutation group, also called the Galois group of the polynomial (over the rational numbers). This can be made much clearer by way of example.

First example — a quadratic equation

Consider the quadratic equation

$$x^2 - 4x + 1 = 0.$$

By using the quadratic formula, we find that the two roots are

$$A = 2 + \sqrt{3}$$

$$B = 2 - \sqrt{3}.$$

Examples of algebraic equations satisfied by A and B include

$$A + B = 4, \quad \text{and}$$

$$AB = 1.$$

Obviously, in either of these equations, if we exchange A and B , we obtain another true statement. For example, the equation $A + B = 4$ becomes simply $B + A = 4$. Furthermore, it is true, but far less obvious, that this holds for *every* possible algebraic equation with rational coefficients satisfied by the roots A and B ; to prove this requires the theory of symmetric polynomials.

We conclude that the Galois group of the polynomial $x^2 - 4x + 1$ consists of two permutations: the identity permutation which leaves A and B untouched, and the

transposition permutation which exchanges A and B . It is a cyclic group of order two, and therefore isomorphic to $\mathbf{Z}/2\mathbf{Z}$.

One might object that A and B are related by yet another algebraic equation,

$$A - B - 2\sqrt{3} = 0$$

which does *not* remain true when A and B are exchanged. However, this equation does not concern us, because it does not have rational coefficients; in particular, $-2\sqrt{3}$ is not rational.

A similar discussion applies to any quadratic polynomial $ax^2 + bx + c$, where a , b and c are rational numbers.

- If the polynomial has only one root, for example $x^2 - 4x + 4 = (x-2)^2$, then the Galois group is trivial; that is, it contains only the identity permutation.
- If it has two distinct *rational* roots, for example $x^2 - 3x + 2 = (x-2)(x-1)$, the Galois group is again trivial.
- If it has two *irrational* roots (including the case where the roots are complex), then the Galois group contains two permutations, just as in the above example.

Second example — somewhat trickier

Consider the polynomial

$$x^4 - 10x^2 + 1,$$

which can also be written as

$$(x^2 - 5)^2 - 24.$$

We wish to describe the Galois group of this polynomial, again over the field of rational numbers. The polynomial has four roots:

$$A = \sqrt{2} + \sqrt{3}$$

$$B = \sqrt{2} - \sqrt{3}$$

$$C = -\sqrt{2} + \sqrt{3}$$

$$D = -\sqrt{2} - \sqrt{3}.$$

There are 24 possible ways to permute these four roots, but not all of these permutations are members of the Galois group. The members of the Galois group must preserve any algebraic equation with rational coefficients involving A , B , C and D . One such equation is

$$A + D = 0.$$

However, since

$$A + C = 2\sqrt{3} \neq 0,$$

the permutation

$$(A, B, C, D) \rightarrow (A, B, D, C)$$

is not permitted (because it transforms the valid equation $A + D = 0$ into the invalid equation $A + C = 0$).

Another equation that the roots satisfy is

$$(A + B)^2 = 8.$$

This will exclude further permutations, such as

$$(A, B, C, D) \rightarrow (A, C, B, D).$$

Continuing in this way, we find that the only permutations (satisfying both equations simultaneously) remaining are

$$(A, B, C, D) \rightarrow (A, B, C, D)$$

$$(A, B, C, D) \rightarrow (C, D, A, B)$$

$$(A, B, C, D) \rightarrow (B, A, D, C)$$

$$(A, B, C, D) \rightarrow (D, C, B, A),$$

and the Galois group is isomorphic to the Klein four-group.

The modern approach by field theory

In the modern approach, one starts with a field extension L/K (read: L over K), and examines the group of field automorphisms of L/K (these are mappings $\alpha: L \rightarrow L$ with $\alpha(x) = x$ for all x in K). See the article on Galois groups for further explanation and examples.

The connection between the two approaches is as follows. The coefficients of the polynomial in question should be chosen from the base field K . The top field L should be the field obtained by adjoining the roots of the polynomial in question to the base field. Any permutation of the roots which respects algebraic equations as described above gives rise to an automorphism of L/K , and vice versa.

In the first example above, we were studying the extension $\mathbb{Q}(\sqrt{3})/\mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers, and $\mathbb{Q}(\sqrt{3})$ is the field obtained from $\mathbb{Q}$ by adjoining $\sqrt{3}$. In the second example, we were studying the extension $\mathbb{Q}(A, B, C, D)/\mathbb{Q}$.

There are several advantages to the modern approach over the permutation group approach.

- It permits a far simpler statement of the fundamental theorem of Galois theory.
- The use of base fields other than $\mathbb{Q}$ is crucial in many areas of mathematics. For example, in algebraic number theory, one often does Galois theory using number fields, finite fields or local fields as the base field.
- It allows one to more easily study infinite extensions. Again this is important in algebraic number theory, where for example one often discusses the absolute Galois group of $\mathbb{Q}$, defined to be the Galois group of $K/\mathbb{Q}$ where K is an algebraic closure of $\mathbb{Q}$.
- It allows for consideration of inseparable extensions. This issue does not arise in the classical framework, since it was always implicitly assumed that arithmetic took place in characteristic zero, but nonzero characteristic arises frequently in number theory and in algebraic geometry.
- It removes the rather artificial reliance on chasing roots of polynomials. That is, different polynomials may yield the same extension fields, and the modern approach recognizes the connection between these polynomials.

Solvable groups and solution by radicals

The notion of a solvable group in group theory allows one to determine whether a polynomial is solvable in the radicals, depending on whether its Galois group has the property of solvability. In essence, each field extension L/K corresponds to a factor group in a composition series of the Galois group. If a factor group in the composition series is cyclic of order n , then if the corresponding field extension is an extension of a field containing a primitive root of unity, then it is a radical extension, and the elements of L can

then be expressed using the n th root of some element of K .

If all the factor groups in its composition series are cyclic, the Galois group is called *solvable*, and all of the elements of the corresponding field can be found by repeatedly taking roots, products, and sums of elements from the base field (usually $\mathbf{Q}$).

One of the great triumphs of Galois Theory was the proof that for every $n > 4$, there exist polynomials of degree n which are not solvable by radicals—the Abel-Ruffini theorem. This is due to the fact that for $n > 4$ the symmetric group S_n contains a simple, non-cyclic, normal subgroup.

A non-solvable quintic example

Van der Waerden cites the polynomial $f(x) = x^5 - x - 1$. By the rational root theorem it has no rational zeros. Neither does it have linear factors modulo 2 or 3.

$f(x)$ has the factorization $(x^2 + x + 1)(x^3 + x^2 + 1)$ modulo 2. That means its Galois group modulo 2 is cyclic of order 6.

$f(x)$ has no quadratic factor modulo 3. Thus its Galois group modulo 3 has order 5.

A permutation group on 5 objects with operations of orders 6 and 5 must be the symmetric group S_5 , which must be the Galois group of $f(x)$. This is one of the simplest examples of a non-solvable quintic polynomial. Serge Lang said that Artin was fond of this example.

The inverse Galois problem

All finite groups do occur as Galois groups. It is easy to construct field extensions with any given finite group as Galois group, as long as one does not also specify the ground field.

For that, choose a field K and a finite group G . Cayley's theorem says that G is (up to isomorphism) a subgroup of the symmetric group S on the elements of G . Choose indeterminates $\{x_\alpha\}$, one for each element α of G , and adjoin them to K to get the field $F = K(\{x_\alpha\})$. Contained within F is the field L of symmetric rational functions in the $\{x_\alpha\}$. The Galois group of F/L is S , by a basic result of Emil Artin. G acts on F by restriction of action of S . If the fixed field of this action is M , then, by the fundamental theorem of Galois theory, the Galois group of F/M is G .

It is an open problem to prove the existence of a field extension of the rational field $\mathbf{Q}$ with a given finite group as Galois group. Hilbert played a part in solving the problem for all symmetric and alternating groups. Igor Shafarevich proved that every solvable finite group is the Galois group of some extension of $\mathbf{Q}$. Various people have solved the inverse Galois problem for selected non-abelian simple groups. Existence of solutions has been shown for all but possibly one (Mathieu group M_{23}) of the 26 sporadic simple groups. There is even a polynomial with integral coefficients whose Galois group is the Monster group.

See also

- Reed–Solomon error correction

Notes

[1] (Funkhouser 1930)

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External links

Some on-line tutorials on Galois theory appear at:

- <http://www.math.niu.edu/~beachy/aaol/galois.html>
- http://nrich.maths.org/public/viewer.php?obj_id=1422
- <http://www.jmilne.org/math/CourseNotes/math594f.html>

Online textbooks in French, German, Italian and English can be found at:

- <http://www.galois-group.net/>

Grothendieck's Galois theory

In mathematics, **Grothendieck's Galois theory** is a highly abstract approach to the Galois theory of fields, developed around 1960 to provide a way to study the fundamental group of algebraic topology in the setting of algebraic geometry. It provides, in the classical setting of field theory, an alternative perspective to that of Emil Artin based on linear algebra, which became standard from about the 1930s.

The approach of Alexander Grothendieck is concerned with the category-theoretic properties that characterise the categories of finite G -sets for a fixed profinite group G . For example, G might be the group denoted $\hat{\mathbb{Z}}$, which is the inverse limit of the cyclic additive groups $\mathbb{Z}/n\mathbb{Z}$ — or equivalently the completion of the infinite cyclic group $\mathbb{Z}$ for the topology of subgroups of finite index. A finite G -set is then a finite set X on which G acts through a quotient finite cyclic group, so that it is specified by giving some permutation of X .

In the above example, a connection with classical Galois theory can be seen by regarding $\hat{\mathbb{Z}}$ as the profinite Galois group $\text{Gal}(\bar{F}/F)$ of the algebraic closure $\bar{F}$ of any finite field F , over F . That is, the automorphisms of $\bar{F}$ fixing F are described by the inverse limit, as we take larger and larger finite splitting fields over F . The connection with geometry can be seen when we look at covering spaces of the unit disk in the complex plane with the origin removed: the finite covering realised by the z^n map of the disk, thought of by means of a complex number variable z , corresponds to the subgroup $n\mathbb{Z}$ of the fundamental group of the punctured disk.

The theory of Grothendieck, published in SGA1, shows how to reconstruct the category of G -sets from a *fibre functor* Φ , which in the geometric setting takes the fibre of a covering above a fixed base point (as a set). In fact there is an isomorphism proved of the type

$$G \cong \text{Aut}(\Phi),$$

the latter being the group of automorphisms (self-natural equivalences) of Φ . An abstract classification of categories with a functor to the category of sets is given, by means of which one can recognise categories of G -sets for G profinite.

To see how this applies to the case of fields, one has to study the tensor product of fields. Later developments in topos theory make this all part of a theory of *atomic toposes*.

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Notes on Grothendieck's Galois Theory http://arxiv.org/PS_cache/math/pdf/0009/0009145v1.pdf

Algebraic Topology

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1. REDIRECT Algebraic topology

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Open set

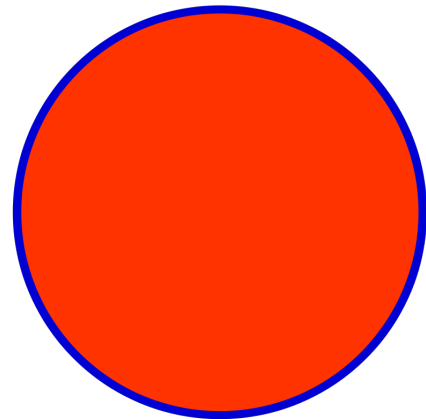
In mathematics, more specifically point-set topology and metric topology, the notion of an open set provides a fundamental way to speak of distance in a topological space, without explicitly defining a metric on the space. In particular, although one cannot obtain concrete values for the distance between two points in a topological space, one may still be able to speak of "nearness" in the space, thus allowing concepts such as continuity to translate into the theory of open sets.

Intuitively (see below for a more intuitive discussion), an open set provides one a method to distinguish two points. For example, if about one point in a topological space, there exists an open set not containing another (distinct) point, the two points are referred to as topologically distinguishable. In this manner, one may speak of whether two subsets of a topological space are "near" without concretely defining a metric on the topological space. Therefore, topological spaces may be seen as a generalization of metric spaces.

Point-set topology is the area of mathematics concerned with general topological spaces, and the relations between them. In the category of topological spaces, morphisms are continuous functions between topological spaces. Continuous functions are readily observed to preserve topological structure, as they map "points close together" to "points close together"; that is, they preserve the structure of open sets defined on the space.

In metric topology, one can concretely define a distance function between two points, and thus metric spaces also have a topology, i.e. a certain structure of open sets defined on them. Thus as opposed to the pure topological invariants, metric topology deals with isometries and the like; that is, distance preserving maps. In this case, the idea of an open set is used as an organizational tool rather than an object of study. From the topological point of view, metric spaces are fairly well understood, although many open problems still remain in metrization theory.

The concept of an open set is of fundamental importance in mathematics due to the numerous areas which exploit it. In algebraic geometry, the Zariski topology is a certain family of open sets which in some sense, reflect the algebraic nature of varieties. In differential topology, the open sets of a topological space are often required to have a simple structure - that is, each point within the space is required to have a neighbourhood homeomorphic to the open ball in Euclidean space. Although open sets are of central importance (and are studied) in point-set topology, they are also used as an organizational tool in other important branches of mathematics.



Example: The points (x, y) satisfying $x^2 + y^2 = r^2$ are colored blue. The points (x, y) satisfying $x^2 + y^2 < r^2$ are colored red. The red points form an open set. The union of the red and blue points is a closed set.

Informal discussion

Intuitively speaking, a set U is open if starting from any point x in U one can move by a small amount in any direction and still be in the set U . In other words, the distance between any point x in U and the edge of U is always greater than zero.

As an example, consider the open interval $(0, 1)$ consisting of all real numbers x with $0 < x < 1$. Here, the topology is the usual topology on the real line. We can look at this in two ways. Since any point in the interval is different from 0 and 1, the distance from that point to the edge is always non-zero. Or equivalently, for any point in the interval we can move by a small enough amount in any direction without touching the edge and still be inside the set. Therefore, the interval $(0, 1)$ is open. However, the interval $(0, 1]$ consisting of all numbers x with $0 < x \leq 1$ is not open in the topology induced from the real line; if one takes $x = 1$ and moves an arbitrarily small amount in the positive direction, one will be outside of $(0, 1]$.

Definitions

The concept of open sets can be formalized with various degrees of generality, for example:

Geometric

A point set in $\mathbf{R}^n$ is called *open* when every point P of the set is an interior point.

Euclidean space

A subset U of the Euclidean n -space $\mathbf{R}^n$ is called *open* if, given any point x in U , there exists a real number $\varepsilon > 0$ such that, given any point y in $\mathbf{R}^n$ whose Euclidean distance from x is smaller than ε , y also belongs to U . Equivalently, U is open if every point in U has a neighbourhood contained in U .

Metric spaces

A subset U of a metric space (M, d) is called *open* if, given any point x in U , there exists a real number $\varepsilon > 0$ such that, given any point y in M with $d(x, y) < \varepsilon$, y also belongs to U . Equivalently, U is open if every point in U has a neighbourhood contained in U .

This generalises the Euclidean space example, since Euclidean space with the Euclidean distance is a metric space.

Topological spaces

If a nonempty set X has a collection of subsets T that is a topological space, then any member of T is an open set.

Note that infinite intersections of open sets need not be open. For example, the intersection of all intervals of the form $(-1/n, 1/n)$, where n is a positive integer, is the set $\{0\}$ which is closed in the real line. Sets that can be constructed as the intersection of countably many open sets are denoted $\mathbf{G}_\delta$ sets.

The topological definition of open sets generalises the metric space definition: If one begins with a metric space and defines open sets as before, then the family of all open sets is a topology on the metric space. Every metric space is therefore, in a natural way, a topological space. There are, however, topological spaces that are not metric spaces.

Properties

- The empty set is both open and closed.
- The union of any number of open sets is open.
- The intersection of a finite number of open sets is open.

Uses

Open sets have a fundamental importance in topology. The concept is required to define and make sense of topological space and other topological structures that deal with the notions of closeness and convergence for spaces such as metric spaces and uniform spaces.

Every subset A of a topological space X contains a (possibly empty) open set; the largest such open set is called the interior of A . It can be constructed by taking the union of all the open sets contained in A .

Given topological spaces X and Y , a function f from X to Y is *continuous* if the preimage of every open set in Y is open in X . The map f is called *open* if the image of every open set in X is open in Y .

An open set on the real line has the characteristic property that it is a countable union of disjoint open intervals.

Note

Note that whether a given set U is open depends on the surrounding space. For instance, if U is defined as the set of rational numbers in the interval $(0, 1)$, then U is open *in the rational numbers*, but not open *in the real numbers*. This is because when U is *in the rational numbers* there are no irrational numbers that can be moved to—the smallest possible displacement is from one rational number to another. Also, no matter how close an element of U is to 0 or 1, there is always another rational number between it and 0 or 1, so from any element of U there is always a way to make a small enough displacement that you can get closer to 0 or 1 while staying inside U . But, when this set is *in the real numbers*, there are irrational numbers between all of the rational numbers and it is possible to move from an element of U to an irrational number (which is not an element of U). So, for any displacement from some beginning element of U to some ending element, there is always a smaller distance from the beginning element to an irrational number which is outside of U . (Even though the irrational number may be between 0 and 1, it is not in U because U contains only rational numbers.)

Some sets are both open and closed (called *clopen sets*); in $\mathbf{R}$ and other connected spaces, only the empty set and the whole space are clopen, while the set of all rational numbers smaller than $\sqrt{2}$ is clopen in the rationals. While others are neither open nor closed, such as $(0, 1]$ in $\mathbf{R}$. In fact, the set $(0, 1]$ is the union of the sets $(0, 1)$ and $\{1\}$, an open set and a closed set respectively. An important point is that an open set is not the opposite of "closed set", rather a closed set is the complement of an open set.

See also

- Closed set
- Clopen set
- Neighbourhood

External links

- Open Set ^[1] on PlanetMath

References

[1] <http://planetmath.org/?op=getobj&from=objects&id=2925>

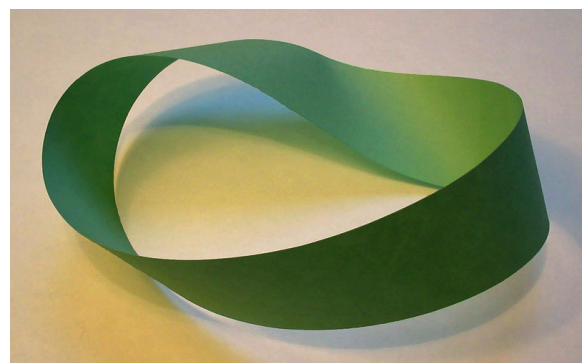
Topology

Topology (from the Greek τόπος, “place”, and λόγος, “study”) is a major area of mathematics concerned with spatial properties that are preserved under bicontinuous deformation; that is, stretching without either tearing or gluing. It emerged through the development of concepts from geometry and set theory, such as those of space, dimension, shape, transformation and others.

Ideas that are now classified as topological were expressed as early as 1736, and toward the end of the 19th century a distinct discipline developed, called in Latin the *geometria situs* (“geometry of place”) or *analysis situs* (Greek-Latin for “picking apart of place”), and later gaining the modern name of topology. In the middle of the 20th century, this was an important growth area within mathematics.

The word *topology* is used both for the mathematical discipline and for a family of sets with certain properties that are used to define a topological space, a basic object of topology. Of particular importance are *homeomorphisms*, which can be defined as continuous functions with a continuous inverse. For instance, the function $y = x^3$ is a homeomorphism of the real line.

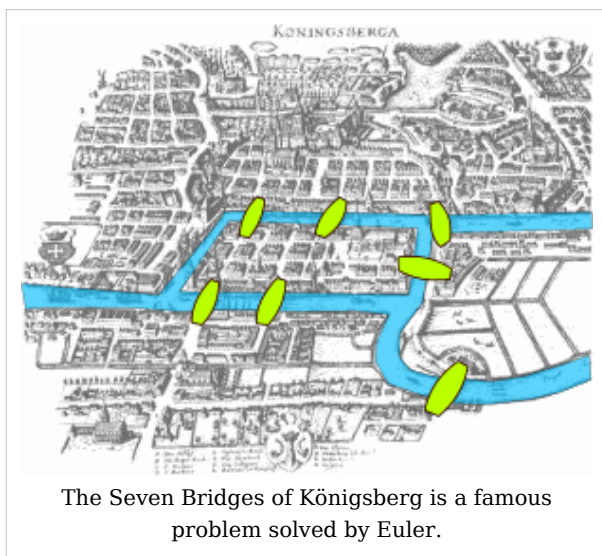
Topology includes many subfields. The most basic and traditional division within topology is **point-set topology**, which establishes the foundational aspects of topology and investigates concepts inherent to topological spaces - basic examples being compactness and connectedness; **algebraic topology**, which generally tries to measure degrees of connectivity using algebraic constructs such as homotopy groups and homology; and **geometric topology**, which primarily studies manifolds and their embeddings (placements) in other manifolds. Some of the most active areas, such as low dimensional topology and graph theory, do not fit neatly in this division.



A Möbius strip, an object with only one surface and one edge. Such shapes are an object of study in topology.

See also: topology glossary for definitions of some of the terms used in topology and topological space for a more technical treatment of the subject.

History



Topology began with the investigation of certain questions in geometry. Euler's 1736 paper on *Seven Bridges of Königsberg* is regarded as one of the first topological results.

The term "Topologie" was introduced in German in 1847 by Johann Benedict Listing in *Vorstudien zur Topologie*, Vandenhoeck und Ruprecht, Göttingen, pp. 67, 1848, who had used the word for ten years in correspondence before its first appearance in print. "Topology," its English form, was introduced in 1883 in the journal *Nature* to distinguish "qualitative geometry from the ordinary geometry in which quantitative

relations chiefly are treated". The term **topologist** in the sense of a specialist in topology was used in 1905 in the magazine *Spectator*. However, none of these uses corresponds exactly to the modern definition of topology.

Modern topology depends strongly on the ideas of set theory, developed by Georg Cantor in the later part of the 19th century. Cantor, in addition to setting down the basic ideas of set theory, considered point sets in Euclidean space, as part of his study of Fourier series.

Henri Poincaré published *Analysis Situs* in 1895, introducing the concepts of homotopy and homology, which are now considered part of algebraic topology.

Maurice Fréchet, unifying the work on function spaces of Cantor, Volterra, Arzelà, Hadamard, Ascoli and others, introduced the metric space in 1906. A metric space is now considered a special case of a general topological space. In 1914, Felix Hausdorff coined the term "topological space" and gave the definition for what is now called a Hausdorff space. In current usage, a topological space is a slight generalization of Hausdorff spaces, given in 1922 by Kazimierz Kuratowski.

For further developments, see point-set topology and algebraic topology.

Elementary introduction

Topology, as a branch of mathematics can be formally defined as "the study of qualitative properties of certain objects (called topological spaces) that are invariant under certain kind of transformations (called continuous maps), especially those properties that are invariant under a certain kind of equivalence (called homeomorphism)."

The term *topology* is also used to refer to a structure imposed upon a set X , a structure which essentially 'characterizes' the set X as a topological space by taking proper care of properties such as convergence, connectedness and continuity, upon transformation.

Topological spaces show up naturally in almost every branch of mathematics. This has made topology one of the great unifying ideas of mathematics.

The motivating insight behind topology is that some geometric problems depend not on the exact shape of the objects involved, but rather on the way they are put together. For example, the square and the circle have many properties in common: they are both one dimensional objects (from a topological point of view) and both separate the plane into two parts, the part inside and the part outside.

One of the first papers in topology was the demonstration, by Leonhard Euler, that it was impossible to find a route through the town of Königsberg (now Kaliningrad) that would cross each of its seven bridges exactly once. This result did not depend on the lengths of the bridges, nor on their distance from one another, but only on connectivity properties: which bridges are connected to which islands or riverbanks. This problem, the *Seven Bridges of Königsberg*, is now a famous problem in introductory mathematics, and led to the branch of mathematics known as graph theory.

Similarly, the hairy ball theorem of algebraic topology says that "one cannot comb the hair flat on a hairy ball without creating a cowlick." This fact is immediately convincing to most people, even though they might not recognize the more formal statement of the theorem, that there is no nonvanishing continuous tangent vector field on the sphere. As with the *Bridges of Königsberg*, the result does not depend on the exact shape of the sphere; it applies to pear shapes and in fact any kind of smooth blob, as long as it has no holes.

In order to deal with these problems that do not rely on the exact shape of the objects, one must be clear about just what properties these problems *do* rely on. From this need arises the notion of homeomorphism. The impossibility of crossing each bridge just once applies to any arrangement of bridges homeomorphic to those in Königsberg, and the hairy ball theorem applies to any space homeomorphic to a sphere.

Intuitively two spaces are homeomorphic if one can be deformed into the other without cutting or gluing. A traditional joke is that a topologist can't distinguish a coffee mug from a doughnut, since a sufficiently pliable doughnut could be reshaped to the form of a coffee



A continuous deformation (homeomorphism) of a coffee cup into a doughnut (torus) and back.

cup by creating a dimple and progressively enlarging it, while shrinking the hole into a handle. A precise definition of homeomorphic, involving a continuous function with a continuous inverse, is necessarily more technical.

Homeomorphism can be considered the most basic *topological equivalence*. Another is homotopy equivalence. This is harder to describe without getting technical, but the essential notion is that two objects are homotopy equivalent if they both result from "squishing" some larger object.

Equivalence classes of the English alphabet in uppercase sans-serif font (Myriad); left - homeomorphism, right - homotopy equivalence

{A, R} {B} {C, G, I, J, L, M, N, S, U, V, W, Z} {D, O} {E, F, T, Y} {H, K} {P, Q} {X}	{A, R, D, O, P, Q} {B} {C, E, F, G, H, I, J, K, L, M, N, S, T, U, V, W, X, Y, Z}
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An introductory exercise is to classify the uppercase letters of the English alphabet according to homeomorphism and homotopy equivalence. The result depends partially on the font used. The figures use a sans-serif font named Myriad. Notice that homotopy equivalence is a rougher relationship than homeomorphism; a homotopy equivalence class can contain several of the homeomorphism classes. The simple case of homotopy equivalence described above can be used here to show two letters are homotopy equivalent, e.g. O fits inside P and the tail of the P can be squished to the "hole" part.

Thus, the homeomorphism classes are: one hole two tails, two holes no tail, no holes, one hole no tail, no holes three tails, a bar with four tails (the "bar" on the K is almost too short to see), one hole one tail, and no holes four tails.

The homotopy classes are larger, because the tails can be squished down to a point. The homotopy classes are: one hole, two holes, and no holes.

To be sure we have classified the letters correctly, we not only need to show that two letters in the same class are equivalent, but that two letters in different classes are not equivalent. In the case of homeomorphism, this can be done by suitably selecting points and showing their removal disconnects the letters differently. For example, X and Y are not homeomorphic because removing the center point of the X leaves four pieces; whatever point in Y corresponds to this point, its removal can leave at most three pieces. The case of homotopy equivalence is harder and requires a more elaborate argument showing an algebraic invariant, such as the fundamental group, is different on the supposedly differing classes.

Letter topology has some practical relevance in stencil typography. The font Braggadocio, for instance, has stencils that are made of one connected piece of material.

Mathematical definition

Let $\mathbf{X}$ be any set and let $\mathbf{T}$ be a family of subsets of $\mathbf{X}$. Then $\mathbf{T}$ is a **topology** on $\mathbf{X}$ if

1. Both the empty set and $\mathbf{X}$ are elements of $\mathbf{T}$.
2. Any union of arbitrarily many elements of $\mathbf{T}$ is an element of $\mathbf{T}$.
3. Any intersection of finitely many elements of $\mathbf{T}$ is an element of $\mathbf{T}$.

If $\mathbf{T}$ is a topology on $\mathbf{X}$, then the pair $(\mathbf{X}, \mathbf{T})$ is called a **topological space**, and the notation $\mathbf{X}_{\mathbf{T}}$ is used to denote a set $\mathbf{X}$ endowed with the particular topology $\mathbf{T}$.

The **open** sets in $\mathbf{X}$ are defined to be the members of $\mathbf{T}$; note that in general not all subsets of $\mathbf{X}$ need be in $\mathbf{T}$. A subset of $\mathbf{X}$ is said to be closed if its complement is in $\mathbf{T}$ (i.e., it is open). A subset of $\mathbf{X}$ may be open, closed, both, or neither.

A function or map from one topological space to another is called **continuous** if the inverse image of any open set is open. If the function maps the real numbers to the real numbers (both spaces with the Standard Topology), then this definition of continuous is equivalent to the definition of continuous in calculus. If a continuous function is one-to-one and onto and if the inverse of the function is also continuous, then the function is called a homeomorphism and the domain of the function is said to be homeomorphic to the range. Another way of saying this is that the function has a natural extension to the topology. If two spaces are homeomorphic, they have identical topological properties, and are considered to be topologically the same. The cube and the sphere are homeomorphic, as are the coffee cup and the doughnut. But the circle is not homeomorphic to the doughnut.

Topology topics

Some theorems in general topology

- Every closed interval in $\mathbf{R}$ of finite length is compact. More is true: In $\mathbf{R}^n$, a set is compact if and only if it is closed and bounded. (See Heine-Borel theorem).
- Every continuous image of a compact space is compact.
- Tychonoff's theorem: The (arbitrary) product of compact spaces is compact.
- A compact subspace of a Hausdorff space is closed.
- Every continuous bijection from a compact space to a Hausdorff space is necessarily a homeomorphism.
- Every sequence of points in a compact metric space has a convergent subsequence.
- Every interval in $\mathbf{R}$ is connected.
- Every compact m -manifold can be embedded in some Euclidean space $\mathbf{R}^n$.
- The continuous image of a connected space is connected.
- A metric space is Hausdorff, also normal and paracompact.
- The metrization theorems provide necessary and sufficient conditions for a topology to come from a metric.
- The Tietze extension theorem: In a normal space, every continuous real-valued function defined on a closed subspace can be extended to a continuous map defined on the whole space.
- Any open subspace of a Baire space is itself a Baire space.
- The Baire category theorem: If X is a complete metric space or a locally compact Hausdorff space, then the interior of every union of countably many nowhere dense sets is empty.

- On a paracompact Hausdorff space every open cover admits a partition of unity subordinate to the cover.
- Every path-connected, locally path-connected and semi-locally simply connected space has a universal cover.

General topology also has some surprising connections to other areas of mathematics. For example:

- In number theory, Furstenberg's proof of the infinitude of primes.

Some useful notions from algebraic topology

See also list of algebraic topology topics.

- Homology and cohomology: Betti numbers, Euler characteristic, degree of a continuous mapping.
- Operations: cup product, Massey product
- Intuitively-attractive applications: Brouwer fixed-point theorem, Hairy ball theorem, Borsuk-Ulam theorem, Ham sandwich theorem.
- Homotopy groups (including the fundamental group).
- Chern classes, Stiefel-Whitney classes, Pontryagin classes.

Generalizations

Occasionally, one needs to use the tools of topology but a "set of points" is not available. In pointless topology one considers instead the lattice of open sets as the basic notion of the theory, while Grothendieck topologies are certain structures defined on arbitrary categories which allow the definition of sheaves on those categories, and with that the definition of quite general cohomology theories.

Topology in art and literature

- Some M. C. Escher works illustrate topological concepts, such as Möbius strips and non-orientable spaces.

See also

- topology glossary
 - List of topology topics
 - List of general topology topics
 - List of geometric topology topics
 - List of algebraic topology topics
 - Publications in topology
-

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Further reading

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- Richeson, David S. (2008) *Euler's Gem: The Polyhedron Formula and the Birth of Topology*. Princeton University Press.

External links

- Elementary Topology: A First Course ^[1] Viro, Ivanov, Netsvetaev, Kharlamov
- An invitation to Topology ^[2] Planar Machines' web site
- Geometry and Topology Index ^[3], MacTutor History of Mathematics archive ^[4]
- ODP category ^[5]
- The Topological Zoo ^[6] at The Geometry Center
- Topology Atlas ^[7]
- Topology Course Lecture Notes ^[8] Aisling McCluskey and Brian McMaster, Topology Atlas
- Topology Glossary ^[9]
- Moscow 1935: Topology moving towards America ^[10], a historical essay by Hassler Whitney.
- "Topologically Speaking" ^[11], a song about topology.
- "The Use of Topology in Dance" ^[12], a review of Alvin Ailey's Memoria on ExploreDance.com in which the use of topologies as a way of structuring choreography is discussed.

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- [1] <http://www.pdmi.ras.ru/~olegviro/topoman/index.html>
- [2] <http://www.math.toronto.edu/~drorbn/People/Eldar/thesis/>
- [3] http://www-groups.dcs.st-and.ac.uk/~history/Indexes/Geometry_Topology.html
- [4] <http://www-groups.dcs.st-and.ac.uk/~history/>
- [5] <http://dmoz.org/Science/Math/Topology/>
- [6] <http://www.geom.uiuc.edu/zoo/>
- [7] <http://at.yorku.ca/topology/>
- [8] <http://at.yorku.ca/i/a/a/b/23.htm>
- [9] <http://www.ornl.gov/sci/ortep/topology/defs.txt>
- [10] http://www.ams.org/online_bks/hmath1/hmath1-whitney10.pdf
- [11] <http://www.montyharper.com/Songs/TS.html>
- [12] <http://www.exploredance.com/article.htm?id=2072>

Topological property

In topology and related areas of mathematics a **topological property** or **topological invariant** is a property of a topological space which is invariant under homeomorphisms. That is, a property of spaces is a topological property if whenever a space X possesses that property every space homeomorphic to X possesses that property. Informally, a topological property is a property of the space that can be expressed using open sets.

A common problem in topology is to decide whether two topological spaces are homeomorphic or not. To prove that two spaces are *not* homeomorphic, it is sufficient to find a topological property which is not shared by them.

Common topological properties

Cardinal functions

- The cardinality $|X|$ of the space X .
- The cardinality $\tau(X)$ of the topology of the space X .
- *Weight* $w(X)$, the least cardinality of a basis of the topology of the space X .
- *Density* $d(X)$, the least cardinality of a subset of X whose closure is X .

Separation

For a detailed treatment, see separation axiom. Some of these terms are defined differently in older mathematical literature; see history of the separation axioms.

- **T_0 or Kolmogorov.** A space is Kolmogorov if for every pair of distinct points x and y in the space, there is at least either an open set containing x but not y , or an open set containing y but not x .
- **T_1 or Fréchet.** A space is Fréchet if for every pair of distinct points x and y in the space, there is an open set containing x but not y . (Compare with T_0 ; here, we are allowed to specify which point will be contained in the open set.) Equivalently, a space is T_1 if all its singletons are closed. T_1 spaces are always T_0 .
- **Sober.** A space is sober if every irreducible closed set C has a unique generic point p . In other words, if C is not the (possibly nondisjoint) union of two smaller closed subsets, then there is a p such that the closure of $\{p\}$ equals C , and p is the only point with this property.
- **T_2 or Hausdorff.** A space is Hausdorff if every two distinct points have disjoint neighbourhoods. T_2 spaces are always T_1 .
- **$T_{2\frac{1}{2}}$ or Urysohn.** A space is Urysohn if every two distinct points have disjoint *closed* neighbourhoods. $T_{2\frac{1}{2}}$ spaces are always T_2 .
- **Regular.** A space is regular if whenever C is a closed set and p is a point not in C , then C and p have disjoint neighbourhoods.
- **T_3 or Regular Hausdorff.** A space is regular Hausdorff if it is a regular T_0 space. (A regular space is Hausdorff if and only if it is T_0 , so the terminology is consistent.)
- **Completely regular.** A space is completely regular if whenever C is a closed set and p is a point not in C , then C and $\{p\}$ are separated by a function.
- **$T_{3\frac{1}{2}}$, Tychonoff, Completely regular Hausdorff or Completely T_3 .** A Tychonoff space is a completely regular T_0 space. (A completely regular space is Hausdorff if and only if it

is T_0 , so the terminology is consistent.) Tychonoff spaces are always regular Hausdorff.

- **Normal.** A space is normal if any two disjoint closed sets have disjoint neighbourhoods. Normal spaces admit partitions of unity.
- **T_4 or Normal Hausdorff.** A normal space is Hausdorff if and only if it is T_1 . Normal Hausdorff spaces are always Tychonoff.
- **Completely normal.** A space is completely normal if any two separated sets have disjoint neighbourhoods.
- **T_5 or Completely normal Hausdorff.** A completely normal space is Hausdorff if and only if it is T_1 . Completely normal Hausdorff spaces are always normal Hausdorff.
- **Perfectly normal.** A space is perfectly normal if any two disjoint closed sets are precisely separated by a function. A perfectly normal space must also be completely normal.
- **Perfectly normal Hausdorff, or perfectly T_4 .** A space is perfectly normal Hausdorff, if it is both perfectly normal and T_1 . A perfectly normal Hausdorff space must also be completely normal Hausdorff.
- **Discrete space.** A space is discrete if all of its points are completely isolated, i.e. if any subset is open.

Countability conditions

- **Separable.** A space is separable if it has a countable dense subset.
- **Lindelöf.** A space is Lindelöf if every open cover has a countable subcover.
- **First-countable.** A space is first-countable if every point has a countable local base.
- **Second-countable.** A space is second-countable if it has a countable base for its topology. Second-countable spaces are always separable, first-countable and Lindelöf.

Connectedness

- **Connected.** A space is connected if it is not the union of a pair of disjoint non-empty open sets. Equivalently, a space is connected if the only clopen sets are the empty set and itself.
- **Locally connected.** A space is locally connected if every point has a local base consisting of connected sets.
- **Totally disconnected.** A space is totally disconnected if it has no connected subset with more than one point.
- **Path-connected.** A space X is path-connected if for every two points x, y in X , there is a path p from x to y , i.e., a continuous map $p: [0,1] \rightarrow X$ with $p(0) = x$ and $p(1) = y$. Path-connected spaces are always connected.
- **Locally path-connected.** A space is locally path-connected if every point has a local base consisting of path-connected sets. A locally path-connected space is connected if and only if it is path-connected.
- **Simply connected.** A space X is simply connected if it is path-connected and every continuous map $f: S^1 \rightarrow X$ is homotopic to a constant map.
- **Locally simply connected.** A space X is locally simply connected if every point x in X has a local base of neighborhoods U that is simply connected.
- **Semi-locally simply connected.** A space X is semi-locally simply connected if every point has a local base of neighborhoods U such that every loop in U is contractible in X . Semi-local simple connectivity, a strictly weaker condition than local simple connectivity, is a necessary condition for the existence of a universal cover.

- **Contractible.** A space X is contractible if the identity map on X is homotopic to a constant map. Contractible spaces are always simply connected.
- **Hyper-connected.** A space is hyper-connected if no two non-empty open sets are disjoint. Every hyper-connected space is connected.
- **Ultra-connected.** A space is ultra-connected if no two non-empty closed sets are disjoint. Every ultra-connected space is path-connected.
- **Indiscrete** or **trivial.** A space is indiscrete if the only open sets are the empty set and itself. Such a space is said to have the trivial topology.

Compactness

- **Compact.** A space is compact if every open cover has a finite subcover. Some authors call these spaces **quasicompact** and reserve compact for Hausdorff spaces where every open cover has finite subcover. Compact spaces are always Lindelöf and paracompact. Compact Hausdorff spaces are therefore normal.
- **Sequentially compact.** A space is sequentially compact if every sequence has a convergent subsequence.
- **Countably compact.** A space is countably compact if every countable open cover has a finite subcover.
- **Pseudocompact.** A space is pseudocompact if every real-valued function on the space is bounded.
- **σ -compact.** A space is σ -compact if it is the union of countably many compact subsets.
- **Paracompact.** A space is paracompact if every open cover has an open locally finite refinement. Paracompact Hausdorff spaces are normal.
- **Locally compact.** A space is locally compact if every point has a local base consisting of compact neighbourhoods. Slightly different definitions are also used. Locally compact Hausdorff spaces are always Tychonoff.
- **Ultraconnected compact.** In an ultra-connected compact space X every open cover must contain X itself. Non-empty ultra-connected compact spaces have a largest proper open subset called a **monolith**.

Metrizability

- **Metrizable.** A space is metrizable if it is homeomorphic to a metric space. Metrizable spaces are always Hausdorff and paracompact (and hence normal and Tychonoff), and first-countable.
- **Polish.** A space is called Polish if it is metrizable with a separable and complete metric.
- **Locally metrizable.** A space is locally metrizable if every point has a metrizable neighbourhood.

Miscellaneous

- **Baire space.** A space X is a Baire space if it is not meagre in itself. Equivalently, X is a Baire space if the intersection of countably many dense open sets is dense.
- **Homogeneous.** A space X is homogeneous if for every x and y in X there is a homeomorphism $f : X \rightarrow X$ such that $f(x) = y$. Intuitively speaking, this means that the space looks the same at every point. All topological groups are homogeneous.
- **Finitely generated** or **Alexandrov.** A space X is Alexandrov if arbitrary intersections of open sets in X are open, or equivalently if arbitrary unions of closed sets are closed.

These are precisely the finitely generated members of the category of topological spaces and continuous maps.

- **Zero-dimensional.** A space is zero-dimensional if it has a base of clopen sets. These are precisely the spaces with a small inductive dimension of 0.
- **Almost discrete.** A space is almost discrete if every open set is closed (hence clopen). The almost discrete spaces are precisely the finitely generated zero-dimensional spaces.
- **Boolean.** A space is Boolean if it is zero-dimensional, compact and Hausdorff (equivalently, totally disconnected, compact and Hausdorff). These are precisely the spaces that are homeomorphic to the Stone spaces of Boolean algebras.
- **Reidemeister torsion**

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Topological space

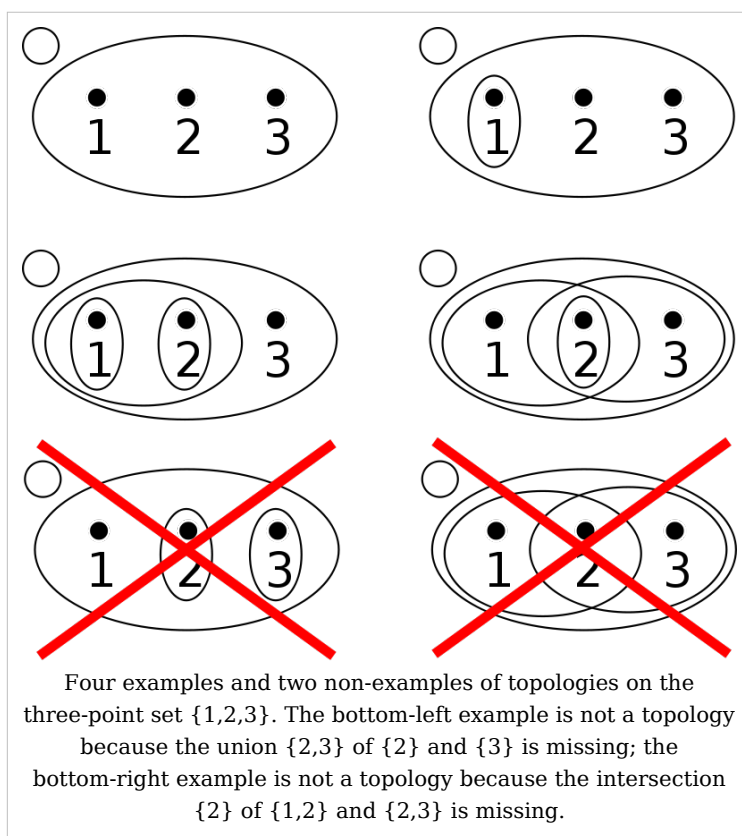
Topological spaces are mathematical structures that allow the formal definition of concepts such as convergence, connectedness, and continuity. They appear in virtually every branch of modern mathematics and are a central unifying notion. The branch of mathematics that studies topological spaces in their own right is called topology.

Definition

A **topological space** is a set X together with T , a collection of subsets of X , satisfying the following axioms:

1. The empty set and X are in T .
2. The union of any collection of sets in T is also in T .
3. The intersection of any finite collection of sets in T is also in T .

The collection T is called a **topology** on X . The elements of X are usually called *points*, though they can be any mathematical objects. A topological space in which the *points* are functions is called a *function space*. The sets in T are the **open sets**, and their complements in X are called **closed sets**. A set may be neither closed nor open, either closed or open, or both. A set that is both closed and open is called a **clopen set**.



Examples

1. $X = \{1, 2, 3, 4\}$ and collection $T =$ of two subsets of X form a trivial topology.
2. $X = \{1, 2, 3, 4\}$ and collection $T =$ of six subsets of X form another topology.
3. $X = \mathbf{Z}$, the set of integers and collection T equal to all finite subsets of the integers plus $\mathbf{Z}$ itself is *not* a topology, because (for example) the union over all finite sets not containing zero is infinite but is not all of $\mathbf{Z}$, and so is not in T .

Equivalent definitions

There are many other equivalent ways to define a topological space. (In other words, each of the following defines a category equivalent to the category of topological spaces above.) For example, using de Morgan's laws, the axioms defining open sets above become axioms defining closed sets:

1. The empty set and X are closed.
2. The intersection of any collection of closed sets is also closed.
3. The union of any pair of closed sets is also closed.

Using these axioms, another way to define a topological space is as a set X together with a collection T of subsets of X satisfying the following axioms:

1. The empty set and X are in T .
2. The intersection of any collection of sets in T is also in T .
3. The union of any pair of sets in T is also in T .

Under this definition, the sets in the topology T are the closed sets, and their complements in X are the open sets.

Another way to define a topological space is by using the Kuratowski closure axioms, which define the closed sets as the fixed points of an operator on the power set of X .

A neighbourhood of a point x is any set that contains an open set containing x . The *neighbourhood system* at x consists of all neighbourhoods of x . A topology can be determined by a set of axioms concerning all neighbourhood systems.

A net is a generalisation of the concept of sequence. A topology is completely determined if for every net in X the set of its accumulation points is specified.

Comparison of topologies

A variety of topologies can be placed on a set to form a topological space. When every set in a topology T_1 is also in a topology T_2 , we say that T_2 is *finer* than T_1 , and T_1 is *coarser* than T_2 . A proof which relies only on the existence of certain open sets will also hold for any finer topology, and similarly a proof that relies only on certain sets not being open applies to any coarser topology. The terms *larger* and *smaller* are sometimes used in place of finer and coarser, respectively. The terms *stronger* and *weaker* are also used in the literature, but with little agreement on the meaning, so one should always be sure of an author's convention when reading.

The collection of all topologies on a given fixed set X forms a complete lattice: if $F = \{T_\alpha : \alpha \text{ in } A\}$ is a collection of topologies on X , then the meet of F is the intersection of F , and the join of F is the meet of the collection of all topologies on X which contain every member of F .

Continuous functions

A function between topological spaces is said to be **continuous** if the inverse image of every open set is open. This is an attempt to capture the intuition that there are no "breaks" or "separations" in the function. A homeomorphism is a bijection that is continuous and whose inverse is also continuous. Two spaces are said to be *homeomorphic* if there exists a homeomorphism between them. From the standpoint of topology, homeomorphic spaces are essentially identical.

In category theory, **Top**, the category of topological spaces with topological spaces as objects and continuous functions as morphisms is one of the fundamental categories in mathematics. The attempt to classify the objects of this category (up to homeomorphism) by invariants has motivated and generated entire areas of research, such as homotopy theory, homology theory, and K-theory, to name just a few.

Examples of topological spaces

A given set may have many different topologies. If a set is given a different topology, it is viewed as a different topological space. Any set can be given the discrete topology in which every subset is open. The only convergent sequences or nets in this topology are those that are eventually constant. Also, any set can be given the trivial topology (also called the indiscrete topology), in which only the empty set and the whole space are open. Every sequence and net in this topology converges to every point of the space. This example shows that in general topological spaces, limits of sequences need not be unique. However, often topological spaces are required to be Hausdorff spaces where limit points are unique.

There are many ways of defining a topology on $\mathbf{R}$, the set of real numbers. The standard topology on $\mathbf{R}$ is generated by the open intervals. The set of all open intervals forms a base or basis for the topology, meaning that every open set is a union of some collection of sets from the base. In particular, this means that a set is open if there exists an open interval of non zero radius about every point in the set. More generally, the Euclidean spaces $\mathbf{R}^n$ can be given a topology. In the usual topology on $\mathbf{R}^n$ the basic open sets are the open balls. Similarly, $\mathbf{C}$ and $\mathbf{C}^n$ have a standard topology in which the basic open sets are open balls.

Every metric space can be given a metric topology, in which the basic open sets are open balls defined by the metric. This is the standard topology on any normed vector space. On a finite-dimensional vector space this topology is the same for all norms.

Many sets of operators in functional analysis are endowed with topologies that are defined by specifying when a particular sequence of functions converges to the zero function.

Any local field has a topology native to it, and this can be extended to vector spaces over that field.

Every manifold has a natural topology since it is locally Euclidean. Similarly, every simplex and every simplicial complex inherits a natural topology from $\mathbf{R}^n$.

The Zariski topology is defined algebraically on the spectrum of a ring or an algebraic variety. On $\mathbf{R}^n$ or $\mathbf{C}^n$, the closed sets of the Zariski topology are the solution sets of systems of polynomial equations.

A linear graph has a natural topology that generalises many of the geometric aspects of graphs with vertices and edges.

Sierpiński space is the simplest non-discrete topological space. It has important relations to the theory of computation and semantics.

There exist numerous topologies on any given finite set. Such spaces are called finite topological spaces. Finite spaces are often used to provide examples or counterexamples to conjectures about topological spaces in general.

Any set can be given the cofinite topology in which the open sets are the empty set and the sets whose complement is finite. This is the smallest T_1 topology on any infinite set.

Any set can be given the cocountable topology, in which a set is defined to be open if it is either empty or its complement is countable. When the set is uncountable, this topology serves as a counterexample in many situations.

The real line can also be given the lower limit topology. Here, the basic open sets are the half open intervals $[a, b)$. This topology on $\mathbf{R}$ is strictly finer than the Euclidean topology defined above; a sequence converges to a point in this topology if and only if it converges from above in the Euclidean topology. This example shows that a set may have many distinct topologies defined on it.

If Γ is an ordinal number, then the set $\Gamma = [0, \Gamma)$ may be endowed with the order topology generated by the intervals (a, b) , $[0, b)$ and (a, Γ) where a and b are elements of Γ .

Topological constructions

Every subset of a topological space can be given the subspace topology in which the open sets are the intersections of the open sets of the larger space with the subset. For any indexed family of topological spaces, the product can be given the product topology, which is generated by the inverse images of open sets of the factors under the projection mappings. For example, in finite products, a basis for the product topology consists of all products of open sets. For infinite products, there is the additional requirement that in a basic open set, all but finitely many of its projections are the entire space.

A quotient space is defined as follows: if X is a topological space and Y is a set, and if $f : X \rightarrow Y$ is a surjective function, then the quotient topology on Y is the collection of subsets of Y that have open inverse images under f . In other words, the quotient topology is the finest topology on Y for which f is continuous. A common example of a quotient topology is when an equivalence relation is defined on the topological space X . The map f is then the natural projection onto the set of equivalence classes.

The **Vietoris topology** on the set of all non-empty subsets of a topological space X , named for Leopold Vietoris, is generated by the following basis: for every n -tuple $U_1, \dots, U_n$ of open sets in X , we construct a basis set consisting of all subsets of the union of the U_i which have non-empty intersection with each U_i .

Classification of topological spaces

Topological spaces can be broadly classified, up to homeomorphism, by their topological properties. A topological property is a property of spaces that is invariant under homeomorphisms. To prove that two spaces are not homeomorphic it is sufficient to find a topological property which is not shared by them. Examples of such properties include connectedness, compactness, and various separation axioms.

See the article on *topological properties* for more details and examples.

Topological spaces with algebraic structure

For any algebraic objects we can introduce the discrete topology, under which the algebraic operations are continuous functions. For any such structure which is not finite, we often have a natural topology which is compatible with the algebraic operations in the sense that the algebraic operations are still continuous. This leads to concepts such as topological groups, topological vector spaces, topological rings and local fields.

Topological spaces with order structure

- **Spectral.** A space is spectral if and only if it is the prime spectrum of a ring (Hochster theorem).
- **Specialization preorder.** In a space the **specialization** (or **canonical**) **preorder** is defined by $x \leq y$ if and only if $\text{cl}\{x\} \subseteq \text{cl}\{y\}$.

Specializations and generalizations

The following spaces and algebras are either more specialized or more general than the topological spaces discussed above.

- Proximity spaces provide a notion of closeness of two sets.
- Metric spaces embody a metric, a precise notion of distance between points.
- Uniform spaces axiomatize ordering the distance between distinct points.
- Cauchy spaces axiomatize the ability to test whether a net is Cauchy. Cauchy spaces provide a general setting for studying completions.
- Convergence spaces capture some of the features of convergence of filters.
- σ -algebras build on the notion of measurable sets.

See also

- T0 space
 - T1 space
 - Hausdorff space (T2)
 - Completely Hausdorff space
 - Urysohn space
 - T3 space
 - Tychonoff space
 - Normal Hausdorff space (T4)
 - Completely normal Hausdorff space (T5)
 - Perfectly normal Hausdorff space (T6)
-

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External links

- Topological space ^[1] on PlanetMath

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Continuous function (topology)

In topology and related areas of mathematics a **continuous function** is a morphism between topological spaces. Intuitively, this is a function f where a set of points near $f(x)$ always contain the image of a set of points near x . For a general topological space, this means a neighbourhood of $f(x)$ always contains the image of a neighbourhood of x .

In a metric space (for example, the real numbers) this means that the points within a given distance of $f(x)$ always contain the images of all the points within some other distance of x , giving the ε - δ definition.

Definitions

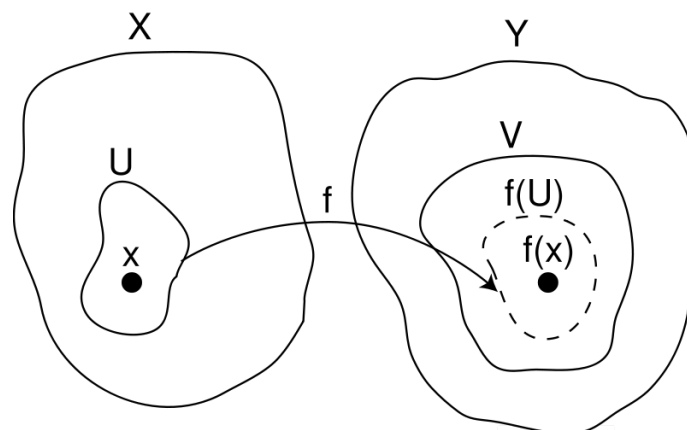
Several equivalent definitions for a topological structure exist and thus there are several equivalent ways to define a continuous function.

Open and closed set definition

The most common notion of continuity in topology defines continuous functions as those functions for which the preimages of open sets are open. Similar to the open set formulation is the **closed set formulation**, which says that preimages of closed sets are closed.

Neighborhood definition

Definitions based on preimages are often difficult to use directly. Instead, suppose we have a function $f : X \rightarrow Y$, where X and Y are topological spaces^[1]. We say f is **continuous at x** for some $x \in X$ if for any neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$. Although this definition appears complicated, the intuition is that no matter how "small" V becomes, we can always find a U containing x that will map inside it. If f is continuous at every $x \in X$, then we simply say f is continuous.



In a metric space, it is equivalent to consider the neighbourhood system of open balls centered at x and $f(x)$ instead of all neighborhoods. This leads to the standard δ - ε definition of a continuous function from real analysis, which says roughly that a function is continuous if all points close to x map to points close to $f(x)$. This only really makes sense in a metric space, however, which has a notion of distance.

Note, however, that if the target space is Hausdorff, it is still true that f is continuous at a if and only if the limit of f as x approaches a is $f(a)$. At an isolated point, every function is

continuous.

Sequences and nets

In several contexts, the topology of a space is conveniently specified in terms of limit points. In many instances, this is accomplished by specifying when a point is the limit of a sequence, but for some spaces that are too large in some sense, one specifies also when a point is the limit of more general sets of points indexed by a directed set, known as nets. A function is continuous only if it takes limits of sequences to limits of sequences. In the former case, preservation of limits is also sufficient; in the latter, a function may preserve all limits of sequences yet still fail to be continuous, and preservation of nets is a necessary and sufficient condition.

In detail, a function $f : X \rightarrow Y$ is **sequentially continuous** if whenever a sequence (x_n) in X converges to a limit x , the sequence $(f(x_n))$ converges to $f(x)$. Thus sequentially continuous functions "preserve sequential limits". Every continuous function is sequentially continuous. If X is a first-countable space, then the converse also holds: any function preserving sequential limits is continuous. In particular, if X is a metric space, sequential continuity and continuity are equivalent. For non first-countable spaces, sequential continuity might be strictly weaker than continuity. (The spaces for which the two properties are equivalent are called sequential spaces.) This motivates the consideration of nets instead of sequences in general topological spaces. Continuous functions preserve limits of nets, and in fact this property characterizes continuous functions.

Closure operator definition

Given two topological spaces (X, cl) and (X', cl') where cl and cl' are two closure operators then a function

$$f : (X, cl) \rightarrow (X', cl')$$

is **continuous** if for all subsets A of X

$$f(cl(A)) \subseteq cl'(f(A)).$$

One might therefore suspect that given two topological spaces (X, int) and (X', int') where int and int' are two interior operators then a function

$$f : (X, int) \rightarrow (X', int')$$

is **continuous** if for all subsets A of X

$$f(int(A)) \subseteq int'(f(A))$$

or perhaps if

$$f(int(A)) \supseteq int'(f(A));$$

however, neither of these conditions is either necessary or sufficient for continuity.

Instead, we must resort to inverse images: given two topological spaces (X, int) and (X', int') where int and int' are two interior operators then a function

$$f : (X, int) \rightarrow (X', int')$$

is **continuous** if for all subsets A of X'

$$f^{-1}(int'(A)) \subseteq int(f^{-1}(A)).$$

We can also write that given two topological spaces (X, cl) and (X', cl') where cl and cl' are two closure operators then a function

$$f : (X, \text{cl}) \rightarrow (X', \text{cl}')$$

is **continuous** if for all subsets A of X'

$$f^{-1}(\text{cl}'(A)) \supseteq \text{cl}(f^{-1}(A)).$$

Closeness relation definition

Given two topological spaces (X, δ) and (X', δ') where δ and δ' are two closeness relations then a function

$$f : (X, \delta) \rightarrow (X', \delta')$$

is **continuous** if for all points x and of X and all subsets A of X' ,

$$x\delta A \Rightarrow f(x)\delta' f(A).$$

This is another way of writing the closure operator definition.

Useful properties of continuous maps

Some facts about continuous maps between topological spaces:

- If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then so is the composition $g \circ f : X \rightarrow Z$.
- If $f : X \rightarrow Y$ is continuous and
 - X is compact, then $f(X)$ is compact.
 - X is connected, then $f(X)$ is connected.
 - X is path-connected, then $f(X)$ is path-connected.
- The identity map $\text{id}_X : (X, \tau_2) \rightarrow (X, \tau_1)$ is continuous if and only if $\tau_1 \sqsubseteq \tau_2$ (see also comparison of topologies).

Other notes

If a set is given the discrete topology, all functions with that space as a domain are continuous. If the domain set is given the indiscrete topology and the range set is at least T_0 , then the only continuous functions are the constant functions. Conversely, any function whose range is indiscrete is continuous.

Given a set X , a partial ordering can be defined on the possible topologies on X . A continuous function between two topological spaces stays continuous if we strengthen the topology of the domain space or weaken the topology of the codomain space. Thus we can consider the continuity of a given function a topological property, depending only on the topologies of its domain and codomain spaces.

For a function f from a topological space X to a set S , one defines the final topology on S by letting the open sets of S be those subsets A of S for which $f^{-1}(A)$ is open in X . If S has an existing topology, f is continuous with respect to this topology if and only if the existing topology is coarser than the final topology on S . Thus the final topology can be characterized as the finest topology on S which makes f continuous. If f is surjective, this topology is canonically identified with the quotient topology under the equivalence relation defined by f . This construction can be generalized to an arbitrary family of functions $X \rightarrow S$.

Dually, for a function f from a set S to a topological space, one defines the initial topology on S by letting the open sets of S be those subsets A of S for which $f(A)$ is open in X . If S has an existing topology, f is continuous with respect to this topology if and only if the existing topology is finer than the initial topology on S . Thus the initial topology can be characterized as the coarsest topology on S which makes f continuous. If f is injective, this

topology is canonically identified with the subspace topology of S , viewed as a subset of X . This construction can be generalized to an arbitrary family of functions $S \rightarrow X$.

Symmetric to the concept of a continuous map is an open map, for which *images* of open sets are open. In fact, if an open map f has an inverse, that inverse is continuous, and if a continuous map g has an inverse, that inverse is open.

If a function is a bijection, then it has an inverse function. The inverse of a continuous bijection is open, but need not be continuous. If it is, this special function is called a homeomorphism. If a continuous bijection has as its domain a compact space and its codomain is Hausdorff, then it is automatically a homeomorphism.

Footnotes

[1] f is a function $f: X \rightarrow Y$ between two topological spaces (X, T_X) and (Y, T_Y) . That is, the function f is defined on the elements of the set X , not on the elements of the topology T_X . However continuity of the function does depend on the topologies used.

Fibration

In mathematics, especially algebraic topology, a **fibration** is a continuous mapping

$$p: E \rightarrow B$$

satisfying the homotopy lifting property with respect to any space. Fiber bundles (over paracompact bases) constitute important examples. In homotopy theory any mapping is 'as good as' a fibration — i.e. any map can be decomposed as a homotopy equivalence into a "mapping path space" followed by a fibration. (See homotopy fiber.)

A fibration with the homotopy lifting property for CW complexes (or equivalently, just cubes I^n) is called a *Serre fibration*, in honor of the part played by the concept in the thesis of Jean-Pierre Serre. This thesis firmly established in algebraic topology the use of spectral sequences, and clearly separated the notions of fiber bundles and fibrations from the notion of sheaf (both concepts together having been implicit in the pioneer treatment of Jean Leray). Because a sheaf (thought of as an *étalé* space) can be considered a local homeomorphism, the notions were closely interlinked at the time.

The *fibers* are by definition the subspaces of E that are the inverse images of points b of B . If the base space B is path connected, it is a consequence of the definition that the fibers of two different points b_1 and b_2 in B are homotopy equivalent. Therefore one usually speaks of "the fiber" F . Fibrations do not necessarily have the local cartesian product structure that defines the more restricted fiber bundle case, but something weaker that still allows "sideways" movement from fiber to fiber. One of the main desirable properties of the Serre spectral sequence is to account for the action of the fundamental group of the base B on the homology of the *total space* E .

The projection map from a product space is very easily seen to be a fibration. Fiber bundles have *local trivializations* — such cartesian product structures exist locally on B , and this is usually enough to show that a fiber bundle is a fibration. More precisely, if there are local trivializations over a "numerable open cover" of B , the bundle is a fibration. Any open cover of a paracompact space — for example any metric space, has a numerable refinement, so any bundle over such a space is a fibration. The local triviality also implies the existence of a well-defined *fiber* (up to homeomorphism), at least on each connected component of B .

Examples

In the following examples a fibration is denoted

$$F \rightarrow E \rightarrow B,$$

where the first map is the inclusion of "the" fiber F into the total space E and the second map is the fibration onto the basis B . This is also referred to as a fibration sequence.

- The Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$ was historically one of the earliest non-trivial examples of a fibration.
- The Serre fibration $SO(2) \rightarrow SO(3) \rightarrow S^2$ comes from the action of the rotation group $SO(3)$ on the 2-sphere S^2 .
- Over complex projective space, there is a fibration $S^1 \rightarrow S^{2n+1} \rightarrow \mathbf{CP}^n$.

Properties

Euler characteristic

The Euler characteristic is multiplicative for fibrations with certain conditions.

If $p: E \rightarrow B$ is a fibration with fiber F , with the base B path-connected, and the fibration is orientable over a field K , then the Euler characteristic with coefficients in the field K satisfies the product property:^[1]

$$\chi(E) = \chi(F) \cdot \chi(B).$$

This includes product spaces and covering spaces as special cases, and can be proven by the Serre spectral sequence on homology of a fibration.

For fiber bundles, this can also be understood in terms of a transfer map $\tau: H_*(B) \rightarrow H_*(E)$ - note that this is a lifting and goes "the wrong way" - whose composition with the projection map $p_*: H_*(E) \rightarrow H_*(B)$ is multiplication by the Euler class of the fiber:^[2] $p_* \circ \tau = \chi(F) \cdot 1$.

Fibrations in closed model categories

Fibrations of topological spaces fit into a more general framework, the so-called closed model categories. In such categories, there are distinguished classes of morphisms, the so-called *fibrations*, *cofibrations* and *weak equivalences*. Certain axioms, such as stability of fibrations under composition and pullbacks, factorization of every morphism into the composition of an acyclic cofibration followed by a fibration or a cofibration followed by an acyclic fibration, where the word "acyclic" indicates that the corresponding arrow is also a weak equivalence, and other requirements are set up to allow the abstract treatment of homotopy theory. (In the original treatment, due to Daniel Quillen, the word "trivial" was used instead of "acyclic.")

It can be shown that the category of topological spaces is in fact a model category, where (abstract) fibrations are just the fibrations introduced above and weak equivalences are homotopy equivalences. See Dwyer, Spaliński (1995).

See also

- Homotopy fiber

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Homeomorphism

***Topological equivalence** redirects here; see also topological equivalence (dynamical systems).*

In the mathematical field of topology, a **homeomorphism** or **topological isomorphism** (from the Greek words *ὁμοιος* (*homoios*) = similar and *μορφή* (*morphē*) = shape = form (Latin deformation of *morphe*)) is a bicontinuous function between two topological spaces. Homeomorphisms are the isomorphisms in the category of topological spaces — that is, they are the mappings which preserve all the topological properties of a given space. Two spaces with a homeomorphism between them are called **homeomorphic**, and from a topological viewpoint they are the same.

Roughly speaking, a topological space is a geometric object, and the homeomorphism is a continuous stretching and bending of the object into a new shape. Thus, a square and a circle are homeomorphic to each other, but a sphere and a donut are not. An often-repeated joke is that topologists can't tell the coffee cup from which they are drinking from the donut they are eating, since a sufficiently pliable donut could be reshaped to the form of a coffee cup by creating a dimple and progressively enlarging it, while shrinking the hole into a handle.

Topology is the study of those properties of objects that do not change when homeomorphisms are applied. As Henri Poincaré famously said, mathematics is not the



A continuous deformation between a coffee mug and a donut illustrating that they are homeomorphic. But there does not need to be a continuous deformation for two spaces to be homeomorphic — only a continuous mapping with a continuous inverse.

study of objects, but instead, the relations (isomorphisms for instance) between them.

Definition

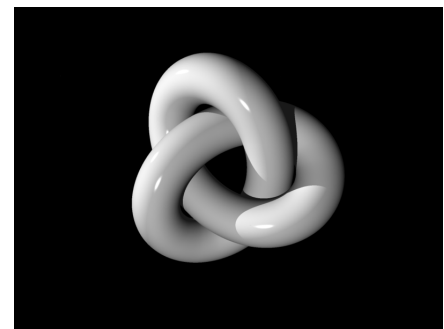
A function $f: X \rightarrow Y$ between two topological spaces (X, T_X) and (Y, T_Y) is called a **homeomorphism** if it has the following properties:

- f is a bijection (one-to-one and onto),
- f is continuous,
- the inverse function f^{-1} is continuous (f is an open mapping).

A function with these three properties is sometimes called **bicontinuous**. If such a function exists, we say X and Y are **homeomorphic**. A **self-homeomorphism** is a homeomorphism of a topological space and itself. The homeomorphisms form an equivalence relation on the class of all topological spaces. The resulting equivalence classes are called **homeomorphism classes**.

Examples

- The unit 2-disc D^2 and the unit square in $\mathbf{R}^2$ are homeomorphic.
- The open interval $(-1, +1)$ is homeomorphic to the real numbers $\mathbf{R}$.
- The product space $S^1 \times S^1$ and the two-dimensional torus are homeomorphic.
- Every uniform isomorphism and isometric isomorphism is a homeomorphism.
- Any 2-sphere with a single point removed is homeomorphic to the set of all points in $\mathbf{R}^2$ (a 2-dimensional plane).



A trefoil knot is homeomorphic to a circle. While this may seem illogical, in four dimensions they can easily be deformed continuously. Here the knot has been thickened to make the image understandable.

- Let A be a commutative ring with unity and let S be a multiplicative subset of A . Then $\text{Spec}(A_S)$ is homeomorphic to $\{p \in \text{Spec} A : p \cap S = \emptyset\}$.
- $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for $n \neq m$.
- An example of a continuous bijection that is not a homeomorphism is the map that takes the half-open interval $[0, 1)$ and wraps it around the circle. In this case the inverse – although it exists – fails to be continuous. The preimages of certain sets which are actually open in the relative topology of the half-open interval are not open in the more natural topology of the circle (they are half-open intervals).

Notes

The third requirement, that f^{-1} be continuous, is essential. Consider for instance the function $f : [0, 2\pi) \rightarrow S^1$ defined by $f(\varphi) = (\cos(\varphi), \sin(\varphi))$. This function is bijective and continuous, but not a homeomorphism.

Homeomorphisms are the isomorphisms in the category of topological spaces. As such, the composition of two homeomorphisms is again a homeomorphism, and the set of all self-homeomorphisms $X \rightarrow X$ forms a group, called the **homeomorphism group** of X , often denoted $\text{Homeo}(X)$; this group can be given a topology, such as the compact-open topology, making it a topological group.

For some purposes, the homeomorphism group happens to be too big, but by means of the isotopy relation, one can reduce this group to the mapping class group.

Similarly, as usual in category theory, given two spaces that are homeomorphic, the space of homeomorphisms between, $\text{Homeo}(X, Y)$ them is a torsor for the homeomorphism groups $\text{Homeo}(X)$ and $\text{Homeo}(Y)$, and given a specific homeomorph between X and Y , all three sets are identified.

Properties

- Two homeomorphic spaces share the same topological properties. For example, if one of them is compact, then the other is as well; if one of them is connected, then the other is as well; if one of them is Hausdorff, then the other is as well; their homology groups will coincide. Note however that this does not extend to properties defined via a metric; there are metric spaces which are homeomorphic even though one of them is complete and the other is not.
- A homeomorphism is simultaneously an open mapping and a closed mapping, that is it maps open sets to open sets and closed sets to closed sets.
- Every self-homeomorphism in S^1 can be extended to a self-homeomorphism of the whole disk D^2 (Alexander's Trick).

Informal discussion

The intuitive criterion of stretching, bending, cutting and gluing back together takes a certain amount of practice to apply correctly — it may not be obvious from the description above that deforming a line segment to a point is impermissible, for instance. It is thus important to realize that it is the formal definition given above that counts.

This characterization of a homeomorphism often leads to confusion with the concept of homotopy, which is actually *defined* as a continuous deformation, but from one *function* to another, rather than one space to another. In the case of a homeomorphism, envisioning a continuous deformation is a mental tool for keeping track of which points on space X correspond to which points on Y — one just follows them as X deforms. In the case of homotopy, the continuous deformation from one map to the other is of the essence, and it is also less restrictive, since none of the maps involved need to be one-to-one or onto. Homotopy does lead to a relation on spaces: homotopy equivalence.

There is a name for the kind of deformation involved in visualizing a homeomorphism. It is (except when cutting and regluing are required) an isotopy between the identity map on X and the homeomorphism from X to Y .

See also

- Local homeomorphism
- Diffeomorphism
- Uniform isomorphism is an isomorphism between uniform spaces
- Isometric isomorphism is an isomorphism between metric spaces
- Dehn twist
- Homeomorphism (graph theory) (closely related to graph subdivision)
- Isotopy
- Mapping class group

External links

- Homeomorphism^[1] on PlanetMath

References

[1] <http://planetmath.org/?op=getobj&from=objects&id=912>

Compact space

In mathematics, a topological space is called **compact** if each of its open covers has a finite subcover. Otherwise it is called **non-compact**.

Note: Some authors such as Bourbaki use the term "**quasi-compact**" for this instead, and reserve the term "compact" for topological spaces that are both Hausdorff and "quasi-compact".

The Heine–Borel theorem shows that this definition is equivalent to "closed and bounded" for subsets of Euclidean space. So a subset of Euclidean space $\mathbf{R}^n$ is called **compact** if it is closed and bounded. For example, in $\mathbf{R}$, the closed unit interval $[0, 1]$ is compact, but the set of integers $\mathbf{Z}$ is not (it is not bounded) and neither is the half-open interval $[0, 1)$ (it is not closed).

The concept of a compact subset of the real numbers can be extended to compact subsets of any topological space and indeed to the concept of a **compact space**. A subset is compact if when endowed with the subspace topology it becomes a compact space.

A single compact set is sometimes referred to as a **compactum**; following the Latin second declension (neuter), the corresponding plural form is **compacta**.

History and motivation

The identity of bounded closed subsets of real numbers and sets whose open covers have finite subcovers was discovered and proved in the late 19th century. See Heine-Borel theorem.

The term *compact* was introduced by Fréchet in 1906.

It has long been recognized that a property like compactness is necessary to prove many useful theorems. It used to be that "compact" meant "sequentially compact" (every sequence has a convergent subsequence). This was when primarily metric spaces were studied. The "covering compact" definition has become more prominent because it allows us to consider general topological spaces, and many of the old results about metric spaces can be generalized to this setting. This generalization is particularly useful in the study of function spaces, many of which are not metric spaces.

One of the main reasons for studying compact spaces is because they are in some ways very similar to finite sets: there are many results which are easy to show for finite sets, whose proofs carry over with minimal change to compact spaces. It is often said that "compactness is the next best thing to finiteness". Here is an example:

- Suppose X is a Hausdorff space, and we have a point x in X and a finite subset A of X not containing x . Then we can separate x and A by neighbourhoods: for each a in A , let $U(x)$ and $V(a)$ be disjoint neighbourhoods containing x and a , respectively. Then the intersection of all the $U(x)$ and the union of all the $V(a)$ are the required neighbourhoods of x and A .

Note that if A is infinite, the proof fails, because the intersection of arbitrarily many neighbourhoods of x might not be a neighbourhood of x . The proof can be "rescued", however, if A is compact: we simply take a finite subcover of the cover $\{V(a) : a \text{ in } A\}$ of A , then intersect the corresponding finitely many $U(x)$. In this way, we see that in a Hausdorff space, any point can be separated by neighbourhoods from any compact set not containing it. In fact, repeating the argument shows that any two disjoint compact sets in a Hausdorff space can be separated by neighbourhoods – note that this is precisely what we get if we replace "point" (i.e. singleton set) with "compact set" in the Hausdorff separation axiom. Many of the arguments and results involving compact spaces follow such a pattern.

Definitions

Compactness of topological spaces

A topological space X is defined as compact if all its open covers have a finite subcover. Formally, this means that

for every arbitrary collection $\{U_\alpha\}_{\alpha \in A}$ of open subsets of X such that $\bigcup_{\alpha \in A} U_\alpha \supseteq X$,
there is a finite subset $J \subset A$ such that $\bigcup_{i \in J} U_i \supseteq X$.

An often used equivalent definition is given in terms of the finite intersection property: if any collection of closed sets satisfying the finite intersection property has nonempty intersection, then the space is compact^[1]. This definition is dual to the usual one stated in terms of open sets.

Some authors require that a compact space also be Hausdorff, and the non-Hausdorff version is then called **quasicompact**.

Compactness of subsets of $\mathbf{R}^n$

For any subset of Euclidean space $\mathbf{R}^n$, the following four conditions are equivalent:

- Every open cover has a finite subcover. This is the topological definition.
- Every sequence in the set has a convergent subsequence, the limit point of which belongs to the set.
- Every infinite subset of the set has at least one accumulation point in the set.
- The set is closed and bounded. This is the condition that is easiest to verify, for example a closed interval or closed n -ball.

In other spaces, these conditions may or may not be equivalent, depending on the properties of the space.

Note that while compactness is a property of the set itself (with its topology), closedness is relative to a space it is in; above "closed" is used in the sense of closed in $\mathbf{R}^n$. A set which is closed in e.g. $\mathbf{Q}^n$ is typically not closed in $\mathbf{R}^n$, hence not compact.

Examples of compact spaces

- Any finite topological space, including the empty set, is compact. Slightly more generally, any space with a finite topology (only finitely many open sets) is compact; this includes in particular the trivial topology.
- The closed unit interval $[0, 1]$ is compact. This follows from the Heine–Borel theorem; the proof of which is about as hard as proving directly that $[0, 1]$ is compact. The open interval $(0, 1)$ is not compact: the open cover $(\frac{1}{n}, 1 - \frac{1}{n})$ for $n = 3, 4, \dots$ does not have a finite subcover. Similarly, the set of *rational numbers* in the closed interval $[0, 1]$ is not compact: the sets of rational numbers in the intervals $\left[0, \frac{1}{\pi} - \frac{1}{n}\right]$ and $\left[\frac{1}{\pi} + \frac{1}{n}, 1\right]$ cover all the rationals in $[0, 1]$ for $n = 4, 5, \dots$ but this cover does not have a finite subcover. (Note that the sets are open in the subspace topology even though they are not open as subsets of $\mathbf{R}$.)
- The set $\mathbf{R}$ of all real numbers is not compact as there is a cover of open intervals that does not have a finite subcover. For example, intervals $(n - 1, n + 1)$, where n takes all integer values in $\mathbf{Z}$, cover $\mathbf{R}$ but there is no finite subcover.
- For every natural number n , the n -sphere is compact. Again from the Heine–Borel theorem, the closed unit ball of any finite-dimensional normed vector space is compact. This is not true for infinite dimensions; in fact, a normed vector space is finite-dimensional if and only if its closed unit ball is compact.
- The Cantor set is compact. Since the p -adic integers are homeomorphic to the Cantor set, they also form a compact set. Since a finite set containing p elements is compact, this shows that the countable product of finite sets is compact, and is thus a special case of Tychonoff's theorem.
- Consider the set K of all functions $f : \mathbb{R} \rightarrow [0, 1]$ from the real number line to the closed unit interval, and define a topology on K so that a sequence $\{f_n\}$ in K converges towards $f \in K$ if and only if $\{f_n(x)\}$ converges towards $f(x)$ for all $x \in \mathbb{R}$. There is only one such topology; it is called the topology of pointwise convergence. Then K is a

compact topological space, again a consequence of Tychonoff's theorem.

- Consider the set K of all functions $f: [0, 1] \rightarrow [0, 1]$ satisfying the Lipschitz condition $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in [0, 1]$. Consider on K the metric induced by the uniform distance $d(f, g) = \sup\{|f(x) - g(x)|: x \in [0, 1]\}$. Then by Arzelà-Ascoli theorem the space K is compact.
- Any space carrying the cofinite topology is compact.
- Any locally compact Hausdorff space can be turned into a compact space by adding a single point to it, by means of Alexandroff one-point compactification. The one-point compactification of $\mathbf{R}$ is homeomorphic to the circle S^1 ; the one-point compactification of $\mathbf{R}^2$ is homeomorphic to the sphere S^2 . Using the one-point compactification, one can also easily construct compact spaces which are not Hausdorff, by starting with a non-Hausdorff space.
- The spectrum of any continuous linear operator on a Hilbert space is a compact subset of the complex numbers $\mathbf{C}$. If the Hilbert space is infinite-dimensional, then any compact subset of $\mathbf{C}$ arises in this manner, as the spectrum of some continuous linear operator on the Hilbert space.
- The spectrum of any commutative ring or Boolean algebra is compact.
- The Hilbert cube is compact; this follows from the Tychonoff theorem.
- The right order topology or left order topology on any bounded totally ordered set is compact. In particular, Sierpinski space is compact.
- The prime spectrum of any commutative ring with the Zariski topology is a compact space, important in algebraic geometry. These prime spectra are almost never Hausdorff spaces.
- $\mathbf{R}$ carrying the lower limit topology satisfies the property that no uncountable set is compact.
- In the cocountable topology on $\mathbf{R}$ (or any uncountable set for that matter), no infinite set is compact.
- Neither of the spaces in the previous two examples are locally compact but both are still Lindelöf

Theorems

Some theorems related to compactness (see the Topology Glossary for the definitions):

- A continuous image of a compact space is compact.^[2]
- The extreme value theorem: a continuous real-valued function on a nonempty compact space is bounded and attains its supremum.
- A closed subset of a compact space is compact.^[3]
- A compact subset of a Hausdorff space is closed.
- A nonempty compact subset of the real numbers has a greatest element and a least element.
- A subset of Euclidean n -space is compact if and only if it is closed and bounded. (Heine-Borel theorem)
- A metric space (or uniform space) is compact if and only if it is complete and totally bounded.
- The product of any collection of compact spaces is compact. (Tychonoff's theorem, which is equivalent to the axiom of choice)

- A compact Hausdorff space is normal.
- Every continuous map from a compact space to a Hausdorff space is closed and proper. It follows that every continuous bijective map from a compact space to a Hausdorff space is a homeomorphism.
- A metric space (or more generally any first-countable uniform space) is compact if and only if every sequence in the space has a convergent subsequence.
- A topological space is compact if and only if every net on the space has a convergent subnet.
- A topological space is compact if and only if every filter on the space has a convergent refinement.
- A topological space is compact if and only if every ultrafilter on the space is convergent.
- A topological space is compact if and only if every infinite subset of the space has a complete accumulation point.
- A topological space can be embedded in a compact Hausdorff space if and only if it is a Tychonoff space.
- Every non-compact topological space X is a dense subspace of a compact space which has at most one point more than X . (Alexandroff one-point compactification)
- If the metric space X is compact and an open cover of X is given, then there exists a number $\delta > 0$ such that every subset of X of diameter $< \delta$ is contained in some member of the cover. (Lebesgue's number lemma)
- If a topological space has a sub-base such that every cover of the space by members of the sub-base has a finite subcover, then the space is compact. (Alexander's sub-base theorem)
- Two compact Hausdorff spaces X_1 and X_2 are homeomorphic if and only if their rings of continuous real-valued functions $C(X_1)$ and $C(X_2)$ are isomorphic. (Gelfand–Naimark theorem)
- Let X be a simply ordered set endowed with the order topology. Then X is compact if and only if X is a complete lattice (i.e. all subsets have suprema and infima).^[4]
- Every compact metric space is separable.

Other forms of compactness

There are a number of topological properties which are equivalent to compactness in metric spaces, but are inequivalent in general topological spaces. These include the following.

- **Sequentially compact:** Every sequence has a convergent subsequence.
- **Countably compact:** Every countable open cover has a finite subcover. (Or, equivalently, every infinite subset has an ω -accumulation point.)
- **Pseudocompact :** Every real-valued continuous function on the space is bounded.
- **Weakly countably compact (or limit point compact):** Every infinite subset has an accumulation point.

While all these conditions are equivalent for metric spaces, in general we have the following implications:

- Compact spaces are countably compact.
- Sequentially compact spaces are countably compact.
- Countably compact spaces are pseudocompact and weakly countably compact.

Not every countably compact space is compact; an example is given by the first uncountable ordinal with the order topology. Not every compact space is sequentially compact; an example is given by $2^{[0,1]}$, with the product topology (Example 5.3, Scarborough and Stone 1966).

A metric space is called pre-compact or totally bounded if any sequence has a Cauchy subsequence; this can be generalised to uniform spaces. For complete metric spaces this is equivalent to compactness. See relatively compact for the topological version.

Another related notion which (by most definitions) is strictly weaker than compactness is local compactness.

Generalizations of compactness include H-closed and the property of being an H-set in a parent space. A space is H-closed if every open cover has a finite subfamily whose union is dense. Whereas we say $\mathbf{X}$ is an H-set of $\mathbf{Z}$ if every cover of $\mathbf{X}$ with open sets of $\mathbf{Z}$ has a finite subfamily whose $\mathbf{Z}$ closure contains $\mathbf{X}$.

See also

- Exhaustion by compact sets
- Lindelöf space
- Metacompact space
- Paracompact space

Notes

- [1] A space is compact if and only if the space has the finite intersection property (<http://planetmath.org/?op=getobj&from=objects&id=4181>) on PlanetMath
- [2] Compactness is preserved under a continuous map (<http://planetmath.org/?op=getobj&from=objects&id=4689>) on PlanetMath
- [3] Closed set in a compact space is compact (<http://planetmath.org/?op=getobj&from=objects&id=4177>) on PlanetMath; Closed subsets of a compact set are compact (<http://planetmath.org/?op=getobj&from=objects&id=4691>) on PlanetMath
- [4] Steen, Lynn Arthur; J. Arthur Seebach, Jr. (1995). *Counterexamples in Topology* (Dover ed.). New York: Springer-Verlag. pp. 67. ISBN 048668735X.

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- Scarborough, C.T.; Stone, A.H. (1966), "Products of nearly compact spaces", *Transactions of the American Mathematical Society* **124**: 131–147, doi: 10.2307/1994440 (<http://dx.doi.org/10.2307/1994440>)
- Steen, Lynn Arthur; Seebach, J. Arthur Jr. (1995) [1978], *Counterexamples in Topology* (Dover reprint of 1978 ed.), Berlin, New York: Springer-Verlag, MR 507446 (<http://www.ams.org/mathscinet-getitem?mr=507446>), ISBN 978-0-486-68735-3
- Countably compact (<http://planetmath.org/?op=getobj&from=objects&id=1233>) on PlanetMath

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Hausdorff space

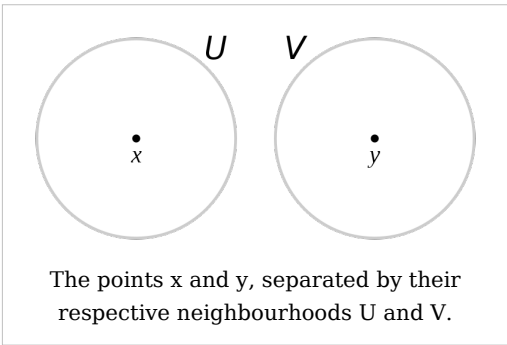
Topological spaces in separation axiom
Kolmogorov (T_0) version
$T_0 \mid T_1 \mid T_2 \mid T_{2\frac{1}{2}} \mid \text{completely } T_2$ $T_3 \mid T_{3\frac{1}{2}} \mid T_4 \mid T_5 \mid T_6$

In topology and related branches of mathematics, a **Hausdorff space**, **separated space** or **T_2 space** is a topological space in which distinct points have disjoint neighbourhoods. Of the many separation axioms that can be imposed on a topological space, the "Hausdorff condition" (T_2) is the most frequently used and discussed. It implies the uniqueness of limits of sequences, nets, and filters. Intuitively, the condition is illustrated by the pun that a space is Hausdorff if any two points can be "housed off" from each other by open sets.

Hausdorff spaces are named for Felix Hausdorff, one of the founders of topology. Hausdorff's original definition of a topological space (in 1914) included the Hausdorff condition as an axiom.

Definitions

Suppose that X is a topological space. Let x and y be points in X . We say that x and y can be *separated by neighbourhoods* if there exists a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint ($U \cap V = \emptyset$). X is a **Hausdorff space** if any two distinct points of X can be separated by neighborhoods. This condition is the third separation axiom (after T_0 and T_1), which is why Hausdorff spaces are also called T_2 spaces. The name *separated space* is also used.



A related, but weaker, notion is that of a **preregular space**. X is a preregular space if any two topologically distinguishable points can be separated by neighbourhoods. Preregular spaces are also called R_1 spaces.

The relationship between these two conditions is as follows. A topological space is Hausdorff if and only if it is both preregular (i.e. topologically distinguishable points are separated) and Kolmogorov (i.e. distinct points are topologically distinguishable). A topological space is preregular if and only if its Kolmogorov quotient is Hausdorff.

Equivalences

For a topological space X , the following are equivalent:

- X is a Hausdorff space.
- Every singleton set contained in X is equal to the intersection of all closed neighbourhoods containing it.
- The diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed as a subset of the product space $X \times X$.

Examples and counterexamples

Almost all spaces encountered in analysis are Hausdorff; most importantly, the real numbers (under the standard metric topology on real numbers) are a Hausdorff space. More generally, all metric spaces are Hausdorff. In fact, many spaces of use in analysis, such as topological groups and topological manifolds, have the Hausdorff condition explicitly stated in their definitions.

A simple example of a topology that is T_1 but is not Hausdorff is the cofinite topology defined by an infinite set.

Pseudometric spaces typically are not Hausdorff, but they are preregular, and their use in analysis is usually only in the construction of Hausdorff gauge spaces. Indeed, when analysts run across a non-Hausdorff space, it is still probably at least preregular, and then they simply replace it with its Kolmogorov quotient, which is Hausdorff.

In contrast, non-preregular spaces are encountered much more frequently in abstract algebra and algebraic geometry, in particular as the Zariski topology on an algebraic variety or the spectrum of a ring. They also arise in the model theory of intuitionistic logic: every complete Heyting algebra is the algebra of open sets of some topological space, but this space need not be preregular, much less Hausdorff.

While convergent sequences in a Hausdorff space have a unique limit, there are non-Hausdorff T_1 spaces in which every convergent sequence has a unique limit.^[1]

Properties

Subspaces and products of Hausdorff spaces are Hausdorff,^[2] but quotient spaces of Hausdorff spaces need not be Hausdorff. In fact, *every* topological space can be realized as the quotient of some Hausdorff space.

Hausdorff spaces are T_1 , meaning that all singletons are closed. Similarly, preregular spaces are R_0 .

Another nice property of Hausdorff spaces is that compact sets are always closed.^[3] This may fail for spaces which are non-Hausdorff (there are examples of T_1 spaces where it fails).

The definition of a Hausdorff space says that points can be separated by neighborhoods. It turns out that this implies something which is seemingly stronger: in a Hausdorff space every pair of disjoint compact sets can also be separated by neighborhoods^[4], in other words there is a neighborhood of one set and a neighborhood of the other, such that the two neighborhoods are disjoint. This is an example of the general rule that compact sets often behave like points.

Compactness conditions together with preregularity often imply stronger separation axioms. For example, any locally compact preregular space is completely regular. Compact

preregular spaces are normal, meaning that they satisfy Urysohn's lemma and the Tietze extension theorem and have partitions of unity subordinate to locally finite open covers. The Hausdorff versions of these statements are: every locally compact Hausdorff space is Tychonoff, and every compact Hausdorff space is normal Hausdorff.

The following results are some technical properties regarding maps (continuous and otherwise) to and from Hausdorff spaces.

Let $f : X \rightarrow Y$ be a continuous function and suppose Y is Hausdorff. Then the graph of f , $\{(x, f(x)) \mid x \in X\}$, is a closed subset of $X \times Y$.

Let $f : X \rightarrow Y$ be a function and let $\ker(f) \triangleq \{(x, x') \mid f(x) = f(x')\}$ be its kernel regarded as a subspace of $X \times X$.

- If f is continuous and Y is Hausdorff then $\ker(f)$ is closed.
- If f is an open surjection and $\ker(f)$ is closed then Y is Hausdorff.
- If f is a continuous, open surjection (i.e. an open quotient map) then Y is Hausdorff if and only if $\ker(f)$ is closed.

If $f, g : X \rightarrow Y$ are continuous maps and Y is Hausdorff then the equalizer $\text{eq}(f, g) = \{x \mid f(x) = g(x)\}$ is closed in X . It follows that if Y is Hausdorff and f and g agree on a dense subset of X then $f = g$. In other words, continuous functions into Hausdorff spaces are determined by their values on dense subsets.

Let $f : X \rightarrow Y$ be a closed surjection such that $f^{-1}(y)$ is compact for all $y \in Y$. Then if X is Hausdorff so is Y .

Let $f : X \rightarrow Y$ be a quotient map with X a compact Hausdorff space. Then the following are equivalent

- Y is Hausdorff
- f is a closed map
- $\ker(f)$ is closed

Preregularity versus regularity

All regular spaces are preregular, as are all Hausdorff spaces. There are many results for topological spaces that hold for both regular and Hausdorff spaces. Most of the time, these results hold for all preregular spaces; they were listed for regular and Hausdorff spaces separately because the idea of preregular spaces came later. On the other hand, those results that are truly about regularity generally don't also apply to nonregular Hausdorff spaces.

There are many situations where another condition of topological spaces (such as paracompactness or local compactness) will imply regularity if preregularity is satisfied. Such conditions often come in two versions: a regular version and a Hausdorff version. Although Hausdorff spaces aren't generally regular, a Hausdorff space that is also (say) locally compact will be regular, because any Hausdorff space is preregular. Thus from a certain point of view, it is really preregularity, rather than regularity, that matters in these situations. However, definitions are usually still phrased in terms of regularity, since this condition is better known than preregularity.

See History of the separation axioms for more on this issue.

Variants

The terms "Hausdorff", "separated", and "preregular" can also be applied to such variants on topological spaces as uniform spaces, Cauchy spaces, and convergence spaces. The characteristic that unites the concept in all of these examples is that limits of nets and filters (when they exist) are unique (for separated spaces) or unique up to topological indistinguishability (for preregular spaces).

As it turns out, uniform spaces, and more generally Cauchy spaces, are always preregular, so the Hausdorff condition in these cases reduces to the T_0 condition. These are also the spaces in which completeness makes sense, and Hausdorffness is a natural companion to completeness in these cases. Specifically, a space is complete if and only if every Cauchy net has at *least* one limit, while a space is Hausdorff if and only if every Cauchy net has at *most* one limit (since only Cauchy nets can have limits in the first place).

See also

- Weak Hausdorff space

Notes

- [1] van Douwen, Eric K. *An anti-Hausdorff Fréchet space in which convergent sequences have unique limits*. Topology and its Applications 51 (1993) 147–158
- [2] Hausdorff property is hereditary (<http://planetmath.org/?op=getobj&from=objects&id=7202>) on PlanetMath
- [3] Proof of A compact set in a Hausdorff space is closed (<http://planetmath.org/?op=getobj&from=objects&id=4203>) on PlanetMath
- [4] Point and a compact set in a Hausdorff space have disjoint open neighborhoods (<http://planetmath.org/?op=getobj&from=objects&id=4193>) on PlanetMath

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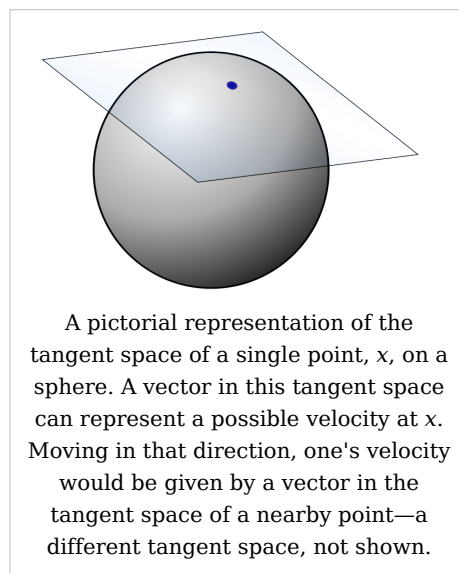
Tangent space

In mathematics, the **tangent space** of a manifold is a concept which facilitates the generalization of vectors from affine spaces to general manifolds, since in the latter case one cannot simply subtract two points to obtain a vector pointing from one to the other.

Informal description

In differential geometry, one can attach to every point x of a differentiable manifold a **tangent space**, a real vector space which intuitively contains the possible "directions" in which one can pass through x . The elements of the tangent space are called **tangent vectors** at x . This is a generalization of the notion of a bound vector in a Euclidean space. All the tangent spaces have the same dimension, equal to the dimension of the manifold.

For example, if the given manifold is a 2-sphere, one can picture the tangent space at a point as the plane which touches the sphere at that point and is perpendicular to the sphere's radius through the point. More generally, if a given manifold is thought of as an embedded submanifold of Euclidean space one can picture the tangent space in this literal fashion.



In algebraic geometry, in contrast, there is an intrinsic definition of **tangent space at a point P** of a variety V , that gives a vector space of dimension at least that of V . The points P at which the dimension is exactly that of V are called the **non-singular** points; the others are **singular** points. For example, a curve that crosses itself doesn't have a unique tangent line at that point. The singular points of V are those where the 'test to be a manifold' fails. See Zariski tangent space.

Once tangent spaces have been introduced, one can define vector fields, which are abstractions of the velocity field of particles moving on a manifold. A vector field attaches to every point of the manifold a vector from the tangent space at that point, in a smooth manner. Such a vector field serves to define a generalized ordinary differential equation on a manifold: a solution to such a differential equation is a differentiable curve on the manifold whose derivative at any point is equal to the tangent vector attached to that point by the vector field.

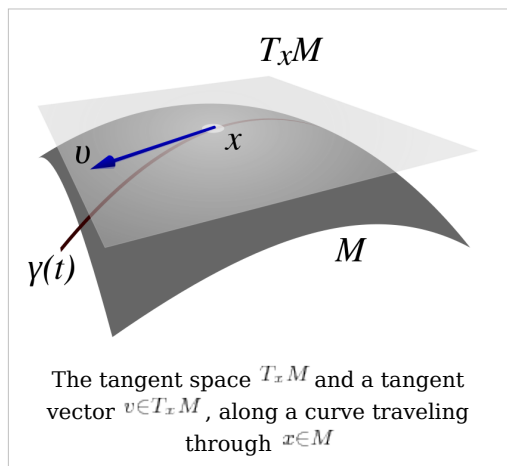
All the tangent spaces can be "glued together" to form a new differentiable manifold of twice the dimension, the tangent bundle of the manifold.

Formal definitions

There are various equivalent ways of defining the tangent spaces of a manifold. While the definition via directions of curves is quite straightforward given the above intuition, it is also the most cumbersome to work with. More elegant and abstract approaches are described below.

Definition as directions of curves

Suppose M is a C^k manifold ($k \geq 1$) and x is a point in M . Pick a chart $\varphi : U \rightarrow \mathbf{R}^n$ where U is an open subset of M containing x . Suppose two curves $\gamma_1 : (-1,1) \rightarrow M$ and $\gamma_2 : (-1,1) \rightarrow M$ with $\gamma_1(0) = \gamma_2(0) = x$ are given such that $\varphi \circ \gamma_1$ and $\varphi \circ \gamma_2$ are both differentiable at 0. Then γ_1 and γ_2 are called *tangent at 0* if the ordinary derivatives of $\varphi \circ \gamma_1$ and $\varphi \circ \gamma_2$ at 0 coincide. This defines an equivalence relation on such curves, and the equivalence classes are known as the tangent vectors of M at x . The equivalence class of the curve γ is written as $\gamma'(0)$. The tangent space of M at x , denoted by $T_x M$, is defined as the set of all tangent vectors; it does not depend on the choice of chart φ .



Definition via derivations

Suppose M is a C^∞ manifold. A real-valued function $f : M \rightarrow \mathbf{R}$ belongs to $C^\infty(M)$ if $f \circ \varphi^{-1}$ is infinitely often differentiable for every chart $\varphi : U \rightarrow \mathbf{R}^n$. $C^\infty(M)$ is a real associative algebra for the pointwise product and sum of functions and scalar multiplication.

Pick a point x in M . A *derivation* at x is a linear map $D : C^\infty(M) \rightarrow \mathbf{R}$ which has the property that for all f, g in $C^\infty(M)$:

$$D(fg) = D(f) \cdot g(x) + f(x) \cdot D(g)$$

modeled on the product rule of calculus. These derivations form a real vector space in a natural manner; this is the tangent space $T_x M$.

The relation between the tangent vectors defined earlier and derivations is as follows: if γ is a curve with tangent vector $\gamma'(0)$, then the corresponding derivation is $D(f) = (f \circ \gamma)'(0)$ (where the derivative is taken in the ordinary sense, since $f \circ \gamma$ is a function from $(-1,1)$ to $\mathbf{R}$).

Generalizations of this definition are possible, for instance to complex manifolds and algebraic varieties. However, instead of examining derivations D from the full algebra of functions, one must instead work at the level of germs of functions. The reason is that the structure sheaf may not be fine for such structures. For instance, let X be an algebraic

variety with structure sheaf F . Then the Zariski tangent space at a point $p \in X$ is the collection of K -derivations $D: F_p \rightarrow K$, where K is the groundfield and F_p is the stalk of F at p .

Definition via the cotangent space

Again we start with a C^∞ manifold M and a point x in M . Consider the ideal I in $C^\infty(M)$ consisting of all functions f such that $f(x) = 0$. Then I and I^2 are real vector spaces, and T_x^*M may be defined as the dual space of the quotient space I / I^2 . This latter quotient space is also known as the cotangent space of M at x .

While this definition is the most abstract, it is also the one most easily transferred to other settings, for instance to the varieties considered in algebraic geometry.

If D is a derivation, then $D(f) = 0$ for every f in I^2 , and this means that D gives rise to a linear map $I / I^2 \rightarrow \mathbf{R}$. Conversely, if $r: I / I^2 \rightarrow \mathbf{R}$ is a linear map, then $D(f) = r(f - f(x)) + I^2$ is a derivation. This yields the correspondence between the tangent space defined via derivations and the tangent space defined via the cotangent space.

Properties

If M is an open subset of $\mathbf{R}^n$, then M is a C^∞ manifold in a natural manner (take the charts to be the identity maps), and the tangent spaces are all naturally identified with $\mathbf{R}^n$.

Tangent vectors as directional derivatives

One way to think about tangent vectors is as directional derivatives. Given a vector v in $\mathbf{R}^n$ one defines the directional derivative of a smooth map $f: \mathbf{R}^n \rightarrow \mathbf{R}$ at a point x by

$$D_v f(x) = \left. \frac{d}{dt} \right|_{t=0} f(x + tv) = \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(x).$$

This map is naturally a derivation. Moreover, it turns out that every derivation of $C^\infty(\mathbf{R}^n)$ is of this form. So there is a one-to-one map between vectors (thought of as tangent vectors at a point) and derivations.

Since tangent vectors to a general manifold can be defined as derivations it is natural to think of them as directional derivatives. Specifically, if v is a tangent vector of M at a point x (thought of as a derivation) then define the directional derivative in the direction v by

$$D_v(f) = v(f)$$

where $f: M \rightarrow \mathbf{R}$ is an element of $C^\infty(M)$. If we think of v as the direction of a curve, $v = \gamma'(0)$, then we write

$$D_v(f) = (f \circ \gamma)'(0).$$

The derivative of a map

Main article: Pushforward (differential)

Every smooth (or differentiable) map $\varphi: M \rightarrow N$ between smooth (or differentiable) manifolds induces natural linear maps between the corresponding tangent spaces:

$$d\varphi_x: T_x M \rightarrow T_{\varphi(x)} N.$$

If the tangent space is defined via curves, the map is defined as

$$d\varphi_x(\gamma'(0)) = (\varphi \circ \gamma)'(0).$$

If instead the tangent space is defined via derivations, then

$$d\varphi_x(X)(f) = X(f \circ \varphi).$$

The linear map $d\varphi_x$ is called variously the *derivative*, *total derivative*, *differential*, or *pushforward* of φ at x . It is frequently expressed using a variety of other notations:

$$D\varphi_x, \quad (\varphi_*)_x, \quad \varphi'(x).$$

In a sense, the derivative is the best linear approximation to φ near x . Note that when $N = \mathbf{R}$, the map $d\varphi_x : T_x M \rightarrow \mathbf{R}$ coincides with the usual notion of the differential of the function φ . In local coordinates the derivative of f is given by the Jacobian.

An important result regarding the derivative map is the following:

Theorem. If $\varphi : M \rightarrow N$ is a local diffeomorphism at x in M then $d\varphi_x : T_x M \rightarrow T_{\varphi(x)} N$ is a linear isomorphism. Conversely, if $d\varphi_x$ is an isomorphism then there is an open neighborhood U of x such that φ maps U diffeomorphically onto its image.

This is a generalization of the inverse function theorem to maps between manifolds.

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External links

- Tangent Planes ^[1] at MathWorld
- Berkeley Mathematics lecture ^[2] on Tangent Planes

References

[1] <http://mathworld.wolfram.com/TangentPlane.html>

[2] <http://academicearth.org/lectures/tangent-planes-and-linear-approximations>

Riemann space

1. redirect metric space

Riemannian manifold

In Riemannian geometry, a **Riemannian manifold** (M, g) (with **Riemannian metric** g) is a real differentiable manifold M in which each tangent space is equipped with an inner product g in a manner which varies smoothly from point to point. The metric g is a positive definite metric tensor. This allows one to define various notions such as angles, lengths of curves, areas (or volumes), curvature, gradients of functions and divergence of vector fields. In other words, a Riemannian manifold is a differentiable manifold in which the tangent space at each point is a finite-dimensional Hilbert space. The terms are named after German mathematician Bernhard Riemann.

Overview

The tangent bundle of a smooth manifold M assigns to each fixed point of M a vector space called the tangent space, and each tangent space can be equipped with an inner product. If such a collection of inner products on the tangent bundle of a manifold varies smoothly as one traverses the manifold, then concepts that were defined only pointwise at each tangent space can be extended to yield analogous notions over finite regions of the manifold. For example, a smooth curve $\alpha(t): [0, 1] \rightarrow M$ has tangent vector $\alpha'(t_0)$ in the tangent space $TM(t_0)$ at any point $t_0 \in (0, 1)$, and each such vector has length $\|\alpha'(t_0)\|$, where $\|\cdot\|$ denotes the norm induced by the inner product on $TM(t_0)$. The integral of these lengths gives the length of the curve α :

$$L(\alpha) = \int_0^1 \|\alpha'(t)\| dt.$$

Smoothness of $\alpha(t)$ for t in $[0, 1]$ guarantees that the integral $L(\alpha)$ exists and the length of this curve is defined.

In many instances, in order to pass from a linear-algebraic concept to a differential-geometric one, the smoothness requirement is very important.

Every smooth submanifold of $\mathbf{R}^n$ has an induced Riemannian metric g : the inner product on each tangent space is the restriction of the inner product on $\mathbf{R}^n$. In fact, as follows from the Nash embedding theorem, all Riemannian manifolds can be realized this way. In particular one could *define* Riemannian manifold as a metric space which is isometric to a smooth submanifold of $\mathbf{R}^n$ with the induced intrinsic metric, where isometry here is meant in the sense of preserving the length of curves. This definition might theoretically not be flexible enough, but it is quite useful to build the first geometric intuitions in Riemannian geometry.

Riemannian manifolds as metric spaces

Usually a Riemannian manifold is defined as a smooth manifold with a smooth section of the positive-definite quadratic forms on the tangent bundle. Then one has to work to show that it can be turned to a metric space:

If $\gamma: [a, b] \rightarrow M$ is a continuously differentiable curve in the Riemannian manifold M , then we define its length $L(\gamma)$ in analogy with the example above by

$$L(\gamma) = \int_a^b \|\gamma'(t)\| dt.$$

With this definition of length, every connected Riemannian manifold M becomes a metric space (and even a length metric space) in a natural fashion: the distance $d(x, y)$ between the points x and y of M is defined as

$$d(x, y) = \inf\{L(\gamma) : \gamma \text{ is a continuously differentiable curve joining } x \text{ and } y\}.$$

Even though Riemannian manifolds are usually "curved", there is still a notion of "straight line" on them: the geodesics. These are curves which locally join their points along shortest paths.

Assuming the manifold is compact, any two points x and y can be connected with a geodesic whose length is $d(x, y)$. Without compactness, this need not be true. For example, in the punctured plane $\mathbf{R}^2 \setminus \{0\}$, the distance between the points $(-1, 0)$ and $(1, 0)$ is 2, but there is no geodesic realizing this distance.

Properties

In Riemannian manifolds, the notions of geodesic completeness, topological completeness and metric completeness are the same: that each implies the other is the content of the Hopf-Rinow theorem.

Riemannian metrics

Let M be a differentiable manifold of dimension n . A **Riemannian metric** on M is a family of (positive definite) inner products

$$g_p : T_p M \times T_p M \longrightarrow \mathbf{R}, \quad p \in M$$

such that, for all differentiable vector fields X, Y on M ,

$$p \mapsto g_p(X(p), Y(p))$$

defines a differentiable function $M \rightarrow \mathbf{R}$. The assignment of an inner product g_p to each point p of the manifold is called a **metric tensor**.

In a system of local coordinates on the manifold M given by n real-valued functions $x_1, x_2, \dots, x_n$, the vector fields

$$\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}$$

give a basis of tangent vectors at each point of M . Relative to this coordinate system, the components of the metric tensor are, at each point p ,

$$g_{ij}(p) := g_p \left(\left(\frac{\partial}{\partial x_i} \right)_p, \left(\frac{\partial}{\partial x_j} \right)_p \right).$$

Equivalently, the metric tensor can be written in terms of the dual basis $\{dx_1, \dots, dx_n\}$ of the cotangent bundle as

$$g = \sum_{i,j} g_{ij} dx_i \otimes dx_j.$$

Endowed with this metric, the differentiable manifold (M, g) is a **Riemannian manifold**.

Examples

- With $\frac{\partial}{\partial x_i}$ identified with $e_i = (0, \dots, 1, \dots, 0)$, the standard metric over an open subset $U \subset \mathbb{R}^n$ is defined by

$$g_p^{\text{can}} : T_p U \times T_p U \longrightarrow \mathbb{R}, \quad \left(\sum_i a_i \frac{\partial}{\partial x_i}, \sum_j b_j \frac{\partial}{\partial x_j} \right) \longmapsto \sum_i a_i b_i.$$

Then g is a Riemannian metric, and

$$g_{ij}^{\text{can}} = \langle e_i, e_j \rangle = \delta_{ij}.$$

Equipped with this metric, $\mathbf{R}^n$ is called **Euclidean space** of dimension n and g_{ij}^{can} is called the **Euclidean metric**.

- Let (M, g) be a Riemannian manifold and $N \subset M$ be a submanifold of M . Then the restriction of g to vectors tangent along N defines a Riemannian metric over N .
- More generally, let $f: M^n \rightarrow N^{n+k}$ be an immersion. Then, if N has a Riemannian metric, f induces a Riemannian metric on M via pullback:

$$g_p^M : T_p M \times T_p M \longrightarrow \mathbb{R}, \quad (u, v) \longmapsto g_p^M(u, v) := g_{f(p)}^N(T_p f(u), T_p f(v)).$$

This is then a metric; the positive definiteness follows of the injectivity of the differential of an immersion.

- Let (M, g^M) be a Riemannian manifold, $h: M^{n+k} \rightarrow N^k$ be a differentiable application and $q \in N$ be a regular value of h (the differential $dh(p)$ is surjective for all $p \in h^{-1}(q)$). Then $N = h^{-1}(q) \subset M$ is a submanifold of M of dimension n . Thus N carries the Riemannian metric induced by inclusion.
- In particular, consider the following application :

$$h : \mathbb{R}^n \longrightarrow \mathbb{R}, \quad (x_1, \dots, x_n) \longmapsto \sum_{i=1}^n x_i^2 - 1.$$

Then, 0 is a regular value of h and

$$h^{-1}(0) = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 = 1\} = S^{n-1}$$

is the unit sphere $S^{n-1} \subset \mathbb{R}^n$. The metric induced from $\mathbb{R}^n$ on S^{n-1} is called the **canonical metric** of S^{n-1} .

- Let M_1 and M_2 be two Riemannian manifolds and consider the cartesian product $M_1 \times M_2$ with the product structure. Furthermore, let $\pi_1 : M_1 \times M_2 \rightarrow M_1$ and $\pi_2 : M_1 \times M_2 \rightarrow M_2$ be the natural projections. For $(p, q) \in M_1 \times M_2$, a Riemannian metric on $M_1 \times M_2$ can be introduced as follows :

$$g_{(p,q)}^{M_1 \times M_2} : T_{(p,q)}(M_1 \times M_2) \times T_{(p,q)}(M_1 \times M_2) \longrightarrow \mathbb{R}, \quad (u, v) \longmapsto g_p^{M_1}(T_{(p,q)}\pi_1(u), T_{(p,q)}\pi_1(v)) + g_q^{M_2}(T_{(p,q)}\pi_2(u), T_{(p,q)}\pi_2(v)).$$

The identification

$$T_{(p,q)}(M_1 \times M_2) \cong T_p M_1 \oplus T_q M_2$$

allows us to conclude that this defines a metric on the product space.

The torus $S^1 \times \dots \times S^1 = T^n$ possesses for example a Riemannian structure obtained by choosing the induced Riemannian metric from $\mathbb{R}^2$ on the circle $S^1 \subset \mathbb{R}^2$ and then taking the product metric. The torus T^n endowed with this metric is called the flat torus.

- Let g_0, g_1 be two metrics on M . Then,

$$\tilde{g} := \lambda g_0 + (1 - \lambda)g_1, \quad \lambda \in [0, 1],$$

is also a metric on M .

The pullback metric

If $f: M \rightarrow N$ is a diffeomorphism and (N, g^N) a Riemannian manifold, then the pullback of g^N along f is a Riemannian metric on M . The pullback is the metric f^*g^N on M defined for $v, w \in T_p M$ by

$$(f^*g^N)(v, w) = g^N(df(v), df(w)).$$

where $df(v)$ is the pushforward of v by f .

Existence of a metric

Every paracompact differentiable manifold admits a Riemannian metric. To prove this result, let M be a manifold and $\{(U_\alpha, \phi(U_\alpha)) | \alpha \in I\}$ a locally finite atlas of open subsets U of M and diffeomorphisms onto open subsets of $\mathbb{R}^n$

$$\phi : U_\alpha \rightarrow \phi(U_\alpha) \subseteq \mathbb{R}^n.$$

Let τ_α be a differentiable partition of unity subordinate to the given atlas. Then define the metric g on M by

$$g := \sum_{\beta} \tau_{\beta} \cdot \tilde{g}_{\beta}, \quad \text{with} \quad \tilde{g}_{\beta} := \phi_{\beta}^* g^{\text{can}}.$$

where g^{can} is the Euclidean metric. This is readily seen to be a metric on M .

Isometries

Let (M, g^M) and (N, g^N) be two Riemannian manifolds, and $f : M \rightarrow N$ be a diffeomorphism. Then, f is called an **isometry**, if

$$g_p^M(u, v) = g_{f(p)}^N(T_p f(u), T_p f(v)) \quad \forall p \in M, \forall u, v \in T_p M.$$

Moreover, a differentiable mapping $f : M \rightarrow N$ is called a **local isometry** at $p \in M$ if there is a neighbourhood $U \subset M$, $U \ni p$, such that $f : U \rightarrow f(U)$ is a diffeomorphism satisfying the previous relation.

Riemannian manifolds as metric spaces

A connected Riemannian manifold carries the structure of a metric space whose distance function is the arclength of a minimizing geodesic.

Specifically, let (M, g) be a connected Riemannian manifold. Let $c : [a, b] \rightarrow M$ be a parametrized curve in M , which is differentiable with velocity vector c' . The length of c is defined as

$$L_a^b(c) := \int_a^b \sqrt{g(c'(t), c'(t))} dt = \int_a^b \|c'(t)\| dt$$

By change of variables, the arclength is independent of the chosen parametrization. In particular, a curve $[a, b] \rightarrow M$ can be parametrized by its arc length. A curve is parametrized by arclength if and only if $\|c'(t)\| = 1$ for all $t \in [a, b]$.

The distance function $d : M \times M \rightarrow [0, \infty)$ is defined by

$$d(p, q) = \inf L(\gamma)$$

where the infimum extends over all differentiable curves γ beginning at $p \in M$ and ending at $q \in M$.

This function d satisfies the properties of a distance function for a metric space. The only property which is not completely straightforward is to show that $d(p, q) = 0$ implies that $p = q$. For this property, one can use a normal coordinate system, which also allows one to show that the topology induced by d is the same as the original topology on M .

Diameter

The **diameter** of a Riemannian manifold M is defined by

$$\text{diam}(M) := \sup_{p, q \in M} d(p, q) \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

The diameter is invariant under global isometries. Furthermore, the Heine-Borel property holds for (finite-dimensional) Riemannian manifolds: M is compact if and only if it is complete and has finite diameter.

Geodesic completeness

A Riemannian manifold M **is geodesically complete** if for all $p \in M$, the exponential map $\exp_p$ is defined for all $v \in T_p M$, i.e. if any geodesic $\gamma(t)$ starting from p is defined for all values of the parameter $t \in \mathbb{R}$. The Hopf-Rinow theorem asserts that M is geodesically complete if and only if it is complete as a metric space.

If M is complete, then M is non-extendable in the sense that it is not isometric to a proper submanifold of any other Riemannian manifold. The converse is not true, however: there exist non-extendable manifolds which are not complete.

See also

- Riemannian geometry
- Finsler manifold
- sub-Riemannian manifold
- pseudo-Riemannian manifold
- Metric tensor
- Hermitian manifold

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Kähler manifold

In mathematics, a **Kähler manifold** is a manifold with unitary structure (a $U(n)$ -structure) satisfying an integrability condition. In particular, it is a Riemannian manifold,^[1] a complex manifold, and a symplectic manifold, with these three structures all mutually compatible.

This threefold structure corresponds to the presentation of the unitary group as an intersection:

$$U(n) = O(2n) \cap GL(n, \mathbb{C}) \cap Sp(2n).$$

Without any integrability conditions, the analogous notion is an almost Hermitian manifold. If the Sp -structure is integrable (but the complex structure need not be), the notion is an almost Kähler manifold; if the complex structure is integrable (but the Sp -structure need not be), the notion is a Hermitian manifold.

Kähler manifolds are named after the mathematician Erich Kähler and are important in algebraic geometry: they are a differential geometric generalization of complex algebraic varieties

Definition

A manifold with a Hermitian metric is an almost Hermitian manifold; a Kähler manifold is a manifold with a Hermitian metric that satisfies an integrability condition, which has several equivalent formulations.

Kähler manifolds can be characterized in many ways: they are often defined as a complex manifold with an additional structure (or a symplectic manifold with an additional structure, or a Riemannian manifold with an additional structure).

One can summarize the connection between the three structures via $h = g + i\omega$, where h is the Hermitian form, g is the Riemannian metric, i is the almost complex structure, and ω is the almost symplectic structure.

A **Kähler metric** on a complex manifold M is a hermitian metric on the tangent bundle TM satisfying a condition that has several equivalent characterizations (the most

geometric being that parallel transport induced by the metric gives rise to complex-linear mappings on the tangent spaces). In terms of local coordinates it is specified in this way: if

$$h = \sum h_{i\bar{j}} dz^i \otimes d\bar{z}^j$$

is the hermitian metric, then the associated **Kähler form** defined (up to a factor of $i/2$) by

$$\omega = \sum h_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

is closed: that is, $d\omega = 0$. If M carries such a metric it is called a Kähler manifold.

The metric on a Kähler manifold locally satisfies

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^j}$$

for some function K , called the Kähler potential.

A Kähler manifold, the associated Kähler form and metric are called **Kähler-Einstein** (or sometimes Einstein-Kähler) iff its Ricci tensor is proportional to the metric tensor, $R = \lambda g$, for some constant λ . This name is a reminder of Einstein's considerations about the cosmological constant. See the article on Einstein manifolds for more details.

Examples

1. Complex Euclidean space $\mathbf{C}^n$ with the standard Hermitian metric is a Kähler manifold.
2. A torus $\mathbf{C}^n/\Lambda$ (Λ a full lattice) inherits a flat metric from the Euclidean metric on $\mathbf{C}^n$, and is therefore a compact Kähler manifold.
3. Every Riemannian metric on a Riemann surface is Kähler, since the condition for ω to be closed is trivial in 2 (real) dimensions.
4. Complex projective space $\mathbf{CP}^n$ admits a homogeneous Kähler metric, the Fubini-Study metric. An Hermitian form in (the vector space) $\mathbf{C}^{n+1}$ defines a unitary subgroup $U(n+1)$ in $GL(n+1, \mathbf{C})$; a Fubini-Study metric is determined up to homothety (overall scaling) by invariance under such a $U(n+1)$ action. By elementary linear algebra, any two Fubini-Study metrics are isometric under a projective automorphism of $\mathbf{CP}^n$, so it is common to speak of "the" Fubini-Study metric.
5. The induced metric on a complex submanifold of a Kähler manifold is Kähler. In particular, any Stein manifold (embedded in $\mathbf{C}^n$) or algebraic variety (embedded in $\mathbf{CP}^n$) is of Kähler type. This is fundamental to their analytic theory.
6. The unit complex ball $\mathbf{B}^n$ admits a Kähler metric called the Bergman metric which has constant holomorphic sectional curvature.
7. Every K3 surface is Kähler (by a theorem of Y.-T. Siu).

An important subclass of Kähler manifolds are Calabi-Yau manifolds.

See also

- Almost complex manifold
- Hyper-Kähler manifold
- Kähler-Einstein metric
- Quaternion-Kähler manifold
- Complex Poisson manifold

Notes

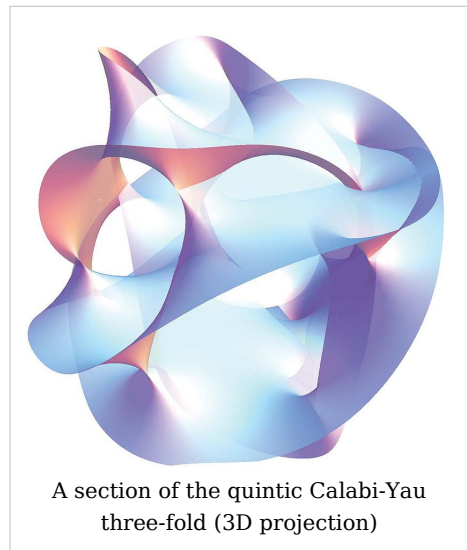
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Calabi-Yau manifold

In mathematics, **Calabi-Yau manifolds** are sometimes defined as compact Kähler manifolds whose canonical bundle is trivial, though many other similar but inequivalent definitions are sometimes used. They were named "Calabi-Yau spaces" by Candelas et al. (1985) after *E. Calabi* (1954, 1957) who first studied them, and *S. T. Yau* (1978) who proved the Calabi conjecture that they have Ricci flat metrics. In superstring theory the extra dimensions of spacetime are sometimes conjectured to take the form of a 6-dimensional Calabi-Yau manifold, which led to the idea of mirror symmetry.



Definitions

There are many different inequivalent definitions of a Calabi-Yau manifold used by different authors. This section summarizes some of the more common definitions and the relations between them.

A **Calabi-Yau n -fold** or **Calabi-Yau manifold** of dimension n is sometimes defined as a compact n -dimensional Kähler manifold M satisfying one of the following equivalent conditions:

- The canonical bundle of M is trivial.
- M has a holomorphic n -form that vanishes nowhere.
- The structure group of M can be reduced from $U(n)$ to $SU(n)$.
- The first **integral** Chern class $c_1(M)$ of M vanishes.

- M has a Kähler metric with **global** holonomy contained in $SU(n)$.

For a compact n -dimensional Kähler manifold M the following conditions are equivalent to each other, but are weaker than the conditions above, and are sometimes used as the definition of a Calabi-Yau manifold:

- M has vanishing first **real** Chern class.
- M has a Kähler metric with vanishing Ricci curvature.
- M has a Kähler metric with **local** holonomy contained in $SU(n)$.
- A positive power of the canonical bundle of M is trivial.
- M has a finite cover that has trivial canonical bundle.
- M has a finite cover that is a product of a torus and a simply connected manifold with trivial canonical bundle.

In particular if a compact Kähler manifold is simply connected then all the weak definition above is equivalent to the stronger definition. Enriques surfaces give examples of complex manifolds that have Ricci-flat metrics, but their canonical bundles are not trivial so they are Calabi-Yau manifolds according to the second but not the first definition above. Their double covers are Calabi-Yau manifolds for both definitions (in fact K3 surfaces).

By far the hardest part of proving the equivalences between the various properties above is proving the existence of Ricci-flat metrics. This follows from Yau's proof of the Calabi conjecture, which implies that a compact Kähler manifold with a vanishing first real Chern class has a Kähler metric in the same class with vanishing Ricci curvature. (The class of a Kähler metric is the cohomology class of its associated 2-form.) Calabi showed such a metric is unique.

There are many other inequivalent definitions of Calabi-Yau manifolds that are sometimes used, which differ in the following ways (among others):

- The first Chern class may vanish as an integral class or as a real class.
- Most definitions assert that Calabi-Yau manifolds are compact, but some allow them to be non-compact. In the generalization to non-compact manifolds, the difference $(\Omega \wedge \bar{\Omega} - J^n/n!)$ must vanish asymptotically. Here, J is the Kähler form associated with the Kähler metric, g (Tian 1990, 1991).
- Some definitions put restrictions on the fundamental group of a Calabi-Yau manifold, such as demanding that it be finite or trivial. Any Calabi-Yau manifold has a finite cover that is the product of a torus and a simply-connected Calabi-Yau manifold.
- Some definitions require that the holonomy be exactly equal to $SU(n)$ rather than a subgroup of it, which implies that the Hodge numbers $h^{i,0}$ vanish for $0 < i < \dim(M)$. Abelian surfaces have a (Ricci) flat metric with holonomy strictly smaller than $SU(2)$ (in fact 0) so are not Calabi-Yau manifolds according such definitions.
- Most definitions assume that a Calabi-Yau manifold has a Riemannian metric, but some treat them as complex manifolds without a metric.
- Most definitions assume the manifold is non-singular, but some allow mild singularities. While the Chern class fails to be well-defined for singular Calabi-Yau's, the canonical bundle and canonical class may still be defined if all the singularities are Gorenstein, and so may be used to extend the definition of a smooth Calabi-Yau manifold to a possibly singular Calabi-Yau variety.

Examples

In one complex dimension, the only compact examples are tori, which form a one-parameter family. The Ricci-flat metric on a torus is actually a flat metric, so that the holonomy is the trivial group $SU(1)$. A one-dimensional Calabi-Yau manifold is a complex elliptic curve, and in particular, algebraic.

In two complex dimensions, the K3 surfaces furnish the only compact simply connected Calabi-Yau manifolds. Non simply-connected examples are given by abelian surfaces. Enriques surfaces and hyperelliptic surfaces have first Chern class that vanishes as an element of the real cohomology group, but not as an element of the integral cohomology group, so Yau's theorem about the existence of a Ricci-flat metric still applies to them but they are sometimes not considered to be Calabi-Yau manifolds. Abelian surfaces are sometimes excluded from the classification of being Calabi-Yau, as their holonomy (again the trivial group) is a proper subgroup of $SU(2)$, instead of being isomorphic to $SU(2)$.

In three complex dimensions, classification of the possible Calabi-Yau manifolds is an open problem, although Yau suspects that there is a finite number of families (albeit a much bigger number than his estimate from 20 years ago). One example of a three-dimensional Calabi-Yau manifold is a non-singular quintic threefold in $\mathbf{CP}^4$, which is the algebraic variety consisting of all of the zeros of a homogeneous quintic polynomial in the homogeneous coordinates of the $\mathbf{CP}^4$. Some discrete quotients of the quintic by various $\mathbf{Z}_5$ actions are also Calabi-Yau and have received a lot of attention in the literature. One of these is related to the original quintic by mirror symmetry.

For every positive integer n , the zero set of a non-singular homogeneous degree $n+2$ polynomial in the homogeneous coordinates of the complex projective space $\mathbf{CP}^{n+1}$ is a compact Calabi-Yau n -fold. The case $n=1$ describes an elliptic curve, while for $n=2$ one obtains a K3 surface.

All hyper-Kähler manifolds are Calabi-Yau.

Applications in superstring theory

Calabi-Yau manifolds are important in superstring theory. In the most conventional superstring models, ten conjectural dimensions in string theory are supposed to come as four of which we are aware, carrying some kind of fibration with fiber dimension six. Compactification on Calabi-Yau n -folds are important because they leave some of the original supersymmetry unbroken. More precisely, in the absence of fluxes, compactification on a Calabi-Yau 3-fold (real dimension 6) leaves one quarter of the original supersymmetry unbroken if the holonomy is the full $SU(3)$.

More generally, a flux-free compactification on an n -manifold with holonomy $SU(n)$ leaves 2^{1-n} of the original supersymmetry unbroken, corresponding to 2^{6-n} supercharges in a compactification of type II supergravity or 2^{5-n} supercharges in a compactification of type I. When fluxes are included the supersymmetry condition instead implies that the compactification manifold be a generalized Calabi-Yau, a notion introduced by Hitchin (2003). These models are known as flux compactifications.

Essentially, Calabi-Yau manifolds are shapes that satisfy the requirement of space for the six "unseen" spatial dimensions of string theory, which may be smaller than our currently observable lengths as they have not yet been detected. A popular alternative known as large extra dimensions, which often occurs in braneworld models, is that the Calabi-Yau is

large but we are confined to a small subset on which it intersects a D-brane.

F-theory compactifications on various Calabi-Yau four-folds provide physicists with the a method to find a large number of classical solution in the so-called string theory landscape.

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External links

- Calabi-Yau Homepage^[19] is an interactive reference which describes many examples and classes of Calabi-Yau manifolds and also the physical theories in which they appear.
- Spinning Calabi-Yau Space video.^[20]
- *Calabi-Yau Space*^[21] by Andrew J. Hanson with additional contributions by Jeff Bryant, Wolfram Demonstrations Project.
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Pseudo-Riemannian manifold

In differential geometry, a **pseudo-Riemannian manifold** (also called a **semi-Riemannian manifold**) is a generalization of a Riemannian manifold. It is one of many mathematical objects named after Bernhard Riemann. The key difference between a Riemannian manifold and a pseudo-Riemannian manifold is that on a pseudo-Riemannian manifold the metric tensor need not be positive-definite. Instead a weaker condition of nondegeneracy is imposed.

Introduction

Manifolds

Main articles: Manifold, differentiable manifolds

In differential geometry a differentiable manifold is a space which is locally similar to a Euclidean space. In an n -dimensional Euclidean space any point can be specified by n real numbers. These are called the coordinates of the point.

An n -dimensional differentiable manifold is a generalisation of n -dimensional Euclidean space. In a manifold it may only be possible to define coordinates *locally*. This is achieved by defining coordinate patches: subsets of the manifold which can be mapped into n -dimensional Euclidean space.

See Manifold, differentiable manifold, coordinate patch for more details.

Tangent spaces and metric tensors

Main articles: Tangent space, metric tensor

Associated with each point p in an n -dimensional differentiable manifold M is a tangent space (denoted $T_p M$). This is an n -dimensional vector space whose elements can be thought of as equivalence classes of curves passing through the point p .

A metric tensor is a non-degenerate, smooth, symmetric, bilinear map which assigns a real number to pairs of tangent vectors at each tangent space of the manifold. Denoting the metric tensor by g we can express this as $g : T_p M \times T_p M \rightarrow \mathbb{R}$.

The map is symmetric and bilinear so if $X, Y, Z \in T_p M$ are tangent vectors at a point p in the manifold M then we have

- $g(X, Y) = g(Y, X)$
- $g(aX + Y, Z) = ag(X, Z) + g(Y, Z)$

for some real number a .

That g is non-degenerate means there are no non-zero $X \in T_p M$ such that $g(X, Y) = 0$ for all $Y \in T_p M$.

Metric signatures

For an n -dimensional manifold the metric tensor (in a fixed coordinate system) has n eigenvalues. If the metric is non-degenerate then none of these eigenvalues are zero. The **signature** of the metric denotes the number of positive and negative eigenvalues, this quantity is independent of the chosen coordinate system by Sylvester's rigidity theorem and locally non-decreasing. If the metric has p positive eigenvalues and q negative eigenvalues

then the metric signature is (p, q) . For a non-degenerate metric $p + q = n$.

Definition

A **pseudo-Riemannian manifold** (M, g) is a differentiable manifold M equipped with a non-degenerate, smooth, symmetric metric tensor g which, unlike a Riemannian metric, need not be positive-definite, but must be non-degenerate. Such a metric is called a **pseudo-Riemannian metric** and its values can be positive, negative or zero.

The signature of a pseudo-Riemannian metric is (p, q) where both p and q are non-negative.

Lorentzian manifold

A **Lorentzian manifold** is an important special case of a pseudo-Riemannian manifold in which the signature of the metric is $(1, n - 1)$ (or sometimes $(n - 1, 1)$, see sign convention). Such metrics are called **Lorentzian metrics**. They are named after the physicist Hendrik Lorentz.

Applications in physics

After Riemannian manifolds, Lorentzian manifolds form the most important subclass of pseudo-Riemannian manifolds. They are important because of their physical applications to the theory of general relativity.

A principal assumption of general relativity is that spacetime can be modeled as a 4-dimensional Lorentzian manifold of signature $(3, 1)$ (or equivalently $(1, 3)$). Unlike Riemannian manifolds with positive-definite metrics, a signature of $(p, 1)$ or $(1, q)$ allows tangent vectors to be classified into *timelike*, *null* or *spacelike* (see Causal structure).

Properties of pseudo-Riemannian manifolds

Just as Euclidean space $\mathbb{R}^n$ can be thought of as the model Riemannian manifold, Minkowski space $\mathbb{R}^{n-1,1}$ with the flat Minkowski metric is the model Lorentzian manifold. Likewise, the model space for a pseudo-Riemannian manifold of signature (p, q) is $\mathbb{R}^{p,q}$ with the metric: $g = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_n^2$

Some basic theorems of Riemannian geometry can be generalized to the pseudo-Riemannian case. In particular, the fundamental theorem of Riemannian geometry is true of pseudo-Riemannian manifolds as well. This allows one to speak of the Levi-Civita connection on a pseudo-Riemannian manifold along with the associated curvature tensor. On the other hand, there are many theorems in Riemannian geometry which do not hold in the generalized case. For example, it is *not* true that every smooth manifold admits a pseudo-Riemannian metric of a given signature; there are certain topological obstructions. Furthermore, a submanifold of a pseudo-Riemannian manifold need not be a pseudo-Riemannian manifold.

See also

- Riemannian manifold
- Causal structure
- Metric (mathematics)
- Metric signature

Hypersurface

For differential geometry usage, see glossary of differential geometry and topology.

In geometry, a **hypersurface** is a generalization of the concept of hyperplane. Suppose an enveloping manifold M has n dimensions; then any submanifold of M of $n - 1$ dimensions is a hypersurface. Equivalently, the codimension of a hypersurface is one.

In algebraic geometry, a hypersurface in projective space of dimension n is an algebraic set that is purely of dimension $n - 1$. It is then defined by a single equation $F = 0$, a homogeneous polynomial in the homogeneous coordinates. (It may have singularities, so not in fact be a submanifold in the strict sense.)

See also

- hypersphere
- hyperspace

CR manifold

In mathematics, a **CR manifold** is a differentiable manifold together with a geometric structure modeled on that of a real hypersurface in a complex vector space, or more generally modeled on an edge of a wedge.

Formally, a **CR manifold** is a differentiable manifold M together with a preferred complex distribution L , or in other words a subbundle of the complexified tangent bundle $\mathbf{CTM} = TM \otimes \mathbf{C}$ such that

- $[L, L] \subseteq L$ (L is **formally integrable**)
- $L \cap \bar{L} = \{0\}$ (L is almost Lagrangian).

The bundle L is called a **CR structure** on the manifold M .

Introduction and Motivation

The notion of a CR structure attempts to describe *intrinsically* the property of being a hypersurface in complex space by studying the properties of holomorphic vector fields which are tangent to the hypersurface.

Suppose for instance that M is the hypersurface of $\mathbf{C}^2$ given by the equation

$$F(z, w) := |z|^2 + |w|^2 = 1,$$

where z and w are the usual complex coordinates on $\mathbf{C}^2$. The *holomorphic tangent bundle* of $\mathbf{C}^2$ consists of all linear combinations of the vectors

$$\frac{\partial}{\partial z}, \frac{\partial}{\partial w}.$$

The distribution L on M consists of all combinations of these vectors which are *tangent* to M . In detail, the tangent vectors must annihilate the defining equation for M , so L consists of complex scalar multiples of

$$\bar{w} \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial w}.$$

Note that L gives a CR structure on M , for $[L, L] = 0$ (since L is one-dimensional) and $L \cap \bar{L} = \{0\}$ since $\partial/\partial z$ and $\partial/\partial w$ are linearly independent of their complex conjugates.

More generally, suppose that M is a real hypersurface in $\mathbf{C}^n$, with defining equation $F(z_1, \dots, z_n) = 0$. Then the CR structure L consists of those linear combinations of the basic holomorphic vectors on $\mathbf{C}^n$:

$$\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$$

which annihilate the defining function. In this case, $L \cap \bar{L} = \{0\}$ for the same reason as before. Moreover, $[L, L] \sqsubset L$ since the commutator of vector fields annihilating F is again a vector field annihilating F .

Embedded and abstract CR manifolds

There is a sharp contrast between the theories of embedded CR manifolds (hypersurface and edges of wedges in complex space) and abstract CR manifolds (those given by the Lagrangian distribution L). Many of the formal geometrical features are similar. These include:

- A notion of convexity (supplied by the **Levi form**)
- A differential operator, analogous to the Dolbeault operator, and an associated cohomology (the **tangential Cauchy-Riemann complex**).

Embedded CR manifolds possess some additional structure, though: a Neumann and Dirichlet problem for the Cauchy-Riemann equations.

This article first treats the geometry of embedded CR manifolds, shows how to defined these structures intrinsically, and then generalizes these to the abstract setting.

Embedded CR manifolds

Preliminaries

Embedded CR manifolds are, first and foremost, submanifolds of $\mathbf{C}^n$. Define a pair of subbundles of the complexified tangent bundle $\mathbf{C} \boxtimes T\mathbf{C}^n$ by:

- $T^{(1,0)}\mathbf{C}^n$ consists of the complex vectors annihilating the antiholomorphic functions. In the holomorphic coordinates:

$$T^{(1,0)}\mathbf{C}^n = \text{span} \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right).$$

- $T^{(0,1)}\mathbf{C}^n$ consists of the complex vectors annihilating the holomorphic functions. In coordinates:

$$T^{(0,1)}\mathbf{C}^n = \text{span} \left(\frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n} \right).$$

Also relevant are the characteristic annihilators from the Dolbeault complex:

- $\Omega^{(1,0)}\mathbf{C}^n = (T^{(0,1)}\mathbf{C}^n)^\perp$. In coordinates,

$$\Omega^{(1,0)}\mathbf{C}^n = \text{span}(dz_1, \dots, dz_n).$$

- $\Omega^{(0,1)}\mathbf{C}^n = (T^{(1,0)}\mathbf{C}^n)^\perp$. In coordinates,

$$\Omega^{(0,1)}\mathbf{C}^n = \text{span}(d\bar{z}_1, \dots, d\bar{z}_n).$$

The exterior products of these are denoted by the self-evident notation $\Omega^{(p,q)}$, and the Dolbeault operator and its complex conjugate map between these spaces via

$$\partial : \Omega^{(p,q)} \rightarrow \Omega^{(p+1,q)}$$

$$\bar{\partial} : \Omega^{(p,q)} \rightarrow \Omega^{(p,q+1)}$$

Furthermore, there is a decomposition of the usual exterior derivative via $d = \partial + \bar{\partial}$.

Real submanifolds of complex space

Let $M \subset \mathbf{C}^n$ be a real submanifold, defined locally as the locus of a system of smooth real-valued functions

$$F_1 = 0, F_2 = 0, \dots, F_k = 0.$$

Suppose that this system has maximal rank, in the sense that the differentials satisfy the following *independence condition*:

$$\partial F_1 \wedge \bar{\partial} F_1 \wedge \dots \wedge \partial F_k \wedge \bar{\partial} F_k \neq 0.$$

Note that this condition is strictly stronger than needed to apply the implicit function theorem: in particular, M is a manifold of real dimension $2n - k$. We say that M is an embedded CR manifold of **CR codimension** k . In most applications, $k = 1$, in which case the manifold is said to be of **hypersurface type**.

Let $L \subset T^{(1,0)}\mathbf{C}^n|_M$ be the subbundle of vectors annihilating all of the defining functions $F_1, \dots, F_k$. Note that, by the usual considerations for integrable distributions on hypersurfaces, L is involutive. Moreover, the independence condition implies that L is a bundle of constant rank $n - k$.

Henceforth, suppose that $k = 1$ (so that the CR manifold is of hypersurface type), unless otherwise noted.

The Levi form

Let M be a CR manifold of hypersurface type with single defining function $F = 0$. The **Levi form** of M is the Hermitian 2-form

$$h = i\partial\bar{\partial}F|_L.$$

This determines a metric on L . M is said to be **strictly pseudoconvex** if h is positive definite (or *pseudoconvex* in case h is positive semidefinite). Many of the analytic existence and uniqueness results in the theory of CR manifolds depend on the strict pseudoconvexity of the Levi form.

This nomenclature comes from the study of pseudoconvex domains: M is the boundary of a (strictly) pseudoconvex domain in $\mathbf{C}^n$ if and only if it is (strictly) pseudoconvex as a CR manifold. (See plurisubharmonic functions and Stein manifold.)

Abstract CR structures

An abstract CR structure on a manifold M of dimension n consists of a subbundle L of the complexified tangent bundle which is formally integrable, in the sense that $[L, L] \subset L$, which is linearly independent of its complex conjugate. The **CR codimension** of the CR structure is $k = n - 2 \dim L$. In case $k = 1$, the CR structure is said to be of **hypersurface type**. Most examples of abstract CR structures are of hypersurface type, unless otherwise made explicit.

The Levi form and pseudoconvexity

Suppose that M is a CR manifold of hypersurface type. The Levi form is the vector valued form, defined on L , with values in the line bundle

$$V = \frac{TM \otimes \mathbb{C}}{L \oplus \bar{L}}$$

given by

$$h(v, w) = \frac{1}{2i} [v, \bar{w}] \mod L \oplus \bar{L}, \quad v, w \in L.$$

h defines a sesquilinear form on L since it does not depend on how v and w are extended to sections of L , by the integrability condition. This form extends to a hermitian form on the bundle $L \oplus \bar{L}$ by the same expression. The extended form is also sometimes referred to as the Levi form.

The Levi form can alternatively be characterized in terms of duality. Consider the line subbundle of the complex cotangent bundle annihilating V

$$H_0 M = V^* = (L \oplus \bar{L})^\perp \subset T^* M \otimes \mathbb{C}.$$

For each local section $\alpha \in \Gamma(H_0 M)$, let

$$h_\alpha(v, w) = d\alpha(v, \bar{w}) = -\alpha([v, \bar{w}]), \quad v, w \in L \oplus \bar{L}.$$

The form h_α is a complex-valued hermitian form associated to α .

Generalizations of the Levi form exist when the manifold is not of hypersurface type, in which case the form no longer assumes values in a line bundle, but rather in a vector bundle. One may then speak, not of a Levi form, but of a collection of Levi forms for the structure.

Characteristic ideals

The tangential Cauchy-Riemann complex

Examples

The canonical example of a CR manifold is the real $2n + 1$ sphere as a submanifold of $\mathbb{C}^{n+1}$. The bundle L described above is given by

$$L = \mathbb{C}TS^{2n+1} \cap T^{1,0}\mathbb{C}^{n+1}$$

where $T^{1,0}\mathbb{C}^{n+1}$ is the bundle of holomorphic vectors. The real form of this is given by $P = \Re(L \oplus \bar{L})$, the bundle given at a point $p \in S^{2n+1}$ concretely in terms of the complex structure, I , on $\mathbb{C}^{n+1}$ by

$$P_p = \{X \in T_p S^{2n+1} : IX \in T_p S^{2n+1} \subset T_p \mathbb{C}^{n+1}\},$$

and the almost complex structure on P is just the restriction of I .

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Symplectic manifold

In mathematics, a **symplectic manifold** is a smooth manifold, M , equipped with a closed, nondegenerate, 2-form, ω , called the symplectic form. The study of symplectic manifolds is called symplectic geometry or symplectic topology. Symplectic manifolds arise naturally in abstract formulations of classical mechanics and analytical mechanics as the cotangent bundles of manifolds, e.g., in the Hamiltonian formulation of classical mechanics, which provides one of the major motivations for the field: The set of all possible configurations of a system is modelled as a manifold, and this manifold's cotangent bundle describes the phase space of the system.

Any real-valued differentiable function, H , on a symplectic manifold can serve as an **energy function** or **Hamiltonian**. Associated to any Hamiltonian is a Hamiltonian vector field; the integral curves of the Hamiltonian vector field are solutions to the Hamilton–Jacobi equations. The Hamiltonian vector field defines a flow on the symplectic manifold, called a **Hamiltonian flow** or symplectomorphism. By Liouville's theorem, Hamiltonian flows preserve the volume form on the phase space.

Definition

A symplectic form on a manifold, M , is a nondegenerate closed two-form, ω . Explicitly, nondegeneracy of the form means that, relative to any given basis, X_i , of the tangent space of M at a point, the matrix

$$\Omega_{ij} = \omega(X_i, X_j)$$

is nonsingular (meaning that its determinant is non-zero). Note that Ω , being a skew-symmetric non-singular matrix, must have an even number of rows and columns. Thus the dimension of M is necessarily an even number $2n$. In intrinsic terms, ω is nondegenerate if and only if its n -th exterior power is non-zero:

$$\omega^{\wedge n} \neq 0.$$

Furthermore, ω is required to be closed, meaning that

$$d\omega = 0$$

where d is the exterior derivative.

Linear symplectic manifold

There is a standard 'local' model, namely $\mathbf{R}^{2n}$ with

$$\begin{aligned} \omega_{i,n+i} &= 1 & \text{and} & \quad \omega_{n+i,i} = -1 \\ \omega_{j,k} &= 0 & \text{for all } j, k &= 1, \dots, n \text{ with } j \neq k - n \text{ and } j \neq k + n. \end{aligned}$$

This is an example of a *linear* symplectic space. See symplectic vector space. A proposition known as Darboux's theorem says that locally any symplectic manifold resembles this simple one.

Volume form

Directly from the definition, one can show that every symplectic manifold, M is of even dimension $2n$ and ω^n is a nowhere vanishing form, the symplectic volume form. It follows that every symplectic manifold is canonically oriented and comes with a canonical measure, the **Liouville measure** (often normalized to be $\omega^n / n!$).

A version of Liouville's theorem asserts that the Liouville measure is invariant under the Hamiltonian flows.

Contact manifolds

Closely related to symplectic manifolds are the odd-dimensional manifolds known as contact manifolds. Any $2n+1$ -dimensional contact manifold, (M, α) , gives rise to a $2n+2$ -dimensional symplectic manifold, $(M \times \mathbf{R}, d(e^t \alpha))$.

Lagrangian and other submanifolds

There are several natural geometric notions of submanifold of a symplectic manifold. There are **symplectic submanifolds** (potentially of any even dimension), where the symplectic form is required to induce a symplectic form on the submanifold. On **isotropic submanifolds**, the symplectic form restricts to zero, i.e. each tangent space is an isotropic subspace of the ambient manifold's tangent space. Similarly, if each tangent subspace to a submanifold is coisotropic (the dual of an isotropic subspace), the submanifold is called **coisotropic**.

The most important case of the above is that of **Lagrangian submanifolds**, which are isotropic submanifolds of maximal dimension, namely half the dimension of the ambient manifold. Lagrangian submanifolds arise naturally in many physical and geometric situations. One major example is that the graph of a symplectomorphism in the product symplectic manifold $(M \times M, \omega \times -\omega)$ is Lagrangian. Their intersections display rigidity properties not possessed by smooth manifolds; the Arnold conjecture gives the sum of the submanifold's Betti numbers as a lower bound for the number of self intersections of a smooth Lagrangian submanifold, rather than the Euler characteristic in the smooth case.

Special cases and generalizations

- A symplectic manifold endowed with a metric that is compatible with the symplectic form is an almost Kähler manifold in the sense that the tangent bundle has an almost complex structure, but this need not be integrable. Symplectic manifolds are special cases of a Poisson manifold. The definition of a symplectic manifold requires that the symplectic form be non-degenerate everywhere, but if this condition is violated, the manifold may still be a Poisson manifold.
- A **multisymplectic manifold** of degree k is a manifold equipped with a closed nondegenerate k -form. See F. Cantrijn, L. A. Ibort and M. de León, J. Austral. Math. Soc. Ser. A 66 (1999), no. 3, 303-330.
- A **polysymplectic manifold** is a Legendre bundle provided with a polysymplectic tangent-valued $(n+2)$ -form; it is utilized in Hamiltonian field theory. See: G. Giachetta, L. Mangiarotti and G. Sardanashvily, Covariant Hamiltonian equations for field theory, Journal of Physics **A32** (1999) 6629-6642; arXiv:

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See also

- Poisson bracket
- Symplectic topology
- Symplectic vector space
- Almost complex manifold
- Almost symplectic manifold
- Symplectic group
- Symplectic matrix
- Tautological one-form
- Wirtinger inequality (2-forms)

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Betti number

In algebraic topology, the **Betti numbers** can be used to distinguish topological spaces. Intuitively, the first Betti number counts the maximum number of cuts that can be made without dividing the space into two pieces.

Each Betti number is a natural number or infinity. For the most reasonable finite-dimensional spaces (such as compact manifolds, finite simplicial complexes or CW complexes), the sequence of Betti numbers is 0 from some points onwards (Betti numbers vanish about the dimension of a space), and they are all finite.

The term "Betti numbers" was coined by Henri Poincaré after Enrico Betti.

Definition

For a non-negative integer k , the k -th Betti number $b_k(X)$ of the space X is defined as the rank of the abelian group $H_k(X)$, the k -th homology group of X . Equivalently, one can define it as the vector space dimension of $H_k(X; \mathbb{Q})$, since the homology group in this case is a vector space over $\mathbb{Q}$. The universal coefficient theorem, in a very simple case, shows that these definitions are the same.

More generally, given a field F one can define $b_k(X, F)$, the k -th Betti number with coefficients in F , as the vector space dimension of $H_k(X, F)$.

Example: the first Betti number in graph theory

In topological graph theory the first Betti number of a graph G with n vertices, m edges and k connected components equals

$$m - n + k.$$

This may be proved straightforwardly by mathematical induction on the number of edges. A new edge either increments the number of 1-cycles or decrements the number of connected components.

See cyclomatic complexity for an application of the first Betti number in software engineering.

Properties

The (rational) Betti numbers $b_k(X)$ do not take into account any torsion in the homology groups, but they are very useful basic topological invariants. In the most intuitive terms, they allow one to count the number of *holes* of different dimensions. For a circle, the first Betti number is 1. For a general pretzel, the first Betti number is twice the number of holes.

In the case of a finite simplicial complex the homology groups $H_k(X, \mathbb{Z})$ are finitely-generated, and so has a finite rank. Also the group is 0 when k exceeds the top dimension of a simplex of X .

For a finite CW-complex K we have

$$\chi(K) = \sum_{i=0}^{\infty} (-1)^i b_i(K, F),$$

where $\chi(K)$ denotes Euler characteristic of K and any field F .

For any two spaces X and Y we have

$$P_{X \times Y} = P_X P_Y,$$

where P_X denotes the **Poincaré polynomial** of X , (more generally, the Poincaré series, for infinite-dimensional spaces), i.e. the generating function of the Betti numbers of X :

$$P_X(z) = b_0(X) + b_1(X)z + b_2(X)z^2 + \cdots,$$

see Künneth theorem.

If X is n -dimensional manifold, there is symmetry interchanging k and $n - k$, for any k :

$$b_k(X) = b_{n-k}(X),$$

under conditions (a *closed* and *oriented* manifold); see Poincaré duality.

The dependence on the field F is only through its characteristic. If the homology groups are torsion-free, the Betti numbers are independent of F . The connection of p -torsion and the Betti number for characteristic p , for p a prime number, is given in detail by the universal coefficient theorem (based on Tor functors, but in a simple case).

Examples

1. The Betti number sequence for a circle is 1, 1, 0, 0, 0, ...;

the Poincaré polynomial is $(1 + x)$.

2. The Betti number sequence for a two-torus is 1, 2, 1, 0, 0, 0, ...;

the Poincaré polynomial is $(1 + x)^2 = 1 + 2x + x^2$.

3. The Betti number sequence for a three-torus is 1, 3, 3, 1, 0, 0, 0,

the Poincaré polynomial is $(1 + x)^3 = 1 + 3x + 3x^2 + x^3$.

Similarly, for an n -torus, the Poincaré polynomial is $(1 + x)^n$ (by the Künneth theorem), so the Betti numbers are the binomial coefficients.

It is possible for spaces that are infinite-dimensional in an essential way to have an infinite sequence of non-zero Betti numbers. An example is the infinite-dimensional complex projective space, with sequence 1, 0, 1, 0, 1, ... that is periodic, with period length 2. In this case the Poincaré function is not a polynomial but rather an infinite series $1 + x^2 + x^4 + \cdots$, which, being a geometric series, can be expressed as the rational function $\frac{1}{(1 - x^2)} = 1 + x^2 + (x^2)^2 + (x^2)^3 + \cdots$.

More generally, any sequence that is periodic can be expressed as a sum of geometric series, generalizing the above (e.g., $a, b, c, a, b, c, \dots$, has generating function $(a + bx + cx^2)/(1 - x^3)$), and more generally linear recursive sequences are exactly the sequences generated by rational functions; thus the Poincaré series is expressible as a rational function if and only if the sequence of Betti numbers is a linear recursive sequence.

Relationship with dimensions of spaces of differential forms

In geometric situations when X is a closed manifold, the importance of the Betti numbers may arise from a different direction, namely that they predict the dimensions of vector spaces of closed differential forms *modulo* exact differential forms. The connection with the definition given above is via three basic results, de Rham's theorem and Poincaré duality (when those apply), and the universal coefficient theorem of homology theory.

There is an alternate reading, namely that the Betti numbers give the dimensions of spaces of harmonic forms. This requires also the use of some of the results of Hodge theory, about the Hodge Laplacian.

In this setting, Morse theory gives a set of inequalities for alternating sums of Betti numbers in terms of a corresponding alternating sum of the number of critical points N_i of a Morse function of a given index:

$$b_i(X) - b_{i-1}(X) + \cdots \geq N_i - N_{i-1} + \cdots.$$

Witten gave an explanation of these inequalities by using the Morse function to modify the exterior derivative in the de Rham complex.

References

- F.W. Warner, Foundations of differentiable manifolds and Lie groups, Springer (1983).
- J.Roe, Elliptic Operators, Topology, and Asymptotic Methods, Second Edition (Research Notes in Mathematics Series **395**), Chapman and Hall (1998).

Euler characteristic

In mathematics, and more specifically in algebraic topology and polyhedral combinatorics, the **Euler characteristic** (or **Euler-Poincaré characteristic**) is a topological invariant, a number that describes a topological space's shape or structure regardless of the way it is bent. It is commonly denoted by χ (Greek letter chi).

The Euler characteristic was originally defined for polyhedra and used to prove various theorems about them, including the classification of the Platonic solids. Leonhard Euler, for whom the concept is named, was responsible for much of this early work. In modern mathematics, the Euler characteristic arises from homology and connects to many other invariants.

Polyhedra

The **Euler characteristic** χ was classically defined for the surfaces of polyhedra, according to the formula

$$\chi = V - E + F$$

where V , E , and F are respectively the numbers of vertices (corners), edges and faces in the given polyhedron. Any *convex* polyhedron's surface has Euler characteristic

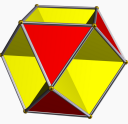
$$\chi = V - E + F = 2.$$

This result is known as **Euler's formula**. A proof is given below.

Name	Image	Vertices <i>V</i>	Edges <i>E</i>	Faces <i>F</i>	Euler characteristic: <i>V</i> − <i>E</i> + <i>F</i>
Tetrahedron		4	6	4	2
Hexahedron or cube		8	12	6	2

Octahedron		6	12	8	2
Dodecahedron		20	30	12	2
Icosahedron		12	30	20	2

The surfaces of nonconvex polyhedra can have various Euler characteristics:

Name	Image	Vertices <i>V</i>	Edges <i>E</i>	Faces <i>F</i>	Euler characteristic: <i>V</i> − <i>E</i> + <i>F</i>
Tetrahemihexahedron		6	12	7	1
Octahemioctahedron		12	24	12	0
Cubohemioctahedron		12	24	10	−2
Great icosahedron		12	30	20	2

Planar graphs

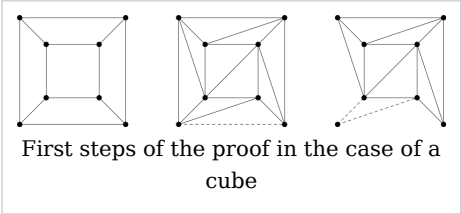
The Euler characteristic can be defined for planar graphs by the same $V - E + F$ formula as for polyhedral surfaces, where F is the number of faces in the graph, including the exterior face.

The Euler characteristic of any planar graph is 2. For via stereographic projection the plane maps to the two-dimensional sphere, such that the graph maps to a polygonal decomposition of the sphere, which has Euler characteristic 2. This viewpoint is implicit in Cauchy's proof of Euler's formula given below.

Proof of Euler's formula

The first rigorous proof of Euler's formula, given by a 20-year-old Cauchy, is as follows.

Remove one face of the polyhedral surface. By pulling the edges of the missing face away from each other, deform all the rest into a planar graph of points and curves, as illustrated by the first of the three graphs for



the special case of the cube. (The assumption that the polyhedral surface is homeomorphic to the sphere at the beginning is what makes this possible.) After this deformation, the

regular faces are generally not regular anymore. The number of vertices and edges has remained the same, but the number of faces has been reduced by 1. As such, proving Euler's formula for the polyhedron reduces to proving $V - E + F = 1$ for this deformed, planar object.

If there is a face with more than three sides, draw a diagonal—that is, a curve through the face connecting two vertices that aren't connected yet. This adds one edge and one face and does not change the number of vertices, so it does not change the quantity $V - E + F$. Continue adding edges in this manner until all of the faces are triangular.

Apply repeatedly either of the following two transformations:

1. Remove a triangle with only one edge adjacent to the exterior, as illustrated by the second graph. This decreases the number of edges and faces by one each and does not change the number of vertices, so it preserves $V - E + F$.
2. Remove a triangle with two edges shared by the exterior of the network, as illustrated by the third graph. Each triangle removal removes a vertex, two edges and one face, so it preserves $V - E + F$.

Repeat these two steps, one after the other, until only one triangle remains.

At this point the lone triangle has $V = 3$, $E = 3$, and $F = 1$, so that $V - E + F = 1$. Since each of the two above transformation steps preserved this quantity, we have shown $V - E + F = 1$ for the deformed, planar object thus demonstrating $V - E + F = 2$ for the polyhedron. This proves the theorem.

For additional proofs, see *Nineteen Proofs of Euler's Formula* ^[1] by David Eppstein. Multiple proofs, including their flaws and limitations, are used as examples in *Proofs and Refutations* by Imre Lakatos. ^[2]

Topological definition

The polyhedral surfaces discussed above are, in modern language, two-dimensional finite CW-complexes. (When only triangular faces are used, they are two-dimensional finite simplicial complexes.) In general, for any finite CW-complex, the **Euler characteristic** can be defined as the alternating sum

$$\chi = k_0 - k_1 + k_2 - k_3 + \dots,$$

where k_n denotes the number of cells of dimension n in the complex.

More generally still, for any topological space, we can define the n th Betti number b_n as the rank of the n -th singular homology group. The **Euler characteristic** can then be defined as the alternating sum

$$\chi = b_0 - b_1 + b_2 - b_3 + \dots$$

This quantity is well-defined if the Betti numbers are all finite and if they are zero beyond a certain index n_0 . This definition subsumes the previous ones.

Properties

Homotopy invariance

Since the homology is a topological invariant (in fact, a homotopy invariant — two topological spaces that are homotopy equivalent have isomorphic homology groups), so is the Euler characteristic.

For example, any convex polyhedron is homeomorphic to the three-dimensional ball, so its surface is homeomorphic (hence homotopy equivalent) to the two-dimensional sphere, which has Euler characteristic 2. This explains why convex polyhedra have Euler characteristic 2.

Inclusion-exclusion principle

If M and N are any two topological spaces, then the Euler characteristic of their disjoint union is the sum of their Euler characteristics, since homology is additive under disjoint union:

$$\chi(M \sqcup N) = \chi(M) + \chi(N).$$

More generally, if M and N are subspaces of a larger space X , then so are their union and intersection. In some cases, the Euler characteristic obeys a version of the inclusion-exclusion principle:

$$\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N).$$

This is true in the following cases:

- if M and N are an excisive couple. In particular, if the interiors of M and N inside the union still cover the union.^[3]
- if X is a locally compact space, and one uses Euler characteristics with compact supports, no assumptions on M or N are needed.
- if X is a stratified space all of whose strata are even dimensional, the inclusion-exclusion principle holds if M and N are unions of strata. This applies in particular if M and N are subvarieties of a complex algebraic variety.^[4]

In general, the inclusion-exclusion principle is false. A counterexample is given by taking X to be the real line, M a subset consisting of one point and N the complement of M .

Product property

Also, the Euler characteristic of any product space $M \times N$ is

$$\chi(M \times N) = \chi(M) \cdot \chi(N).$$

These addition and multiplication properties are also enjoyed by cardinality of sets. In this way, the Euler characteristic can be viewed as a generalisation of cardinality; see [5].

Covering spaces

Similarly, for an k -sheeted covering space $\tilde{M} \rightarrow M$, one has

$$\chi(\tilde{M}) = k \cdot \chi(M).$$

More generally, for a ramified covering space, the Euler characteristic of the cover can be computed from the above, with a correction factor for the ramification points, which yields the Riemann-Hurwitz formula.

Fibration property

The product property holds much more generally, for fibrations with certain conditions.

If $p: E \rightarrow B$ is a fibration with fiber F , with the base B path-connected, and the fibration is orientable over a field K , then the Euler characteristic with coefficients in the field K satisfies the product property:^[6]

$$\chi(E) = \chi(F) \cdot \chi(B).$$

This includes product spaces and covering spaces as special cases, and can be proven by the Serre spectral sequence on homology of a fibration.

For fiber bundles, this can also be understood in terms of a transfer map $\tau: H_*(B) \rightarrow H_*(E)$ - note that this is a lifting and goes "the wrong way" - whose composition with the projection map $p_*: H_*(E) \rightarrow H_*(B)$ is multiplication by the Euler class of the fiber:^[7] $p_* \circ \tau = \chi(F) \cdot 1$.

Other properties

As a corollary of Poincaré duality, the Euler characteristic of any closed odd-dimensional manifold is zero. This applies more generally to any compact stratified space all of whose strata are odd-dimensional.

Relations to other invariants

The Euler characteristic of a closed orientable surface can be calculated from its genus g (the number of tori in a connected sum decomposition of the surface; intuitively, the number of "handles") as

The Euler characteristic of a closed non-orientable surface can be calculated from its non-orientable genus k (the number of real projective planes in a connected sum decomposition of the surface) as

For closed smooth manifolds, the Euler characteristic coincides with the **Euler number**, i.e., the Euler class of its tangent bundle evaluated on the fundamental class of a manifold. The Euler class, in turn, relates to all other characteristic classes of vector bundles.

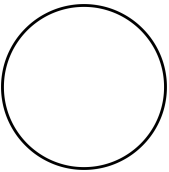
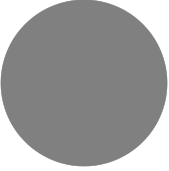
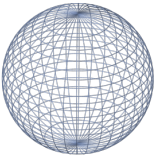
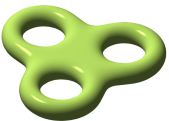
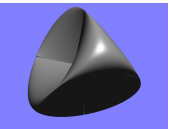
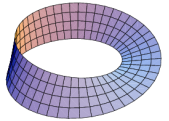
For closed Riemannian manifolds, the Euler characteristic can also be found by integrating the curvature; see the Gauss-Bonnet theorem for the two-dimensional case and the generalized Gauss-Bonnet theorem for the general case.

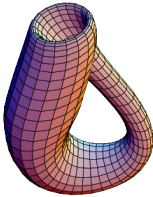
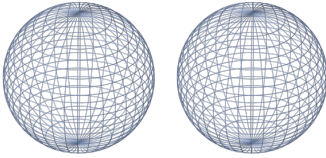
A discrete analog of the Gauss-Bonnet theorem is Descartes' theorem that the "total defect" of a polyhedron, measured in full circles, is the Euler characteristic of the polyhedron; see defect (geometry).

Hadwiger's theorem characterizes the Euler characteristic as the *unique* (up to scalar multiplication) translation-invariant, finitely additive, not-necessarily-nonnegative set function defined on finite unions of compact convex sets in $\mathbf{R}^n$ that is "homogeneous of degree 0".

Examples

The Euler characteristic can be calculated easily for general surfaces by finding a polygonization of the surface (that is, a description as a CW-complex) and using the above definitions.

Name	Image	Euler characteristic
Interval		1
Circle		0
Disk		1
Sphere		2
Torus (Product of two circles)		0
Double torus		-2
Triple torus		-4
Real projective plane		1
Möbius strip		0

Klein bottle		0
Two spheres (not connected) (Disjoint union of two spheres)		$2 + 2 = 4$

Any contractible space (that is, one homotopy equivalent to a point) has trivial homology, meaning that the 0th Betti number is 1 and the others 0. Therefore its Euler characteristic is 1. This case includes Euclidean space $\mathbb{R}^n$ of any dimension, as well as the solid unit ball in any Euclidean space — the one-dimensional interval, the two-dimensional disk, the three-dimensional ball, etc.

The n -dimensional sphere has Betti number 1 in dimensions 0 and n , and all other Betti numbers 0. Hence its Euler characteristic is $1 + (-1)^n$ — that is, either 0 or 2.

The n -dimensional real projective space is the quotient of the n -sphere by the antipodal map. It follows that its Euler characteristic is exactly half that of the corresponding sphere — either 0 or 1.

The n -dimensional torus is the product space of n circles. Its Euler characteristic is 0, by the product property.

An application: Why does a football/soccer ball have 12 pentagons?

An application of the Euler characteristic is the answer to the question: How many pentagons and hexagons does it take to make a football/soccer ball? Assume we use N hexagons and L pentagons; then we have $F = N + L$ faces. Every pentagon (hexagon) has 5 vertices (6 vertices), and each one is shared between 3 faces, hence we have $V = (5L + 6N)/3$ vertices. Similarly, every pentagon (hexagon) has 5 edges (6 edges), and each one is shared between 2 faces, hence we have $E = (5L + 6N)/2$ edges. The Euler characteristic is therefore

$$\chi(M) = V - E + F = \frac{5L + 6N}{3} - \frac{5L + 6N}{2} + (L + N) = \frac{L}{6}.$$

Since the sphere M has Euler characteristic 2, it must be that $L = 12$. The result is that we always need 12 pentagons on a football/soccer ball; the number of hexagons is in principle unconstrained (but for a real football/soccer ball one obviously chooses a number that makes the ball as spherical as possible). One can also apply this result to Fullerenes.

Generalizations

For every combinatorial cell complex, one defines the Euler characteristic as the number of 0-cells, minus the number of 1-cells, plus the number of 2-cells, etc., if this alternating sum is finite. In particular, the Euler characteristic of a finite set is simply its cardinality, and the Euler characteristic of a graph is the number of vertices minus the number of edges.

More generally, one can define the Euler characteristic of any chain complex to be the alternating sum of the ranks of the homology groups of the chain complex.

A version used in algebraic geometry is as follows. For any sheaf $\mathcal{F}$ on a projective scheme X , one defines its Euler characteristic

$$\chi(\mathcal{F}) = \sum (-1)^i h^i(X, \mathcal{F}),$$

where $h^i(X, \mathcal{F})$ is the dimension of the i -th sheaf cohomology group of $\mathcal{F}$.

Another generalization of the concept of Euler characteristic on manifolds comes from orbifolds. While every manifold has an integer Euler characteristic, an orbifold can have a fractional Euler characteristic. For example, the teardrop orbifold has Euler characteristic $1 + 1/p$, where p is a prime number corresponding to the cone angle $2\pi / p$.

The concept of Euler characteristic of a bounded finite poset is another generalization, important in combinatorics. A poset is "bounded" if it has smallest and largest elements; call them 0 and 1. The Euler characteristic of such a poset is defined as the integer $\mu(0,1)$, where μ is the Möbius function in that poset's incidence algebra.

This can be further generalized by defining a $\mathbf{Q}$ -valued Euler characteristic for certain finite categories, a notion compatible with the Euler characteristics of graphs, orbifolds and posets mentioned above. In this setting, the Euler characteristic of a finite group or monoid G is $1/|G|$, and the Euler characteristic of a finite groupoid is the sum of $1/|G_i|$, where we picked one representative group G_i for each connected component of the groupoid.^[8]

See also

- List of uniform polyhedra
- List of topics named after Leonhard Euler

Notes

- [1] <http://www.ics.uci.edu/~eppstein/junkyard/euler/>
- [2] Imre Lakatos: Proofs and Refutations, Cambridge Technology Press, 1976
- [3] Edwin Spanier: Algebraic Topology, Springer 1966, p. 205.
- [4] William Fulton: Introduction to toric varieties, 1993, Princeton University Press, p. 141.
- [5] <http://math.ucr.edu/home/baez/counting/>
- [6] Spanier, Edwin Henry (1982), <http://books.google.com/books?id=h-wc3TnZMCcC> *Algebraic Topology*, Springer, ISBN 978 0 38794426 5, <http://books.google.com/books?id=h-wc3TnZMCcC>, Applications of the homology spectral sequence, p. 481 (<http://books.google.com/books?id=h-wc3TnZMCcC&pg=PA481>)
- [7] Gottlieb, Daniel Henry (1975), "<http://www.math.purdue.edu/~gottlieb/Bibliography/17FibreBundlesAndtheEulerCharacteristic.pdf> [Fibre bundles and the Euler characteristic]", *Journal of Differential Geometry* **10** (1): 39–48, <http://www.math.purdue.edu/~gottlieb/Bibliography/17FibreBundlesAndtheEulerCharacteristic.pdf>
- [8] Tom Leinster, " The Euler characteristic of a category (<http://www.math.uiuc.edu/documenta/vol-13/02.pdf>)", *Documenta Mathematica*, 13 (2008), pp.21-49

Further Reading

- Richeson, David S. (2008) *Euler's Gem: The Polyhedron Formula and the Birth of Topology*. Princeton University Press.
- H. Graham Flegg: *From Geometry to Topology*. Dover 2001, p. 40

External links

- Eric W. Weisstein, *Euler characteristic* (<http://mathworld.wolfram.com/EulerCharacteristic.html>) at MathWorld.
- Eric W. Weisstein, *Polyhedral formula* (<http://mathworld.wolfram.com/PolyhedralFormula.html>) at MathWorld.
- Euler characteristic (<http://eom.springer.de/E/e036400.htm>) in the Encyclopaedia of Mathematics

Presheaf (category theory)

In category theory, a branch of mathematics, a V -valued **presheaf** F on a category C is a functor $F : C^{\text{op}} \rightarrow V$. Often *presheaf* is defined to be a **Set**-valued presheaf. If C is the poset of open sets in a topological space, interpreted as a category, then one recovers the usual notion of presheaf on a topological space.

A morphism of presheaves is defined to be a natural transformation of functors. This makes the collection of all presheaves into a category, often written $\hat{C}$. A functor into $\hat{C}$ is sometimes called a profunctor.

Properties

- A category C embeds fully and faithfully into the category $\hat{C}$ of set-valued presheaves via the Yoneda embedding Y_c which to every object A of C associates the hom-set $C(-, A)$.
- The presheaf category $\hat{C}$ is (up to equivalence of categories) the free colimit completion of the category C .



This category theory-related article is a stub. You can help Wikipedia by expanding it
 [1].

References

- [1] http://en.wikipedia.org/w/index.php?stub&title=Presheaf_%28category_theory%29&action=edit

Sheaf (mathematics)

In mathematics, a **sheaf** is a tool for systematically tracking locally defined data attached to the open sets of a topological space. The data can be restricted to smaller open sets, and the data assigned to an open set is equivalent to all collections of compatible data assigned to collections of smaller open sets covering the original one. For example, such data can consist of the rings of continuous or smooth real-valued functions defined on each open set. Sheaves are by design quite general and abstract objects, and their correct definition is rather technical. They exist in several varieties such as sheaves of sets or sheaves of rings, depending on the type of data assigned to open sets.

There are also maps (or morphisms) from one sheaf to another; sheaves (of a specific type, such as sheaves of abelian groups) with their morphisms on a fixed topological space form a category. On the other hand, to each continuous map there is associated both a *direct image* functor, taking sheaves and their morphisms on the domain to sheaves and morphisms on the codomain, and an *inverse image* functor operating in the opposite direction. These functors, and certain variants of theirs, are essential parts of sheaf theory.

Due to their general nature and versatility, sheaves have several applications in topology and especially in algebraic and differential geometry. First, several geometric structures such as that of a differentiable manifold or a scheme can be expressed in terms of a sheaf of rings on the space. In such contexts several geometric constructions such as vector bundles or divisors are naturally specified in terms of sheaves. Second, sheaves provide the framework for a very general cohomology theory, which encompasses also the "usual" topological cohomology theories such as singular cohomology. Especially in algebraic geometry and the theory of complex manifolds, sheaf cohomology provides a powerful link between topological and geometric properties of spaces. Sheaves also provide the basis for the theory of D-modules, which provide applications to the theory of differential equations. In addition, generalisations of sheaves to more general settings than topological spaces have provided applications to mathematical logic and number theory.

Introduction

In topology, differential geometry and algebraic geometry, several structures defined on a topological space (which can additionally be, e.g., a differentiable manifold) can be naturally *localised* or *restricted* to open subsets of the space: typical examples include continuous real or complex-valued functions, n times differentiable (real or complex-valued) functions, bounded real-valued functions, vector fields, and sections of any vector bundle on the space.

Presheaves formalise the situation common to the examples above: a presheaf (of sets) on a topological space is a structure which associates to each open set U of the space a set $F(U)$ of "*sections*" on U , and to each open set V included in another open set U a map $F(U) \rightarrow F(V)$ giving *restrictions* of sections over U to V . Each of the examples above defines a presheaf with restrictions of functions, vector fields and sections of a vector bundle having the obvious meaning. Moreover, in each of these examples the sets of sections have additional algebraic structure: pointwise operations make them abelian groups, and in the examples of real and complex-valued functions the sets of sections have even a ring structure. In addition, in each example the restriction maps are homomorphisms of the corresponding algebraic structure. This observation leads to the natural definition of

presheaves with additional algebraic structure such as presheaves of groups, of abelian groups, of rings: section sets are required to have the specified algebraic structure, and the restrictions are required to be homomorphisms. Thus for example continuous real-valued functions on a topological space form a presheaf of rings on the space.

Given a presheaf, a natural question to ask is to what extent its sections over an open set U are specified by their restrictions to smaller open sets V_i of an open cover of U . A presheaf is *separated* if its sections are "locally determined": whenever two sections over U coincide when restricted to each of V_i , the two sections are identical. All examples of presheaves discussed above are separated, since in each case the sections are specified by their values at the points of the underlying space. Finally, a separated presheaf is a **sheaf** if *compatible sections can be glued together*, i.e., whenever there is a section of the presheaf over each of the covering sets V_i , chosen so that they match on the overlaps of the covering sets, these sections correspond to a (unique) section on U , of which they are restrictions. It is easy to verify that all examples above except the presheaf of bounded functions are in fact sheaves: in all cases the criterion of being a section of the presheaf is *local* in a sense that it is enough to verify it in an arbitrary neighbourhood of each point.

On the other hand, it is clear that a function can be bounded on each set of an (infinite) open cover of a space without being bounded on all of the space; thus bounded functions provide an example of a presheaf that in general fails to be a sheaf. Another example of a presheaf that fails to be a sheaf is the *constant presheaf* that associates the same fixed set (or abelian group, or a ring,...) to each open set: it follows from the gluing property of sheaves that sections on a disjoint union of two open sets is the Cartesian product of the sections over the two open sets. The correct way to define the constant sheaf F_A (associated to for instance a set A) on a topological space is to require sections on an open set U to be continuous maps from U to A equipped with the discrete topology; then in particular $F_A(U) = A$ for connected U .

Maps between presheaves and sheaves (called morphisms) consist of maps between the sets of sections over each open set of the underlying space, compatible with restrictions of sections. If the presheaves or sheaves considered are provided with additional algebraic structure, these maps are assumed to be homomorphisms.

Presheaves and sheaves are typically denoted by capital letters, F being particularly common, presumably for the French word for sheaves, *faisceau*. Use of script letters such as $\mathcal{F}$ is also common.

Formal definitions

The first step in defining a sheaf is to define a *presheaf*, which captures the idea of associating data and restriction maps to the open sets of a topological space. The second step is to require the normalization and gluing axioms. A presheaf which satisfies these axioms is a sheaf.

Presheaves

Let X be a topological space, and let $\mathbf{C}$ be a category. Usually $\mathbf{C}$ is the category of sets, the category of groups, the category of abelian groups, or the category of commutative rings. A **presheaf** F on X with values in $\mathbf{C}$ is given by the following data:

- For each open set U of X , an object $F(U)$ in $\mathbf{C}$

- For each inclusion of open sets $V \sqsubseteq U$, a morphism $\text{res}_{V,U} : F(U) \rightarrow F(V)$ in the category $\mathbf{C}$. The morphisms $\text{res}_{V,U}$ are called **restriction morphisms**. The restriction morphisms are required to satisfy two properties.
- For every open set U of X , the restriction morphism $\text{res}_{U,U} : F(U) \rightarrow F(U)$ is the identity morphism on $F(U)$.
- If we have three open sets $W \sqsubseteq V \sqsubseteq U$, then $\text{res}_{W,V} \circ \text{res}_{V,U} = \text{res}_{W,U}$

Informally, the second axiom says it doesn't matter whether we restrict to W in one step or restrict first to V , then to W .

There is a compact way to express the notion of a presheaf in terms of category theory. First we define the category of open sets on X to be the category $O(X)$ whose objects are the open sets of X and whose morphisms are inclusions. Then a $\mathbf{C}$ -valued presheaf on X is the same as a contravariant functor from $O(X)$ to $\mathbf{C}$. This definition can be generalized to the case when the source category is not of the form $O(X)$ for any X ; see presheaf (category theory).

If F is a $\mathbf{C}$ -valued presheaf on X , and U is an open subset of X , then $F(U)$ is called the **sections of F over U** . If $\mathbf{C}$ is a concrete category, then each element of $F(U)$ is called a **section**. A section over X is called a **global section**. A common notation (used also below) for the restriction $\text{res}_{V,U}(s)$ of a section is $s|_V$. This terminology and notation is by analogy with sections of fiber bundles or sections of the étale space of a sheaf; see below. $F(U)$ is also often denoted $\Gamma(U, F)$, especially in contexts such as sheaf cohomology where U tends to be fixed and F tends to be variable.

Sheaves

The definition is first given for a sheaf with values in a concrete category (one in which the object can be interpreted as sets with possibly additional structure). This definition is more intuitive than the general one, and applies to the most common examples such as sheaves of sets, abelian groups and rings.

A *sheaf with values in a concrete category $\mathbf{C}$* is a presheaf with values in $\mathbf{C}$ that satisfies the following three axioms:

1. (Normalisation) $F(\square)$ is the terminal object of $\mathbf{C}$;
2. (Local identity) If (U_i) is an open covering of an open set U , and if $s, t \in F(U)$ are such that $s|_{U_i} = t|_{U_i}$ for each set U_i of the covering, then $s = t$; and
3. (Gluing) If (U_i) is an open covering of an open set U , and if for each i there is a section s_i of F over U_i such that for each pair U_i, U_j of the covering sets the restrictions of s_i and s_j agree on the overlaps: $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there is a section $s \in F(U)$ such that $s|_{U_i} = s_i$ for each i .

The section s whose existence is guaranteed by axiom 3 is called the **gluing**, **concatenation**, or **collation** of the sections s_i . By axiom 2 it is unique. Sections s_i satisfying the condition of axiom 3 are often called *compatible*; thus axioms 2 and 3 together state that *compatible sections can be uniquely glued together*.

The general definition of a sheaf with values in category $\mathbf{C}$, not necessarily concrete but assumed to have products, can be expressed in a compact manner by means of an exact sequence: consider in $\mathbf{C}$ the diagram

$$F(U) \rightarrow \prod_{\alpha \in A} F(V_\alpha) \rightrightarrows \prod_{\alpha, \beta \in A} F(V_\alpha \cap V_\beta),$$

where the first map is the product of the restriction maps

$$\text{res}_{V_\alpha, U}: F(U) \rightarrow F(V_\alpha)$$

and the pair of arrows the products of the two sets of restrictions

$$\text{res}_{V_\alpha \cap V_\beta, V_\alpha}: F(V_\alpha) \rightarrow F(V_\alpha \cap V_\beta)$$

and

$$\text{res}_{V_\alpha \cap V_\beta, V_\beta}: F(V_\beta) \rightarrow F(V_\alpha \cap V_\beta).$$

A presheaf F is a sheaf precisely when for each open covering of an open set U by a family V_α of open subsets of U , the first arrow in the diagram above is an equalizer. For a separated presheaf, the first arrow need only be injective. ^[1]

It can be shown that to specify a sheaf, it is enough to specify its restriction to the open sets of a basis for the topology of the underlying space. Moreover, it can also be shown that it is enough to verify the sheaf axioms above relative to the open sets of a covering. Thus a sheaf can often be defined by giving its values on the open sets of a basis, and verifying the sheaf axioms relative to the basis.

Morphisms

Heuristically speaking, a morphism of sheaves is analogous to a function between them. However, because sheaves contain data relative to every open set of a topological space, a morphism of sheaves is defined as a collection of functions, one for each open set, which satisfy a compatibility condition.

Let F and G be two sheaves on X with values in the category $\mathbf{C}$. A *morphism* $\varphi: G \rightarrow F$ consists of a morphism $\varphi(U): G(U) \rightarrow F(U)$ for each open set U of X , subject to the condition that this morphism is compatible with restrictions. In other words, for every open subset U of an open set V , the following diagram

$$\begin{array}{ccc} G(V) & \xrightarrow{\varphi_V} & F(V) \\ r_{V,U} \downarrow & & \downarrow r_{V,U} \\ G(U) & \xrightarrow{\varphi_U} & F(U) \end{array}$$

is commutative.

Recall that we could also express a sheaf as a special kind of functor. In this language, a morphism of sheaves is a natural transformation of the corresponding functors. With this notion of morphism, there is a category of $\mathbf{C}$ -valued sheaves on X for any $\mathbf{C}$. The objects are the $\mathbf{C}$ -valued sheaves, and the morphisms are morphisms of sheaves. An *isomorphism* of sheaves is an isomorphism in this category.

It can be proved that an isomorphism of sheaves is an isomorphism on each open set U . In other words, φ is an isomorphism if and only if for each U , $\varphi(U)$ is an isomorphism. The same is true of monomorphisms, but not of epimorphisms. See sheaf cohomology.

Notice that we did not use the gluing axiom in defining a morphism of sheaves. Consequently, the above definition makes sense for presheaves as well. The category of $\mathbf{C}$ -valued presheaves is then a functor category, the category of contravariant functors from $O(X)$ to $\mathbf{C}$.

Examples

Because sheaves encode exactly the data needed to pass between local and global situations, there are many examples of sheaves occurring throughout mathematics. Here are some additional examples of sheaves:

- Any continuous map of topological spaces determines a sheaf of sets. Let $f: Y \rightarrow X$ be a continuous map. We define a sheaf $\Gamma(Y/X)$ on X by setting $\Gamma(Y/X)(U)$ equal to the sections $U \rightarrow Y$, that is, $\Gamma(Y/X)(U)$ is the set of all functions $s: U \rightarrow Y$ such that $fs = id_U$. Restriction is given by restriction of functions. This sheaf is called the **sheaf of sections** of f , and it is especially important when f is the projection of a fiber bundle onto its base space. Notice that if the image of f does not contain U , then $\Gamma(Y/X)(U)$ is empty. For a concrete example, take $X = \mathbb{C} \setminus \{0\}$, $Y = \mathbb{C}$, and $f(z) = \exp(z)$. $\Gamma(Y/X)(U)$ is the set of branches of the logarithm on U .
- Fix a point x in X and an object S in a category $\mathbf{C}$. The **skyscraper sheaf over x with stalk S** is the sheaf S_x defined as follows: If U is an open set containing x , then $S_x(U) = S$. If U does not contain x , then $S_x(U)$ is the terminal object of $\mathbf{C}$. The restriction maps are either the identity on S , if both open sets contain x , or the unique map from S to the terminal object of $\mathbf{C}$.

Sheaves on manifolds

In the following examples M is an n -dimensional C^k -manifold. The table lists the values of certain sheaves over open subsets U of M and their restriction maps.

Sheaf	Sections over an open set U	Restriction maps	Remarks
Sheaf of j-times continuously differentiable functions $\mathcal{O}_M^j, j \leq k$	C^j -functions $U \rightarrow \mathbf{R}$	Restriction of functions.	This is a sheaf of rings with addition and multiplication given by pointwise addition and multiplication. When $j = k$, this sheaf is called the structure sheaf and is denoted $\mathcal{O}_M$.
Sheaf of nonzero k-times continuously differentiable functions $\mathcal{O}_M^\times$	Nowhere zero C^k -functions $U \rightarrow \mathbf{R}$	Restriction of functions.	A sheaf of groups under pointwise multiplication.
Cotangent sheaves Ω_M^p	Differential forms of degree p on U	Restriction of differential forms.	Ω_M^1 and Ω_M^n are commonly denoted Ω_M and ω_M , respectively.
Sheaf of distributions $\mathcal{D}'_M$	Distributions on U	The dual map to extension of smooth compactly supported functions by zero.	Here M is assumed to be smooth.
Sheaf of differential operators $\mathcal{D}_M$	Finite-order differential operators on U	Restriction of differential operators.	Here M is assumed to be smooth.

Presheaves which are not sheaves

Here are two examples of presheaves which are not sheaves:

- Let X be the two-point topological space $\{x, y\}$ with the discrete topology. Define a presheaf F as follows: $F(\emptyset) = \emptyset$, $F(\{x\}) = \mathbf{R}$, $F(\{y\}) = \mathbf{R}$, $F(\{x, y\}) = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$. The restriction map $F(\{x, y\}) \rightarrow F(\{x\})$ is the projection of $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$ onto its first coordinate, and the restriction map $F(\{x, y\}) \rightarrow F(\{y\})$ is the projection of $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$ onto its second coordinate. F is a presheaf which is not separated: A global section is determined by three numbers, but the values of that section over $\{x\}$ and $\{y\}$ determine only two of those numbers. So while we can glue any two sections over $\{x\}$ and $\{y\}$, we cannot glue them uniquely.
- Let X be the real line, and let $F(U)$ be the set of bounded continuous functions on U . This is not a sheaf because it is not always possible to glue. For example, let U_i be the set of all x such that $|x| < i$. The identity function $f(x) = x$ is bounded on each U_i . Consequently we get a section s_i on U_i . However, these sections do not glue, because the function f is not bounded on the real line. Consequently F is a presheaf, but not a sheaf. In fact, F is separated because it is a sub-presheaf of the sheaf of continuous functions.

Turning a presheaf into a sheaf

It is frequently useful to take the data contained in a presheaf and to express it as a sheaf. It turns out that there is a best possible way to do this. It takes a presheaf F and produces a new sheaf aF called the **sheaving**, **sheafification** or **sheaf associated to the presheaf** F . a is called the **sheaving functor**, **sheafification functor**, or **associated sheaf functor**. There is a natural morphism of presheaves $i : F \rightarrow aF$ which has the universal property that for any sheaf G and any morphism of presheaves $f : F \rightarrow G$, there is a unique morphism of sheaves $\tilde{f} : aF \rightarrow G$ such that $f = \tilde{f}i$. In fact a is the adjoint functor to the inclusion functor from the category of sheaves to the category of presheaves, and i is the unit of the adjunction.

Images of sheaves

Image functors for sheaves
direct image f_* inverse image f^* direct image with compact support $f_!$ exceptional inverse image $Rf^!$ $f^* \hookrightarrow f_*$ $(R)f_! \hookrightarrow (R)f^!$

The definition of a morphism on sheaves makes sense only for sheaves on the same space X . This is because the data contained in a sheaf is indexed by the open sets of the space. If we have two sheaves on different spaces, then their data is indexed differently. There is no way to go directly from one set of data to the other.

However, it is possible to move a sheaf from one space to another using a continuous function. Let $f : X \rightarrow Y$ be a continuous function from a topological space X to a topological space Y . If we have a sheaf on X , we can move it to Y , and vice versa. There are four ways in which sheaves can be moved.

- A sheaf $\mathcal{F}$ on X can be moved to Y using the direct image functor f_* or the direct image with proper support functor $f_!$.
- A sheaf $\mathcal{G}$ on Y can be moved to X using the inverse image functor f^{-1} or the twisted inverse image functor $f^!$.

The twisted inverse image functor $f^!$ is, in general, only defined as a functor between derived categories. These functors come in adjoint pairs: f^{-1} and f_* are left and right adjoints of each other, and $Rf_!$ and $f^!$ are left and right adjoints of each other. The functors are intertwined with each other by Grothendieck duality and Verdier duality.

There is a different inverse image functor for sheaves of modules over sheaves of rings. This functor is usually denoted f^* and it is distinct from f^{-1} . See inverse image functor.

Stalks of a sheaf

The **stalk** $\mathcal{F}_x$ of a sheaf $\mathcal{F}$ captures the properties of a sheaf "around" a point $x \in X$. Here, "around" means that, conceptually speaking, one looks at smaller and smaller neighborhood of the point. Of course, no single neighborhood will be small enough, so we will have to take a limit of some sort.

The stalk is defined by

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U),$$

the direct limit being over all open subsets of X containing the given point x . In other words, an element of the stalk is given by a section over some open neighborhood of x , and two such sections are considered equivalent if their restrictions agree on a smaller neighborhood.

The natural morphism $\mathcal{F}(U) \rightarrow \mathcal{F}_x$ takes a section s in $\mathcal{F}(U)$ to its *germ*. This generalises the usual definition of a germ.

A different way of defining the stalk is

$$\mathcal{F}_x := i^{-1}\mathcal{F}(\{x\}),$$

where i is the inclusion of the one-point space $\{x\}$ into X . The equivalence follows from the definition of the inverse image.

In many situations, knowing the stalks of a sheaf is enough to control the sheaf itself. For example, whether or not a morphism of sheaves is a monomorphism, epimorphism, or isomorphism can be tested on the stalks. They also find use in constructions such as Godement resolutions.

Ringed spaces and locally ringed spaces

A pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings on X is called a **ringed space**. Many types of spaces can be defined as certain types of ringed spaces. The sheaf $\mathcal{O}_X$ is called the **structure sheaf** of the space. A very common situation is when all the stalks of the structure sheaf are local rings, in which case the pair is called a **locally ringed space**. Here are examples of definitions made in this way:

- An n -dimensional C^k manifold M is a locally ringed space whose structure sheaf is an $\underline{\mathbf{R}}$ -algebra and is locally isomorphic to the sheaf of C^k real-valued functions on $\mathbf{R}^n$.
- A complex analytic space is a locally ringed space whose structure sheaf is a $\underline{\mathbf{C}}$ -algebra and is locally isomorphic to the vanishing locus of a finite set of holomorphic functions

together with the restriction (to the vanishing locus) of the sheaf of holomorphic functions on $\mathbf{C}^n$ for some n .

- A scheme is a locally ringed space which is locally isomorphic to the spectrum of a ring.
- A semialgebraic space is a locally ringed space which is locally isomorphic to a semialgebraic set in Euclidean space together with its sheaf of semialgebraic functions.

Sheaves of modules

Let $(X, \mathcal{O}_X)$ be a ringed space. A **sheaf of modules** is a sheaf $\mathcal{M}$ such that on every open set U of X , $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$ -module and for every inclusion of open sets $V \sqsubset U$, the restriction map $\mathcal{M}(U) \rightarrow \mathcal{M}(V)$ is a homomorphism of $\mathcal{O}_X(U)$ -modules.

Most important geometric objects are sheaves of modules. For example, there is a one-to-one correspondence between vector bundles and locally free sheaves of $\mathcal{O}_X$ -modules. Sheaves of solutions to differential equations are D-modules, that is, modules over the sheaf of differential operators.

A particularly important case are abelian sheaves, which are modules over the constant sheaf $\underline{\mathbb{Z}}$. Every sheaf of modules is an abelian sheaf.

The étale space of a sheaf

In the examples above it was noted that some sheaves occur naturally as sheaves of sections. In fact, all sheaves of sets can be represented as sheaves of sections of a topological space called the *étale space*. If F is a sheaf over X , then the **étale space** of F is a topological space E together with a local homeomorphism $\pi: E \rightarrow X$; the sheaf of sections of π is F . E is usually a very strange space, and even if the sheaf F arises from a natural topological situation, E may not have any clear topological interpretation. For example, if F is the sheaf of sections of a continuous function $f: Y \rightarrow X$, then $E = Y$ if and only if f is a covering map.

The étale space E is constructed from the stalks of F over X . As a set, it is their disjoint union and π is the obvious map which takes the value x on the stalk of F over $x \in X$. The topology of E is defined as follows. For each element s of $F(U)$ and each x in U , we get a germ of s at x . These germs determine points of E . For any U and $s \in F(U)$, the union of these points (for all $x \in U$) is declared to be open in E . Notice that each stalk has the discrete topology. Two morphisms between sheaves determine a continuous map of the corresponding étale spaces which is compatible with the projection maps (in the sense that every germ is mapped to a germ over the same point). This makes the construction into a functor.

This gives an example of an étale space over X . An *étale space* is a topological space E together with a continuous map $\pi: E \rightarrow X$ which is a local homeomorphism such that each fiber of π has the discrete topology. The construction above determines an equivalence of categories between the category of sheaves of sets on X and the category of étale spaces over X . The construction of an étale space can also be applied to a presheaf, in which case the sheaf of sections of the étale space recovers the sheaf associated to the given presheaf.

The map π is an example of what is sometimes called an *étale map*. "Étale" here means the same thing as "local homeomorphism". However, the terminology "étale map" is more common in contexts where the right analogue of a local homeomorphism of manifolds is not characterized by the property of being a local homeomorphism. This is the case in algebraic

geometry. For more information see the article *étale morphism*.

This construction makes all sheaves into representable functors on certain categories of topological spaces. As above, let F be a sheaf on X , let E be its étale space, and let $\pi: E \rightarrow X$ be the natural projection. Consider the category **Top**/ X of topological spaces over X , that is, the category of topological spaces together with fixed continuous maps to X . Every object of this space is a continuous map $f: Y \rightarrow X$, and a morphism from $Y \rightarrow X$ to $Z \rightarrow X$ is a continuous map $Y \rightarrow Z$ which commutes with the two maps to X . There is a functor Γ from **Top**/ X to the category of sets which takes an object $f: Y \rightarrow X$ to $(f^{-1}F)(Y)$. For example, if $i: U \rightarrow X$ is the inclusion of an open subset, then $\Gamma(i) = (i^{-1}F)(U)$ agrees with the usual $F(U)$, and if $i: \{x\} \rightarrow X$ is the inclusion of a point, then $\Gamma(\{x\}) = (i^{-1}F)(\{x\})$ is the stalk of F at x . There is a natural isomorphism

$$(f^{-1}F)(Y) \cong \operatorname{Hom}_{\mathbf{Top}/X}(f, \pi)$$

which shows that E represents the functor Γ .

The definition of sheaves by étale spaces is older than the definition given earlier in the article. It is still common in some areas of mathematics such as mathematical analysis.

Sheaf cohomology

It was noted above that the functor $\Gamma(U, -)$ preserves isomorphisms and monomorphisms, but not epimorphisms. If F is a sheaf of abelian groups, or more generally a sheaf with values in an abelian category, then $\Gamma(U, -)$ is actually a left exact functor. This means that it is possible to construct derived functors of $\Gamma(U, -)$. These derived functors are called the *cohomology groups* (or *modules*) of F and are written $H^i(U, -)$. Grothendieck proved in his *Tohoku* paper that every category of sheaves of abelian groups contains enough injective objects, so these derived functors always exist.

However, computing sheaf cohomology using injective resolutions is nearly impossible. In practice, it is much more common to find a different and more tractable resolution of F . A general construction is provided by Godement resolutions, and particular resolutions may be constructed using soft sheaves, fine sheaves, and flabby sheaves (also known as *flasque sheaves* from the French *flasque* meaning flabby). As a consequence, it can become possible to compare sheaf cohomology with other cohomology theories. For example, the de Rham complex is a resolution of the constant sheaf $\underline{\mathbf{R}}$ on any smooth manifold, so the sheaf cohomology of $\underline{\mathbf{R}}$ is equal to its de Rham cohomology. In fact, comparing sheaf cohomology to de Rham cohomology and singular cohomology provides a proof of de Rham's theorem that the two cohomology theories are isomorphic.

A different approach is by Čech cohomology. Čech cohomology was the first cohomology theory developed for sheaves and it is well-suited to concrete calculations. It relates sections on open subsets of the space to cohomology classes on the space. In most cases, Čech cohomology computes the same cohomology groups as the derived functor cohomology. However, for some pathological spaces, Čech cohomology will give the correct H^1 but incorrect higher cohomology groups. To get around this, Jean-Louis Verdier developed hypercoverings. Hypercoverings not only give the correct higher cohomology groups but also allow the open subsets mentioned above to be replaced by certain morphisms from another space. This flexibility is necessary in some applications, such as the construction of Pierre Deligne's mixed Hodge structures.

A much cleaner approach to the computation of some cohomology groups is the Borel-Bott-Weil theorem, which identifies the cohomology groups of some line bundles on flag manifolds with irreducible representations of Lie groups. This theorem can be used, for example, to easily compute the cohomology groups of all line bundles on projective space.

In many cases there is a duality theory for sheaves which generalizes Poincaré duality. See Grothendieck duality and Verdier duality.

Sites and topoi

André Weil's Weil conjectures stated that there was a cohomology theory for algebraic varieties over finite fields which would give an analogue of the Riemann hypothesis. The only natural topology on such a variety, however, is the Zariski topology, but sheaf cohomology in the Zariski topology is badly behaved because there are very few open sets. Alexandre Grothendieck solved this problem by introducing Grothendieck topologies, which axiomatize the notion of *covering*. Grothendieck's insight was that the definition of a sheaf depends only on the open sets of a topological space, not on the individual points. Once he had axiomatized the notion of covering, open sets could be replaced by other objects. A presheaf takes each one of these objects to data, just as before, and a sheaf is a presheaf that satisfies the gluing axiom with respect to our new notion of covering. This allowed Grothendieck to define étale cohomology and l -adic cohomology, which eventually were used to prove the Weil conjectures.

A category with a Grothendieck topology is called a *site*. A category of sheaves on a site is called a *topos* or a *Grothendieck topos*. The notion of a topos was later abstracted by William Lawvere and Miles Tierney to define an elementary topos, which has connections to mathematical logic.

History

The first origins of **sheaf theory** are hard to pin down — they may be co-extensive with the idea of analytic continuation. It took about 15 years for a recognisable, free-standing theory of sheaves to emerge from the foundational work on cohomology.

- 1936 Eduard Čech introduces the *nerve* construction, for associating a simplicial complex to an open covering.
- 1938 Hassler Whitney gives a 'modern' definition of cohomology, summarizing the work since J. W. Alexander and Kolmogorov first defined *cochains*.
- 1943 Norman Steenrod publishes on homology *with local coefficients*.
- 1945 Jean Leray publishes work carried out as a prisoner of war, motivated by proving fixed point theorems for application to PDE theory; it is the start of sheaf theory and spectral sequences.
- 1947 Henri Cartan reproves the de Rham theorem by sheaf methods, in correspondence with André Weil (see De Rham-Weil theorem). Leray gives a sheaf definition in his courses via closed sets (the later *carapaces*).
- 1948 The Cartan seminar writes up sheaf theory for the first time.
- 1950 The "second edition" sheaf theory from the Cartan seminar: the sheaf space (*espace étalé*) definition is used, with stalkwise structure. Supports are introduced, and cohomology with supports. Continuous mappings give rise to spectral sequences. At the same time Kiyoshi Oka introduces an idea (adjacent to that) of a sheaf of ideals, in several complex variables.

- 1951 The Cartan seminar proves the Theorems A and B based on Oka's work.
- 1953 The finiteness theorem for coherent sheaves in the analytic theory is proved by Cartan and Jean-Pierre Serre, as is Serre duality.
- 1954 Serre's paper *Faisceaux algébriques cohérents* (published in 1955) introduces sheaves into algebraic geometry. These ideas are immediately exploited by Hirzebruch, who writes a major 1956 book on topological methods.
- 1955 Alexander Grothendieck in lectures in Kansas defines abelian category and *presheaf*, and by using injective resolutions allows direct use of sheaf cohomology on all topological spaces, as derived functors.
- 1956 Oscar Zariski's report *Algebraic sheaf theory*
- 1957 Grothendieck's *Tohoku* paper rewrites homological algebra; he proves Grothendieck duality (i.e., Serre duality for possibly singular algebraic varieties).
- 1957 onwards: Grothendieck extends sheaf theory in line with the needs of algebraic geometry, introducing: schemes and general sheaves on them, local cohomology, derived categories (with Verdier), and Grothendieck topologies. There emerges also his influential schematic idea of 'six operations' in homological algebra.
- 1958 Godement's book on sheaf theory is published. At around this time Mikio Sato proposes his hyperfunctions, which will turn out to have sheaf-theoretic nature.

At this point sheaves had become a mainstream part of mathematics, with use by no means restricted to algebraic topology. It was later discovered that the logic in categories of sheaves is intuitionistic logic (this observation is now often referred to as Kripke-Joyal semantics, but probably should be attributed to a number of authors). This shows that some of the facets of sheaf theory can also be traced back as far as Leibniz.

See also

- Coherent sheaf
- Gerbe
- Holomorphic sheaf
- Stack (descent theory)

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External links

- Sheaf (<http://planetmath.org/?op=getobj&from=objects&id=5648>) on PlanetMath

Postnikov system

In homotopy theory, a branch of algebraic topology, a **Postnikov system** (or **Postnikov tower**) is a way of constructing a topological space from its homotopy groups. Postnikov systems were introduced, and named after, Mikhail Postnikov.

The Postnikov system of a path-connected space X is a tower of spaces $\dots \rightarrow X_n \rightarrow \dots \rightarrow X_1 \rightarrow X_0$ with the following properties:

- each map $X_n \rightarrow X_{n-1}$ is a fibration;
- $\pi_k(X_n) = \pi_k(X)$ for $k \leq n$;
- $\pi_k(X_n) = 0$ for $k > n$.

Every path-connected space has such a Postnikov system, and it is unique up to homotopy. The space X can be reconstructed from the Postnikov system as its inverse limit: $X = \lim_n X_n$. By the long exact sequence for the fibration $X_n \rightarrow X_{n-1}$, the fiber (call it K_n) has a single homotopy group in degree n ; it is thus an Eilenberg-Mac Lane space of type $K(\pi_n(X), n)$. The Postnikov system can be thought of as a way of constructing X out of Eilenberg-Mac Lane spaces.

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Grothendieck topology

In category theory, a branch of mathematics, a **Grothendieck topology** is a structure on a category C which makes the objects of C act like the open sets of a topological space. A category together with a choice of Grothendieck topology is called a **site**.

Grothendieck topologies axiomatize the notion of an open cover. Using the notion of covering provided by a Grothendieck topology, it becomes possible to define sheaves on a category and their cohomology. This was first done in algebraic geometry and algebraic number theory by Alexander Grothendieck to define the étale cohomology of a scheme. It has been used to define other cohomology theories since then, such as l -adic cohomology, flat cohomology, and crystalline cohomology. While Grothendieck topologies are most often used to define cohomology theories, they have found other applications as well, such as to John Tate's theory of rigid analytic geometry.

There is a natural way to associate a site to an ordinary topological space, and Grothendieck's theory is loosely regarded as a generalization of classical topology. Under meager point-set hypotheses, namely sobriety, this is completely accurate—it is possible to recover a sober space from its associated site. However simple examples such as the indiscrete topological space show that not all topological spaces can be expressed using Grothendieck topologies. Conversely, there are Grothendieck topologies which do not come from topological spaces.

Introduction

André Weil's famous Weil conjectures proposed that certain properties of equations with integral coefficients should be understood as geometric properties of the algebraic variety that they defined. His conjectures postulated that there should be a cohomology theory of algebraic varieties which gave number-theoretic information about their defining equations. This cohomology theory was known as the "Weil cohomology", but using the tools he had available, Weil was unable to construct it.

In the early 1960s, Alexander Grothendieck introduced étale maps into algebraic geometry as algebraic analogues of local analytic isomorphisms in analytic geometry. He used étale coverings to define an algebraic analogue of the fundamental group of a topological space. Soon Jean-Pierre Serre noticed that some properties of étale coverings mimicked those of open immersions, and that consequently it was possible to make constructions which imitated the cohomology functor H^1 . Grothendieck saw that it would be possible to use Serre's idea to define a cohomology theory which he suspected would be the Weil cohomology. To define this cohomology theory, Grothendieck needed to replace the usual, topological notion of an open covering with one that would use étale coverings instead. Grothendieck also saw how to phrase the definition of covering abstractly; this is where the definition of a Grothendieck topology comes from.

Definition

Motivation

The classical definition of a sheaf begins with a topological space X . A sheaf associates information to the open sets of X . This information can be phrased abstractly by letting $O(X)$ be the category whose objects are the open subsets U of X and whose morphisms are the inclusion maps $U \rightarrow V$ of open sets U and V of X . We will call such maps *open immersions*, just as in the context of schemes. Then a presheaf on X is a contravariant functor from $O(X)$ to the category of sets, and a sheaf is a presheaf which satisfies the gluing axiom. The gluing axiom is phrased in terms of pointwise covering, i.e., $\{U_i\}$ covers U if and only if $\bigcup_i U_i = U$. Because the value of a presheaf on an open set determines the value on any smaller set by restriction, it is equivalent to work with the collection of all open subsets V contained in some U_i . A Grothendieck topology encodes the information about collections of open subsets of U and which collections cover U without any reference to the space itself.

Sieves

In a Grothendieck topology, the notion of a collection of open subsets of U stable under inclusion is replaced by the notion of a sieve. If c is any given object in C , a **sieve** on c is a subfunctor of the functor $\text{Hom}(-, c)$; (this is the Yoneda embedding applied to c). In the case of $O(X)$, a sieve S on an open set U corresponds to a collection of open subsets of U "selected" by S . More precisely, consider that for any open subset V of U , $S(V)$ will be a subset of $\text{Hom}(V, U)$, which has only one element, the open immersion $V \rightarrow U$. Then V will be considered "selected" by S if and only if $S(V)$ is nonempty.

If S is a sieve on X , and $f: Y \rightarrow X$ is a morphism, then left composition by f gives a sieve on Y called the **pullback of S along f** , denoted by f^*S . It is defined as the fibered product $S \times_{\text{Hom}(-, X)} \text{Hom}(-, Y)$ together with its natural embedding in $\text{Hom}(-, Y)$. More concretely, for each object Z of C , $f^*S(Z) = \{ g: Z \rightarrow Y \mid fg \in S(Z) \}$, and f^*S inherits its action on morphisms by being a subfunctor of $\text{Hom}(-, Y)$. In the classical example, the pullback of a collection $\{V_i\}$ of subsets of U along an inclusion $W \rightarrow U$ is the collection $\{V_i \cap W\}$.

Covering Sieves

A classical topology on a set X is a collection of distinguished subsets, called open sets. This selection is subject to certain conditions: the axioms of a topological space. By comparison, a Grothendieck topology J on a category C is a collection, *for each object c of C* , of distinguished sieves on c , called **covering sieves** of c and denoted by $J(c)$. This selection will be subject to certain axioms, stated below. Continuing the previous example, a sieve S on an open set U in $O(X)$ will be a covering sieve if and only if the union of all the open sets V for which $S(V)$ is nonempty equals U ; in other words, if and only if S gives us a collection of open sets which cover U in the classical sense.

Axioms

The conditions we impose on a **Grothendieck topology** are:

- (T 1) (Base change) If S is a covering sieve on X , and $f: Y \rightarrow X$ is a morphism, then the pullback $f^* S$ is a covering sieve on Y .
- (T 2) (Local character) Let S be a covering sieve on X , and let T be any sieve on X . Suppose that for each object Y of C and each arrow $f: Y \rightarrow X$ in $S(Y)$, the pullback sieve $f^* T$ is a covering sieve on Y . Then T is a covering sieve on X .
- (T 3) (Identity) $\text{Hom}(-, X)$ is a covering sieve on X for any object X in C .

The base change axiom corresponds to the idea that if $\{U_i\}$ covers U , then $\{U_i \cap V\}$ should cover $U \cap V$. The local character axiom corresponds to the idea that if $\{U_i\}$ covers U and $\{V_{ij}\}_{j \in J_i}$ covers U_i for each i , then the collection $\{V_{ij}\}$ for all i and j should cover U . Lastly, the identity axiom corresponds to the idea that any set is covered by all its possible subsets.

Alternative Axioms

In fact, it is possible to put these axioms in another form where their geometric character is more apparent, assuming that the underlying category C contains certain fibered products. In this case, instead of specifying sieves, we can specify that certain collections of maps with a common codomain should cover their codomain. These collections are called **covering families**. If the collection of all covering families satisfies certain axioms, then we say that they form a **Grothendieck pretopology**. These axioms are:

- (PT 0) (Existence of fibered products) For all objects X of C , and for all morphisms $X_0 \rightarrow X$ which appear in some covering family of X , and for all morphisms $Y \rightarrow X$, the fibered product $X_0 \times_X Y$ exists.
- (PT 1) (Stability under base change) For all objects X of C , all morphisms $Y \rightarrow X$, and all covering families $\{X_\alpha \rightarrow X\}$, the family $\{X_\alpha \times_X Y \rightarrow Y\}$ is a covering family.
- (PT 2) (Local character) If $\{X_\alpha \rightarrow X\}$ is a covering family, and if for all α , $\{X_{\beta\alpha} \rightarrow X_\alpha\}$ is a covering family, then the family of composites $\{X_{\beta\alpha} \rightarrow X_\alpha \rightarrow X\}$ is a covering family.
- (PT 3) (Isomorphisms) If $f: Y \rightarrow X$ is an isomorphism, then $\{f\}$ is a covering family.

For any pretopology, the collection of all sieves that contain a covering family from the pretopology is always a Grothendieck topology.

For categories with fibered products, there is a converse. Given a collection of arrows $\{X_\alpha \rightarrow X\}$, we construct a sieve S by letting $S(Y)$ be the set of all morphisms $Y \rightarrow X$ that factor through some arrow $X_\alpha \rightarrow X$. This is called the sieve **generated by** $\{X_\alpha \rightarrow X\}$. Now choose a topology. Say that $\{X_\alpha \rightarrow X\}$ is a covering family if and only if the sieve that it generates is a covering sieve for the given topology. It is easy to check that this defines a pretopology.

(PT 3) is sometimes replaced by a weaker axiom:

- (PT 3') (Identity) If $1_X: X \rightarrow X$ is the identity arrow, then $\{1_X\}$ is a covering family.

(PT 3) implies (PT 3'), but not conversely. However, suppose that we have a collection of covering families that satisfies (PT 0) through (PT 2) and (PT 3'), but not (PT 3). These families generate a pretopology. The topology generated by the original collection of covering families is then the same as the topology generated by the pretopology, because the sieve generated by an isomorphism $Y \rightarrow X$ is $\text{Hom}(-, X)$. Consequently, if we restrict our attention to topologies, (PT 3) and (PT 3') are equivalent.

Sites and sheaves

Let C be a category and let J be a Grothendieck topology on C . The pair (C, J) is called a **site**.

A **presheaf** on a category is a contravariant functor from C to the category of all sets. Note that for this definition C is not required to have a topology. A sheaf on a site, however, should allow gluing, just like sheaves in classical topology. Consequently, we define a **sheaf** on a site to be a presheaf F such that for all objects X and all covering sieves S on X , the natural map $\text{Hom}(\text{Hom}(-, X), F) \rightarrow \text{Hom}(S, F)$ induced by the inclusion of S into $\text{Hom}(-, X)$ is a bijection. Halfway in between a presheaf and a sheaf is the notion of a **separated presheaf**, where the natural map above is required to be only an injection, not a bijection, for all sieves S .

Using the Yoneda lemma, it is possible to show that a presheaf on the category $O(X)$ is a sheaf on the topology defined above if and only if it is a sheaf in the classical sense.

Sheaves on a pretopology have a particularly simple description: For each covering family $\{X_\alpha \rightarrow X\}$, the diagram

$$F(X) \rightarrow \prod_{\alpha \in A} F(X_\alpha) \rightrightarrows \prod_{\alpha, \beta \in A} F(X_\alpha \times_X X_\beta)$$

must be an equalizer. For a separated presheaf, the first arrow need only be injective.

Similarly, one can define presheaves and sheaves of abelian groups, rings, modules, and so on. One can require either that a presheaf F is a contravariant functor to the category of abelian groups (or rings, or modules, etc.), or that F be an abelian group (ring, module, etc.) object in the category of all contravariant functors from C to the category of sets. These two definitions are equivalent.

Examples

The discrete and indiscrete topologies

Let $\mathbf{C}$ be any category. To define the **discrete topology**, we declare all sieves to be covering sieves. If $\mathbf{C}$ has all fibered products, this is equivalent to declaring all families to be covering families. To define the **indiscrete topology**, we declare only the sieves of the form $\text{Hom}(-, X)$ to be covering sieves. The indiscrete topology is also known as the **biggest** or **chaotic** topology, and it is generated by the pretopology which has only isomorphisms for covering families. A sheaf on the indiscrete site is the same thing as a presheaf.

The canonical topology

Let $\mathbf{C}$ be any category. The Yoneda embedding gives a functor $\text{Hom}(-, X)$ for each object X of $\mathbf{C}$. The **canonical topology** is the biggest topology such that every representable presheaf $\text{Hom}(-, X)$ is a sheaf. A covering sieve or covering family for this site is said to be *strictly universally epimorphic*. A topology which is less fine than the canonical topology, that is, for which every covering sieve is strictly universally epimorphic, is called **subcanonical**. Subcanonical sites are exactly the sites for which every presheaf of the form $\text{Hom}(-, X)$ is a sheaf. Most sites encountered in practice are subcanonical.

Small site associated to a topological space

We repeat the example which we began with above. Let X be a topological space. We defined $O(X)$ to be the category whose objects are the open sets of X and whose morphisms are inclusions of open sets. The covering sieves on an object U of $O(X)$ were those sieves S satisfying the following condition:

- If W is the union of all the sets V such that $S(V)$ is non-empty, then $W = U$.

This topology can also naturally be expressed as a pretopology. We say that a family of inclusions $\{V_\alpha \subseteq U\}$ is a covering family if and only if the union $\bigcup V_\alpha$ equals U . This site is called the **small site associated to a topological space X** .

Big site associated to a topological space

Let Spc be the category of all topological spaces. Given any family of functions $\{u_\alpha : V_\alpha \rightarrow X\}$, we say that it is a **surjective family** or that the morphisms u_α are **jointly surjective** if $\bigcup u_\alpha(V_\alpha)$ equals X . We define a pretopology on Spc by taking the covering families to be surjective families all of whose members are open immersions. Let S be a sieve on Spc . S is a covering sieve for this topology if and only if:

- For all Y and every morphism $f : Y \rightarrow X$ in $S(Y)$, there exists a V and a $g : V \rightarrow X$ such that g is an open immersion, g is in $S(V)$, and f factors through g .
- If W is the union of all the sets $f(Y)$, where $f : Y \rightarrow X$ is in $S(Y)$, then $W = X$.

Fix a topological space X . Consider the comma category Spc/X of topological spaces with a fixed continuous map to X . The topology on Spc induces a topology on Spc/X . The covering sieves and covering families are almost exactly the same; the only difference is that now all the maps involved commute with the fixed maps to X . This is the **big site associated to a topological space X** . Notice that Spc is the big site associated to the one point space. This site was first considered by Jean Giraud.

The big and small sites of a manifold

Let M be a manifold. M has a category of open sets $O(M)$ because it is a topological space, and it gets a topology as in the above example. For two open sets U and V of M , the fiber product $U \times_M V$ is the open set $U \cap V$, which is still in $O(M)$. This means that the topology on $O(M)$ is defined by a pretopology, the same pretopology as before.

Let Mfd be the category of all manifolds and continuous maps. (Or smooth manifolds and smooth maps, or real analytic manifolds and analytic maps, etc.) Mfd is a subcategory of Spc , and open immersions are continuous (or smooth, or analytic, etc.), so Mfd inherits a topology from Spc . This lets us construct the big site of the manifold M as the site Mfd/M . We can also define this topology using the same pretopology we used above. Notice that to satisfy (PT 0), we need to check that for any continuous map of manifolds $X \rightarrow Y$ and any open subset U of Y , the fibered product $U \times_Y X$ is in Mfd/M . This is just the statement that the preimage of an open set is open. Notice, however, that not all fibered products exist in Mfd because the preimage of a smooth map at a critical value need not be a manifold.

Topologies and schemes

Fix a scheme X . There is more than one natural site associated to X . All of the following sites are subcanonical, and they are ordered from coarsest to finest.

The big and small Zariski sites

All schemes are topological spaces. We get the **small Zariski site of X** by considering X as a topological space and looking at the site $O(X)$. To define the big Zariski site, let Zar be the category whose objects are schemes and whose morphisms are morphisms of schemes. We define a pretopology on Zar by taking the covering families to be surjective families of scheme-theoretic open immersions. This defines a topology whose covering sieves S are characterized by the following two properties:

- For all Y and every morphism $f : Y \rightarrow X$ in $S(Y)$, there exists a V and a $g : V \rightarrow X$ such that g is an open immersion, g is in $S(V)$, and f factors through g .
- If W is the union of all the sets $f(Y)$, where $f : Y \rightarrow X$ is in $S(Y)$, then $W = X$.

Despite their outward similarities, the topology on Zar is *not* the restriction of the topology on Spc ! This is because there are morphisms of schemes which are topologically open immersions but which are not scheme-theoretic open immersions. For example, let A be a non-reduced ring and let N be its ideal of nilpotents. The quotient map $A \rightarrow A/N$ induces a map $\text{Spec } A/N \rightarrow \text{Spec } A$ which is the identity on underlying topological spaces. To be a scheme-theoretic open immersion it must also induce an isomorphism on structure sheaves, which this map does not do. In fact, this map is a closed immersion.

We call Zar/X the **big Zariski site of X** .

The big and small Nisnevich sites

A family of morphisms $\{u_\alpha : X_\alpha \rightarrow X\}$ is a **Nisnevich cover** if the family is jointly surjective and each u_α is an étale morphism with the following additional property: For every point $x \in X$, there exists an α and a point $u \in X_\alpha$ such that the induced map of residue fields $k(x) \rightarrow k(u)$ is an isomorphism. (We call such an étale morphism with this property a *Nisnevich morphism*.) We define a pretopology on the category of schemes and morphisms of schemes by declaring covering families to be exactly the Nisnevich covers. This generates a topology called the **Nisnevich topology**. We write Nis for the category of schemes with the Nisnevich topology.

The **small Nisnevich site of X** is the category $O(X_{Nis})$ whose objects are schemes U with a fixed Nisnevich morphism $U \rightarrow X$. The morphisms are morphisms of schemes compatible with the fixed maps to X . The **big Nisnevich site of X** is the category Nis/X , that is, the category of schemes with a fixed map to X , considered with the Nisnevich topology.

Nisnevich called this topology the *completely decomposed topology*. He introduced it in order to provide a cohomological interpretation of the class set of an affine group scheme (originally defined in adelic terms) and used it to partially prove the Grothendieck-Serre conjecture on rationally trivial torsors. This topology has also found important applications in algebraic K-theory, A^1 homotopy theory and the theory of motives.

The big and small étale sites

We say that a family of morphisms $\{u_\alpha : X_\alpha \rightarrow X\}$ is an **étale cover** if the family is jointly surjective and each u_α is an étale morphism. We define a pretopology on the category of schemes and morphisms of schemes by declaring covering families to be exactly the étale covers. This generates a topology called the **étale topology**. We write $\mathcal{E}t$ for the category of schemes with the étale topology.

The **small étale site of X** is the category $O(X_{\mathcal{E}t})$ whose objects are schemes U with a fixed étale morphism $U \rightarrow X$. The morphisms are morphisms of schemes compatible with the fixed maps to X . The **big étale site of X** is the category $\mathcal{E}t/X$, that is, the category of schemes with a fixed map to X , considered with the étale topology.

We can define the étale topology using less data. First, we notice that the étale topology is finer than the Zariski topology. Consequently, to define an étale cover of a scheme X , it suffices to first cover X by open affine subschemes, that is, to take a Zariski cover, and then to define an étale cover of an affine scheme. We define an étale cover of an affine scheme X to be a surjective family $\{u_\alpha : X_\alpha \rightarrow X\}$ such that the set of all α is finite, each X_α is affine, and each u_α is étale. Then an étale cover of X is a family $\{u_\alpha : X_\alpha \rightarrow X\}$ which becomes an étale cover after base changing to any open affine subscheme of X .

Flat sites

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Groupoid

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In abstract algebra, a branch of mathematics, especially in category theory and homotopy theory, a **groupoid** generalises the notion of group and of category in several equivalent ways. A groupoid can be seen as a:

- *Group* with a partial function replacing the binary operation;
- *Category* in which every morphism is an isomorphism. A category of this sort can be viewed as augmented with a unary operation, called *inverse* by analogy with group theory.

Special cases include:

- *Setoids*, that is: sets which come with an equivalence relation;
- *G-sets*, sets equipped with an action of a group G .

Groupoids are often used to reason about geometrical objects such as manifolds. Heinrich Brandt introduced groupoids in 1926.

Definitions

Algebraic

A groupoid is a set G with a unary operation $^{-1} : G \rightarrow G$, and a partial function $* : G \times G \rightarrow G$. $*$ is not a binary operation because it is not necessarily defined for all possible pairs of G -elements. The precise conditions under which $*$ is defined are not articulated here and vary by situation.

$*$ and $^{-1}$ have the following axiomatic properties. Let a , b , and c be elements of G . Then:

- *Associativity*: If $a * b$ and $b * c$ are defined, then $(a * b) * c$ and $a * (b * c)$ are defined and equal. Conversely, if either of these last two expressions is defined, then so is the other (and again they are equal).
- *Inverse*: $a^{-1} * a$ and $a * a^{-1}$ are always defined.
- *Identity*: If $a * b$ is defined, then $a * b * b^{-1} = a$, and $a^{-1} * a * b = b$. (The previous two axioms already show that these expressions are defined and unambiguous.)

In short:

- $(a * b) * c = a * (b * c)$;
- $(a * b) * b^{-1} = a$;
- $a^{-1} * (a * b) = b$.

From these axioms, two easy and convenient theorems follow:

- $(a^{-1})^{-1} = a$;
 - If $a * b$ is defined, then $(a * b)^{-1} = b^{-1} * a^{-1}$.
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Category theoretic

A groupoid is a small category in which every morphism is an isomorphism, and hence invertible. More precisely, a groupoid G is:

- A set G_0 of *objects*;
- For each pair of objects x and y in G_0 , there exists a (possibly empty) set $G(x,y)$ of *morphisms* (or *arrows*) from x to y . We write $f : x \rightarrow y$ to indicate that f is an element of $G(x,y)$.

The objects and morphisms have the properties:

- For every object x , there exists the element id_x of $G(x,x)$;
- For each triple of objects x, y , and z , there exists the function $\text{comp}_{x,y,z} : G(x,y) \times G(y,z) \rightarrow G(x,z)$. We write gf for $\text{comp}_{x,y,z}(f, g)$, where $f \in G(x,y)$, and $g \in G(y,z)$;
- There exists the function $\text{inv}_{x,y} : G(x,y) \rightarrow G(y,x)$.

Moreover, if $f : x \rightarrow y$, $g : y \rightarrow z$, and $h : z \rightarrow w$, then:

- $f\text{id}_x = f$ and $\text{id}_y f = f$;
- $(hg)f = h(gf)$;
- $f\text{inv}(f) = \text{id}_y$ and $\text{inv}(f)f = \text{id}_x$.

Comparing the definitions

The algebraic and category-theoretic definitions are equivalent, as follows. Given a groupoid in the category-theoretic sense, let G be the disjoint union of all of the sets $G(x,y)$. Then comp and inv become partially defined operations on G , and inv will in fact be defined everywhere; so we define $*$ to be comp and $^{-1}$ to be inv . Thus we have a groupoid in the algebraic sense. Explicit reference to G_0 (and hence to id) can be dropped.

Conversely, given a groupoid G in the algebraic sense, with typical element f , let G_0 be the set of all elements of the form $f*f^{-1}$. In other words, the objects are identified with the identity morphisms, so that id_x is just x . Let $G(x,y)$ be the set of all elements f such that yfx is defined. Then $^{-1}$ and $*$ break up into several functions on the various $G(x,y)$, which may be called inv and comp , respectively.

Sets in the definitions above may be replaced with classes, as is generally the case in category theory.

Groupoid Category

The category whose objects are groupoids and whose morphisms are groupoid homomorphisms is called the **groupoid category**, or the **category of groupoids**.

Examples

Linear algebra

Given a field K , the corresponding **general linear groupoid** $GL_*(K)$ consists of all invertible matrices whose entries range over K . Matrix multiplication interprets composition. If $G = GL_*(K)$, then the set of natural numbers is a proper subset of G_0 , since for each natural number n , there is a corresponding identity matrix of dimension n . $G(m,n)$ is empty unless $m=n$, in which case it is the set of all $n \times n$ invertible matrices.

Topology

Given a topological space X , let G_0 be the set X . The morphisms from the point p to the point q are equivalence classes of continuous paths from p to q , with two paths being equivalent if they are homotopic. Two such morphisms are composed by first following the first path, then the second; the homotopy equivalence guarantees that this composition is associative. This groupoid is called the fundamental groupoid of X , denoted $\pi_1(X)$.

An important extension of this idea is to consider the fundamental groupoid $\pi_1(X, A)$ where A is a set of "base points" and a subset of X . Here, one considers only paths whose endpoints belong to A . $\pi_1(X, A)$ is a sub-groupoid of $\pi_1(X)$. The set A may be chosen according to the geometry of the situation at hand.

Equivalence relation

If X is a set with an equivalence relation denoted by infix $\sim$, then a groupoid "representing" this equivalence relation can be formed as follows:

- The objects of the groupoid are the elements of X ;
- For any two elements x and y in X , there is a single morphism from x to y if and only if $x \sim y$.

Group action

If the group G acts on the set X , then we can form a groupoid representing this group action as follows:

- The objects are the elements of X ;
- For any two elements x and y in X , there is a morphism from x to y corresponding to every element g of G such that $gx = y$;
- Composition of morphisms interprets the binary operation of G .

Another way to describe G -sets is the functor category $[\text{Gr}, \text{Set}]$, where Gr is the groupoid (category) with one element and isomorphic to the group G . Indeed, every functor F of this category defines a set $X = F(\text{Gr})$ and for every g in G (i.e. for every morphism in Gr) induces a bijection $F_g : X \rightarrow X$. The categorical structure of the functor F assures us that F defines a G -action on the set X . The (unique) representable functor $F : \text{Gr} \rightarrow \text{Set}$ is the Cayley representation of G . In fact, this functor is isomorphic to $\text{Hom}(\text{Gr}, -)$ and so sends $\text{ob}(\text{Gr})$ to the set $\text{Hom}(\text{Gr}, \text{Gr})$ which is by definition the "set" G and the morphism g of Gr (i.e. the element g of G) to the permutation F_g of the set G . We deduce from the Yoneda embedding that the group G is isomorphic to the group $\{F_g \mid g \in G\}$, a subgroup of the group of permutations of G .

Fifteen puzzle

The symmetries of the Fifteen puzzle form a groupoid (not a group, as not all moves can be composed). This groupoid acts on configurations.

Relation to groups

Group-like structures				
	Totality	Associativity	Identity	Inverses
Group	Yes	Yes	Yes	Yes

Monoid	Yes	Yes	Yes	No
Semigroup	Yes	Yes	No	No
Loop	Yes	No	Yes	Yes
Quasigroup	Yes	No	No	Yes
Magma	Yes	No	No	No
Groupoid	No	Yes	Yes	Yes
Category	No	Yes	Yes	No

If a groupoid has only one object, then the set of its morphisms forms a group. Using the algebraic definition, such a groupoid is literally just a group. Many concepts of group theory generalize to groupoids, with the notion of functor replacing that of group homomorphism.

If x is an object of the groupoid G , then the set of all morphisms from x to x forms a group $G(x)$. If there is a morphism f from x to y , then the groups $G(x)$ and $G(y)$ are isomorphic, with an isomorphism given by the mapping $g \rightarrow fgf^{-1}$.

Every connected groupoid (that is, one in which any two objects are connected by at least one morphism) is isomorphic to a groupoid of the following form. Pick a group G and a set (or class) X . Let the objects of the groupoid be the elements of X . For elements x and y of X , let the set of morphisms from x to y be G . Composition of morphisms is the group operation of G . If the groupoid is not connected, then it is isomorphic to a disjoint union of groupoids of the above type (possibly with different groups G for each connected component). Thus any groupoid may be given (up to isomorphism) by a set of ordered pairs (X, G) .

Note that the isomorphism described above is not unique, and there is no natural choice. Choosing such an isomorphism for a connected groupoid essentially amounts to picking one object x_0 , a group isomorphism h from $G(x_0)$ to G , and for each x other than x_0 , a morphism in G from x_0 to x .

In category-theoretic terms, each connected component of a groupoid is equivalent (but not isomorphic) to a groupoid with a single object, that is, a single group. Thus any groupoid is equivalent to a multiset of unrelated groups. In other words, for equivalence instead of isomorphism, one need not specify the sets X , only the groups G .

Consider the examples in the previous section. The general linear groupoid is both equivalent and isomorphic to the disjoint union of the various general linear groups $GL_n(F)$. On the other hand:

- The fundamental groupoid of X is equivalent to the collection of the fundamental groups of each path-connected component of X , but an isomorphism requires specifying the set of points in each component;
- The set X with the equivalence relation $\sim$ is equivalent (as a groupoid) to one copy of the trivial group for each equivalence class, but an isomorphism requires specifying what each equivalence class is;
- The set X equipped with an action of the group G is equivalent (as a groupoid) to one copy of G for each orbit of the action, but an isomorphism requires specifying what set each orbit is.

The collapse of a groupoid into a mere collection of groups loses some information, even from a category-theoretic point of view, because it is not natural. Thus when groupoids arise in terms of other structures, as in the above examples, it can be helpful to maintain

the full groupoid. Otherwise, one must choose a way to view each $G(x)$ in terms of a single group, and this choice can be arbitrary. In our example from topology, you would have to make a coherent choice of paths (or equivalence classes of paths) from each point p to each point q in the same path-connected component.

As a more illuminating example, the classification of groupoids with one endomorphism does not reduce to purely group theoretic considerations. This is analogous to the fact that the classification of vector spaces with one endomorphism is nontrivial.

Morphisms of groupoids come in more kinds than those of groups: we have, for example, fibrations, covering morphisms, universal morphisms, and quotient morphisms. Thus a subgroup H of a group G yields an action of G on the set of cosets of H in G and hence a covering morphism p from, say, K to G , where K is a groupoid with vertex groups isomorphic to H . In this way, presentations of the group G can be "lifted" to presentations of the groupoid K , and this is a useful way of obtaining information about presentations of the subgroup H . For further information, see the books by Higgins and by Brown in the References.

Another useful fact is that the category of groupoids, unlike that of groups, is cartesian closed.

Lie groupoids and Lie algebroids

When studying geometrical objects, the arising groupoids often carry some differentiable structure, turning them into Lie groupoids. These can be studied in terms of Lie algebroids, in analogy to the relation between Lie groups and Lie algebras.

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Lie groupoid

In mathematics, a **Lie groupoid** is a groupoid where the set Ob of objects and the set Mor of morphisms are both manifolds, the source and target operations

$$s, t : Mor \rightarrow Ob$$

are submersions, and all the category operations (source and target, composition, and identity-assigning map) are smooth.

A Lie groupoid can thus be thought of as a "many-object generalization" of a Lie group, just as a groupoid is a many-object generalization of a group. Just as every Lie group has a Lie algebra, every Lie groupoid has a Lie algebroid.

Examples

- Any Lie group gives a Lie groupoid with one object, and conversely. So, the theory of Lie groupoids includes the theory of Lie groups.
 - Given any manifold M , there is a Lie groupoid called the pair groupoid, with M as the manifold of objects, and precisely one morphism from any object to any other. In this Lie groupoid the manifold of morphisms is thus $M \times M$.
 - Given a Lie group G acting on a manifold M , there is a Lie groupoid called the translation groupoid with one morphism for each triple $g \in G, x, y \in M$ with $gx = y$.
 - Any foliation gives a Lie groupoid.
 - Any principal bundle $P \rightarrow M$ with structure group G gives a groupoid, namely $P \times P/G$ over M , where G acts on the pairs componentwise. Composition is defined via compatible representatives as in the pair groupoid.
-

Morita Morphisms and Smooth Stacks

Beside isomorphism of groupoids there is a more coarse notation of equivalence, the so called Morita equivalence. A quite general example is the Morita-morphism of the **Čech groupoid** which goes as follows. Let M be a smooth manifold and $\{U_\alpha\}$ an open cover of M . Define $G_0 := \bigsqcup_\alpha U_\alpha$ the disjoint union with the obvious submersion $p : G_0 \rightarrow M$. In order to encode the structure of the manifold M define the set of morphisms $G_1 := \bigsqcup_{\alpha, \beta} U_{\alpha\beta}$ where $U_{\alpha\beta} = U_\alpha \cap U_\beta \subset M$. The source and target map are defined as the embeddings $s : U_{\alpha\beta} \rightarrow U_\alpha$ and $t : U_{\alpha\beta} \rightarrow U_\beta$. And multiplication is the obvious one if we read the $U_{\alpha\beta}$ as subsets of M (compatible points in $U_{\alpha\beta}$ and $U_{\beta\gamma}$ actually are the same in M and also lie in $U_{\alpha\gamma}$).

This Čech groupoid is in fact the pullback groupoid of $M \rightrightarrows M$, i.e. the trivial groupoid over M , under p . That is what makes it Morita-morphism.

In order to get the notion of an equivalence relation we need to make the construction symmetric and show that it is also transitive. In this sense we say that 2 groupoids $G_1 \rightrightarrows G_0$ and $H_1 \rightrightarrows H_0$ are Morita equivalent iff there exists a third groupoid $K_1 \rightrightarrows K_0$ together with 2 Morita morphisms from G to K and H to K . Transitivity is an interesting construction in the category of groupoid principal bundles and left to the reader.

It arises the question of what is preserved under the Morita equivalence. There are 2 obvious things, one the coarse quotient/ orbit space of the groupoid $G_0/G_1 = H_0/H_1$ and secondly the stabilizer groups $G_p \cong H_q$ for corresponding points $p \in G_0$ and $q \in H_0$.

The further question of what is the structure of the coarse quotient space leads to the notion of a smooth stack. We can expect the coarse quotient to be a smooth manifold if for example the stabilizer groups are trivial (as in the example of the Čech groupoid). But if the stabilizer groups change we cannot expect a smooth manifold any longer. The solution is to revert the problem and to define:

A **smooth stack** is a Morita-equivalence class of Lie groupoids. The natural geometric objects living on the stack are the geometric objects on Lie groupoids invariant under Morita-equivalence. As an example consider the Lie groupoid cohomology.

Examples

- The notion of smooth stack is quite general, obviously all smooth manifolds are smooth stacks.
- But also orbifolds are smooth stacks, namely (equivalence classes of) étale groupoids.
- Orbit spaces of foliations are another class of examples

External links

Alan Weinstein, Groupoids: unifying internal and external symmetry, *AMS Notices*, **43** (1996), 744-752. Also available as arXiv:math/9602220 ^[1]

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Lie algebroid

In mathematics, Lie algebroids serve the same role in the theory of Lie groupoids that Lie algebras serve in the theory of Lie groups: reducing global problems to infinitesimal ones. Just as a Lie groupoid can be thought of as a "Lie group with many objects", a Lie algebroid is like a "Lie algebra with many objects".

More precisely, a **Lie algebroid** is a triple $(E, [\cdot, \cdot], \rho)$ consisting of a vector bundle E over a manifold M , together with a Lie bracket $[\cdot, \cdot]$ on its module of sections $\Gamma(E)$ and a morphism of vector bundles $\rho : E \rightarrow TM$ called the **anchor**. Here TM is the tangent bundle of M . The anchor and the bracket are to satisfy the Leibniz rule:

$$[X, fY] = \rho(X)f \cdot Y + f[X, Y]$$

where $X, Y \in \Gamma(E)$, $f \in C^\infty(M)$ and $\rho(X)f$ is the derivative of f along the vector field $\rho(X)$. It follows that

$$\rho([X, Y]) = [\rho(X), \rho(Y)]$$

for all $X, Y \in \Gamma(E)$.

Examples

- Every Lie algebra is a Lie algebroid over the one point manifold.
- The tangent bundle TM of a manifold M is a Lie algebroid for the Lie bracket of vector fields and the identity of TM as an anchor.
- Every integrable subbundle of the tangent bundle — that is, one whose sections are closed under the Lie bracket — also defines a Lie algebroid.
- Every bundle of Lie algebras over a smooth manifold defines a Lie algebroid where the Lie bracket is defined pointwise and the anchor map is equal to zero.
- To every Lie groupoid is associated a Lie algebroid, generalizing how a Lie algebra is associated to a Lie group (see also below). For example, the Lie algebroid TM comes from the pair groupoid whose objects are M , with one isomorphism between each pair of objects. Unfortunately, going back from a Lie algebroid to a Lie groupoid is not always possible^[1], but every Lie algebroid gives a stacky Lie groupoid^{[2] [3]}.
- Given the action of a Lie algebra $\mathfrak{g}$ on a manifold M , the set of $\mathfrak{g}$ -invariant vector fields on M is a Lie algebroid over the space of orbits of the action.
- **Atiyah algebroid.** Given a vector bundle V over a smooth manifold M consider its derivations, i.e. smooth $\mathbb{R}$ -linear maps $\psi : \Gamma(V) \rightarrow \Gamma(V)$ for which exists a vector field X such that they fulfill the Leibniz rule $\psi(fv) = X[f]v + f\psi(v)$ for all smooth functions f and all sections v into the vector bundle. The association $\psi \rightarrow X$ is clearly linear and thus comes from a map of vector bundles $\rho : A(V) \rightarrow TM$ (if you find the bundle whose sections give the derivations). The Atiyah algebroid is further characterized by fitting into the following short exact sequence: $0 \rightarrow \text{End}_M(V) \rightarrow A(V) \rightarrow TM \rightarrow 0$. To see that the Atiyah algebroid exists for every vector bundle note that it is the Lie algebroid associated to the Lie groupoid coming from the frame bundle of the vector bundle V .

Lie algebroid associated to a Lie groupoid

To describe the construction let us fix some notation. G is the space of morphisms of the Lie groupoid, M the space of objects, $e : M \rightarrow G$ the units and $t : G \rightarrow M$ the target map.

$T^t G = \bigcup_{p \in M} T(t^{-1}(p)) \subset TG$ the t -fiber tangent space. The Lie algebroid is now the vector

bundle $A := e^* T^t G$. This inherits a bracket from G , because we can identify the M -sections into A with left-invariant vector fields on G . Further these sections act on the smooth functions of M by identifying these with left-invariant functions on G .

As a more explicit example consider the Lie algebroid associated to the pair groupoid $G := M \times M$. The target map is $t : G \rightarrow M : (p, q) \mapsto p$ and the units $e : M \rightarrow G : p \mapsto (p, p)$. The t -fibers are $p \times M$ and therefore

$T^t G = \bigcup_{p \in M} p \times TM \subset TM \times TM$. So the Lie algebroid is the vector bundle

$A := e^* T^t G = \bigcup_{p \in M} T_p M = TM$. The extension of sections X into A to left-invariant vector

fields on G is simply $\tilde{X}(p, q) = 0 \oplus X(q)$ and the extension of a smooth function f from M to a left-invariant function on G is $\tilde{f}(p, q) = f(q)$. Therefore the bracket on A is just the Lie bracket of tangent vector fields and the anchor map is just the identity.

Of course you could do an analog construction with the source map and right-invariant vector fields/ functions. However you get an isomorphic Lie algebroid, with the explicit isomorphism i_* , where $i : G \rightarrow G$ is the inverse map.

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Algebroid

In mathematics, **algebroid** may mean

- algebroid branch, a formal power series branch of an algebraic curve
- algebroid multifunction
- Lie algebroid in the theory of Lie groupoids
- algebroid cohomology

Fundamental group

In mathematics, more specifically algebraic topology, the **fundamental group** or **Poincaré group** is a group associated to any given pointed topological space that provides a way of determining when two paths, starting and ending at a fixed base point, can be continuously deformed into each other. Intuitively, it records information about the basic shape, or *holes*, of the topological space. The fundamental group is the first and simplest of the homotopy groups.

Fundamental groups can be studied using the theory of covering spaces, since a fundamental group coincides with the group of deck transformations of the associated universal covering space. Its abelianisation can be identified with the first homology group of the space. When the topological space is homeomorphic to a simplicial complex, its fundamental group can be described explicitly in terms of generators and relations.

Historically, the concept of fundamental group first emerged in the theory of Riemann surfaces, in the work of Bernhard Riemann, Henri Poincaré and Felix Klein, where it describes the monodromy properties of complex functions, as well as providing a complete topological classification of closed surfaces.

Intuition and definition

Before giving a precise definition of the fundamental group, we try to describe the general idea in non-mathematical terms. Take some space, and some point in it, and consider all the loops both starting and ending at this point — paths which start at this point, wander around as much as they like and eventually return to the starting point. Two loops can be combined together in an obvious way: travel along the first loop, then along the second. The set of all the loops with this method of combining them is the fundamental group, except that for technical reasons it is necessary to consider two loops to be the same if one can be deformed into the other without breaking.

For the precise definition, let X be a topological space, and let x_0 be a point of X . We are interested in the set of continuous functions $f : [0,1] \rightarrow X$ with the property that $f(0) = x_0 = f(1)$. These functions are called **loops** with **base point** x_0 . Any two such loops, say f and g , are considered equivalent if there is a continuous function $h : [0,1] \times [0,1] \rightarrow X$ with the property that, for all $0 \leq t \leq 1$, $h(t,0) = f(t)$, $h(t,1) = g(t)$ and $h(0,t) = x_0 = h(1,t)$. Such an h is called a **homotopy** from f to g , and the corresponding equivalence classes are called **homotopy classes**.

The product $f \square g$ of two loops f and g is defined by setting $(f \square g)(t) := f(2t)$ if $0 \leq t \leq 1/2$ and $(f \square g)(t) := g(2t - 1)$ if $1/2 \leq t \leq 1$. Thus the loop $f \square g$ first follows the loop f with

"twice the speed" and then follows g with twice the speed. The product of two homotopy classes of loops $[f]$ and $[g]$ is then defined as $[f \square g]$, and it can be shown that this product does not depend on the choice of representatives.

With the above product, the set of all homotopy classes of loops with base point x_0 forms the **fundamental group** of X at the point x_0 and is denoted

$$\pi_1(X, x_0),$$

or simply $\pi(X, x_0)$. The identity element is the constant map at the basepoint, and the inverse of a loop f is the loop g defined by $g(t) = f(1 - t)$. That is, g follows f backwards.

Although the fundamental group in general depends on the choice of base point, it turns out that, up to isomorphism, this choice makes no difference so long as the space X is path-connected. For path-connected spaces, therefore, we can write $\pi_1(X)$ instead of $\pi_1(X, x_0)$ without ambiguity whenever we care about the isomorphism class only.

Examples

In many spaces, such as $\mathbf{R}^n$, or any convex subset of $\mathbf{R}^n$, there is only one homotopy class of loops, and the fundamental group is therefore trivial, i.e. $(\{0\}, +)$. A path-connected space with a trivial fundamental group is said to be simply connected.

A more interesting example is provided by the circle. It turns out that each homotopy class consists of all loops which wind around the circle a given number of times (which can be positive or negative, depending on the direction of winding). The product of a loop which winds around m times and another that winds around n times is a loop which winds around $m + n$ times. So the fundamental group of the circle is isomorphic to $(\mathbb{Z}, +)$, the additive group of integers. This fact can be used to give proofs of the Brouwer fixed point theorem and the Borsuk-Ulam theorem in dimension 2.

Since the fundamental group is a homotopy invariant, the theory of the winding number for the complex plane minus one point is the same as for the circle.

Unlike the homology groups and higher homotopy groups associated to a topological space, the fundamental group need not be abelian. For example, the fundamental group of the figure eight is the free group on two letters. More generally, the fundamental group of any graph G is a free group. Here the rank of the free group is equal to $1 - \chi(G)$: one minus the Euler characteristic of G , when G is connected. A somewhat more sophisticated example of a space with a non-abelian fundamental group is the complement of a trefoil knot in $\mathbf{R}^3$.

Functoriality

If $f : X \rightarrow Y$ is a continuous map, $x_0 \in X$ and $y_0 \in Y$ with $f(x_0) = y_0$, then every loop in X with base point x_0 can be composed with f to yield a loop in Y with base point y_0 . This operation is compatible with the homotopy equivalence relation and with composition of loops. The resulting group homomorphism, called the induced homomorphism, is written as $\pi(f)$ or, more commonly,

$$f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0).$$

We thus obtain a functor from the category of topological spaces with base point to the category of groups.

It turns out that this functor cannot distinguish maps which are homotopic relative the base point: if f and $g : X \rightarrow Y$ are continuous maps with $f(x_0) = g(x_0) = y_0$, and f and g are

homotopic relative to $\{x_0\}$, then $f_* = g_*$. As a consequence, two homotopy equivalent path-connected spaces have isomorphic fundamental groups:

$$X \simeq Y \Rightarrow \pi_1(X, x_0) \cong \pi_1(Y, y_0).$$

The fundamental group functor takes products to products and coproducts to coproducts. That is, if X and Y are path connected, then

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$$

and

$$\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y).$$

(In the latter formula, $\vee$ denotes the wedge sum of topological spaces, and $*$ the free product of groups.) Both formulas generalize to arbitrary products. Furthermore the latter formula is a special case of the Seifert-van Kampen theorem which states that the fundamental group functor takes pushouts along inclusions to pushouts.

Fibrations

A generalization of a product of spaces is given by a fibration,

$$F \rightarrow E \rightarrow B.$$

Here the total space E is a sort of "twisted product" of the base space B and the fiber F . In general the fundamental groups of B , E and F are terms in a long exact sequence involving higher homotopy groups. When all the spaces are connected, this has the following consequences for the fundamental groups:

- $\pi_1(B)$ and $\pi_1(E)$ are isomorphic if F is simply connected
- $\pi_{n+1}(B)$ and $\pi_n(F)$ are isomorphic if E is contractible

The latter is often applied to the situation $E = \text{path space of } B$, $F = \text{loop space of } B$ or $B = \text{classifying space } BG$ of a topological group G , $B = \text{universal } G\text{-bundle } EG$.

Relationship to first homology group

The fundamental groups of a topological space X are related to its first singular homology group, because a loop is also a singular 1-cycle. Mapping the homotopy class of each loop at a base point x_0 to the homology class of the loop gives a homomorphism from the fundamental group $\pi(X, x_0)$ to the homology group $H_1(X)$. If X is path-connected, then this homomorphism is surjective and its kernel is the commutator subgroup of $\pi(X, x_0)$, and $H_1(X)$ is therefore isomorphic to the abelianization of $\pi(X, x_0)$. This is a special case of the Hurewicz theorem of algebraic topology.

Universal covering space

If X is a topological space that is path connected, locally path connected and locally simply connected, then it has a simply connected universal covering space on which the fundamental group $\pi(X, x_0)$ acts freely by deck transformations with quotient space X . This space can be constructed analogously to the fundamental group by taking pairs (x, γ) , where x is a point in X and γ is a homotopy class of paths from x_0 to x and the action of $\pi(X, x_0)$ is by concatenation of paths. It is uniquely determined as a covering space.

Examples

Let G be a connected, simply connected compact Lie group, for example the special unitary group SU_n , and let Γ be a finite subgroup of G . Then the homogeneous space $X=G/\Gamma$ has fundamental group Γ , which acts by right multiplication on the universal covering space G . Among the many variants of this construction, one of the most important is given by locally symmetric spaces $X=G/\Gamma K$, where

- G is a non-compact simply connected, connected Lie group (often semisimple),
- K is a maximal compact subgroup of G
- Γ is a discrete countable torsion-free subgroup of G .

In this case the fundamental group is Γ and the universal covering space G/K is actually contractible (by the Cartan decomposition for Lie groups).

As an example take $G=SL_2(\mathbf{R})$, $K=SO_2$ and Γ any torsion-free congruence subgroup of the modular group $SL_2(\mathbf{Z})$.

An even simpler example is given by $G=\mathbf{R}$ (so that K is trivial) and $\Gamma=\mathbf{Z}$: in this case $X=\mathbf{R}/\mathbf{Z}=S^1$.

From the explicit realization, it also follows that the universal covering space of a path connected topological group H is again a path connected topological group G . Moreover the covering map is a continuous open homomorphism of G onto H with kernel Γ , a closed discrete normal subgroup of G :

$$1 \rightarrow \Gamma \rightarrow G \rightarrow H \rightarrow 1.$$

Since G is a connected group with a continuous action by conjugation on a discrete group Γ , it must act trivially, so that Γ has to be a subgroup of the center of G . In particular $\pi_1(H) = \Gamma$ is an Abelian group; this can also easily be seen directly without using covering spaces. The group G is called the *universal covering group* of H .

As the universal covering group suggests, there is an analogy between the fundamental group of a topological group and the center of a group; this is elaborated at Lattice of covering groups.

Edge-path group of a simplicial complex

If X is a connected simplicial complex, an *edge-path* in X is defined to be a chain of vertices connected by edges in X . Two edge-paths are said to be *edge-equivalent* if one can be obtained from the other by successively switching between an edge and the two opposite edges of a triangle in X . If v is a fixed vertex in X , an *edge-loop* at v is an edge-path starting and ending at v . The **edge-path group** $E(X,v)$ is defined to be the set of edge-equivalence classes of edge-loops at v , with product and inverse defined by concatenation and reversal of edge-loops.

The edge-path group is naturally isomorphic to $\pi_1(|X|,v)$, the fundamental group of the geometric realisation $|X|$ of X . Since it depends only on the 2-skeleton X^2 of X (i.e. the vertices, edges and triangles of X), the groups $\pi_1(|X|,v)$ and $\pi_1(|X^2|,v)$ are isomorphic.

The edge-path group can be described explicitly in terms of generators and relations. If T is a maximal spanning tree in the 1-skeleton of X , then $E(X,v)$ is canonically isomorphic to the group with generators the oriented edges of X not occurring in T and relations the edge-equivalences corresponding to triangles in X containing one or more edge not in T . A similar result holds if T is replaced by any simply connected—in particular

contractible—subcomplex of X . This often gives a practical way of computing fundamental groups and can be used to show that every finitely presented group arises as the fundamental group of a finite simplicial complex. It is also one of the classical methods used for topological surfaces, which are classified by their fundamental groups.

The *universal covering space* of a finite connected simplicial complex X can also be described directly as a simplicial complex using edge-paths. Its vertices are pairs (w, γ) where w is a vertex of X and γ is an edge-equivalence class of paths from v to w . The k -simplices containing (w, γ) correspond naturally to the k -simplices containing w . Each new vertex u of the k -simplex gives an edge wu and hence, by concatenation, a new path γ_u from v to u . The points (w, γ) and (u, γ_u) are the vertices of the "transported" simplex in the universal covering space. The edge-path group acts naturally by concatenation, preserving the simplicial structure, and the quotient space is just X .

It is well-known that this method can also be used to compute the fundamental group of an arbitrary topological space. This was doubtless known to Čech and Leray and explicitly appeared as a remark in a paper by Weil (1960); various other authors such as L. Calabi, W-T. Wu and N. Berikashvili have also published proofs. In the simplest case of a compact space X with a finite open covering in which all non-empty finite intersections of open sets in the covering are contractible, the fundamental group can be identified with the edge-path group of the simplicial complex corresponding to the nerve of the covering.

Realizability

Every group can be realized as the fundamental group of a connected CW-complex of dimension 2 (or higher). As noted above, though, only free groups can occur as fundamental groups of 1-dimensional CW-complexes (that is, graphs).

Every finitely presented group can be realized as the fundamental group of a compact, connected, smooth manifold of dimension 4 (or higher). But there are severe restrictions on which groups occur as fundamental groups of low-dimensional manifolds. For example, no free abelian group of rank 4 or higher can be realized as the fundamental group of a manifold of dimension 3 or less.

Related concepts

The fundamental group measures the 1-dimensional hole structure of a space. For studying "higher-dimensional holes", the homotopy groups are used. The elements of the n -th homotopy group of X are homotopy classes of (basepoint-preserving) maps from S^n to X .

The set of loops at a particular base point can be studied without regarding homotopic loops as equivalent. This larger object is the loop space.

For topological groups, a different group multiplication may be assigned to the set of loops in the space, with pointwise multiplication rather than concatenation. The resulting group is the loop group.

Fundamental groupoid

Rather than singling out one point and considering the loops based at that point up to homotopy, one can also consider *all* paths in the space up to homotopy (fixing the initial and final point). This yields not a group but a groupoid, the **fundamental groupoid** of the space.

See also

- Homotopy group, generalization of fundamental group

There are also similar notions of fundamental group for algebraic varieties (the étale fundamental group) and for orbifolds (the orbifold fundamental group).

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- Fundamental group^[2] on PlanetMath
- Fundamental groupoid^[3] on PlanetMath

External links

- Dylan G.L. Allegretti, *Simplicial Sets and van Kampen's Theorem*^[4] (*An elementary discussion of the fundamental groupoid of a topological space and the fundamental groupoid of a simplicial set*).
- Animations to introduce to the fundamental group by Nicolas Delanoue^[5]

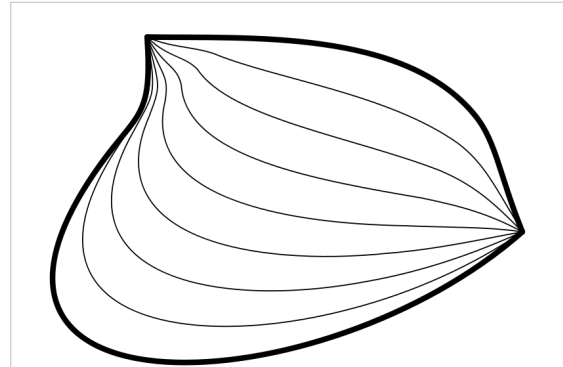
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- [4] <http://www.math.uchicago.edu/~may/VIGRE/VIGREREU2008.html>
- [5] http://www.istia.univ-angers.fr/~delanoue/topo_alg/

Homotopy

In topology, two continuous functions from one topological space to another are called **homotopic** (Greek *homos* = identical and *topos* = place) if one can be "continuously deformed" into the other, such a deformation being called a **homotopy** between the two functions. An outstanding use of homotopy is the definition of homotopy groups and cohomotopy groups, important invariants in algebraic topology.

In practice, there are technical difficulties in using homotopies with certain pathological spaces. Consequently most algebraic topologists work with compactly generated spaces, CW complexes, or spectra.



The two bold paths shown above are homotopic relative to their endpoints. Thin lines mark isocontours of one possible homotopy.

Formal definition

Formally, a homotopy between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function $H: X \times [0,1] \rightarrow Y$ from the product of the space X with the unit interval $[0,1]$ to Y such that, for all points x in X , $H(x,0)=f(x)$ and $H(x,1)=g(x)$.

If we think of the second parameter of H as "time", then H describes a "continuous deformation" of f into g : at time 0 we have the function f , at time 1 we have the function g .

An alternative notation is to say that a homotopy between two continuous functions $f, g: X \rightarrow Y$ is a family of continuous functions $h_t: X \rightarrow Y$ for $t \in [0,1]$

such that $h_0 = f$ and $h_1 = g$ and the map $t \rightarrow h_t$ is continuous from $[0,1]$ to the space of all continuous functions $X \rightarrow Y$. The two versions coincide by setting $h_t(x) = H(x,t)$.



A homotopy of a coffee cup into a doughnut (torus).

Properties

Continuous functions f and g (both from topological space X to Y) are said to be homotopic if and only if there is a homotopy H taking f to g as described above. Being homotopic is an equivalence relation on the set of all continuous functions from X to Y . This homotopy relation is compatible with function composition in the following sense: if $f_1, g_1: X \rightarrow Y$ are homotopic, and $f_2, g_2: Y \rightarrow Z$ are homotopic, then their compositions $f_2 \circ f_1$ and $g_2 \circ g_1: X \rightarrow$

Z are homotopic as well.

Homotopy equivalence and null-homotopy

Given two spaces X and Y , we say they are **homotopy equivalent** or of the same **homotopy type** if there exist continuous maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $g \circ f$ is homotopic to the identity map id_X and $f \circ g$ is homotopic to id_Y .

The maps f and g are called **homotopy equivalences** in this case. Clearly, every homeomorphism is a homotopy equivalence, but the converse is not true: for example, a solid disk is not homeomorphic to a single point, although the disk and the point are homotopy equivalent.

Intuitively, two spaces X and Y are homotopy equivalent if they can be transformed into one another by bending, shrinking and expanding operations. For example, a solid disk or solid ball is homotopy equivalent to a point, and $\mathbf{R}^2 - \{(0,0)\}$ is homotopy equivalent to the unit circle S^1 . Spaces that are homotopy equivalent to a point are called contractible.

A function f is said to be **null-homotopic** if it is homotopic to a constant function. (The homotopy from f to a constant function is then sometimes called a **null-homotopy**.) For example, a map from the circle S^1 is null-homotopic precisely when it can be extended to a map of the disc D^2 .

It follows from these definitions that a space X is contractible if and only if the identity map from X to itself—which is always a homotopy equivalence—is null-homotopic.

Homotopy invariance

Homotopy equivalence is important because in algebraic topology many concepts are **homotopy invariant**, that is, they respect the relation of homotopy equivalence. For example, if X and Y are homotopy equivalent spaces, then:

- if X is path-connected, then so is Y
- if X is simply connected, then so is Y
- the (singular) homology and cohomology groups of X and Y are isomorphic
- if X and Y are path-connected, then the fundamental groups of X and Y are isomorphic, and so are the higher homotopy groups. Without the path-connectedness assumption, one has $\pi_1(X, x_0)$ isomorphic to $\pi_1(Y, f(x_0))$ where $f: X \rightarrow Y$ is a homotopy equivalence and x_0 a given point in X .

An example of an algebraic invariant of topological spaces which is not homotopy-invariant is compactly supported homology (which is, roughly speaking, the homology of the compactification, and compactification is not homotopy-invariant).

Relative homotopy

In order to define the fundamental group, one needs the notion of **homotopy relative to a subspace**. These are homotopies which keep the elements of the subspace fixed. Formally: if f and g are continuous maps from X to Y and K is a subset of X , then we say that f and g are homotopic relative to K if there exists a homotopy $H: X \times [0,1] \rightarrow Y$ between f and g such that $H(k,t) = f(k) = g(k)$ for all $k \in K$ and $t \in [0,1]$. Also, if g is a retract from X to K and f is the identity map, this is known as a strong deformation retract of X to K . When K is a point, the term **pointed homotopy** is used.

Homotopy groups

Since the relation of two functions $f, g : X \rightarrow Y$ being homotopic relative to a subspace is an equivalence relation, we can look at the equivalence classes of maps between a fixed X and Y . If we fix $X = [0,1]^n$, the unit interval $[0,1]$ crossed with itself n times, and we take our subspace to be its boundary $\partial([0,1]^n)$ then the equivalence classes form a group, denoted $\pi_n(Y, y_0)$, where y_0 is the image of the subspace $\partial([0,1]^n)$.

We can define the action of one equivalence class on another, and so we get a group. These groups are called the homotopy groups. In the case $n = 1$, it is also called the fundamental group.

Homotopy category

The idea of homotopy can be turned into a formal category of category theory. The **homotopy category** is the category whose objects are topological spaces, and whose morphisms are homotopy equivalence classes of continuous maps. Two topological spaces X and Y are isomorphic in this category if and only if they are homotopy-equivalent. Then a functor on the category of topological spaces is homotopy invariant if it can be expressed as a functor on the homotopy category.

For example, homology groups are a *functorial* homotopy invariant: this means that if f and g from X to Y are homotopic, then the group homomorphisms induced by f and g on the level of homology groups are the same: $H_n(f) = H_n(g) : H_n(X) \rightarrow H_n(Y)$ for all n . Likewise, if X and Y are in addition path-connected, then the group homomorphisms induced by f and g on the level of homotopy groups are also the same: $\pi_n(f) = \pi_n(g) : \pi_n(X) \rightarrow \pi_n(Y)$.

Timelike homotopy

On a Lorentzian manifold, certain curves are distinguished as timelike. A timelike homotopy between two timelike curves is a homotopy such that each intermediate curve is timelike. No closed timelike curve (CTC) on a Lorentzian manifold is timelike homotopic to a point (that is, null timelike homotopic); such a manifold is therefore said to be multiply connected by timelike curves. A manifold such as the 3-sphere can be simply connected (by any type of curve), and yet be multiply timelike connected.

Homotopy lifting property

If we have a homotopy $H : X \times [0, 1] \rightarrow Y$ and a cover $p : \tilde{Y} \rightarrow Y$ and we are given a map $\tilde{h}_0 : X \rightarrow \tilde{Y}$ such that $\tilde{h}_0 = p \circ h_0$ ($\tilde{h}_0$ is called a lift of h_0), then we can lift all of H to a map $\tilde{H} : X \times [0, 1] \rightarrow \tilde{Y}$ such that $p \circ \tilde{H} = H$. The homotopy lifting property is used to characterize fibrations.

Homotopy extension property

Another useful property involving homotopy is the homotopy extension property, which characterizes the extension of a homotopy between two functions from a subset of some set to the set itself. It is useful when dealing with cofibrations.

Isotopy

In case the two given continuous functions f and g from the topological space X to the topological space Y are homeomorphisms, one can ask whether they can be connected 'through homeomorphisms'. This gives rise to the concept of **isotopy**, which is a homotopy, H , in the notation used before, such that for each fixed t , $H(x,t)$ gives a homeomorphism.

Requiring that two homeomorphisms be isotopic really is a stronger requirement than that they be homotopic. For example, the map of the unit disc in R^2 defined by $f(x,y) = (-x, -y)$ is equivalent to a 180-degree rotation around the origin, and so the identity map and f are isotopic because they can be connected by rotations. However, the map on the interval $[-1,1]$ in R defined by $f(x) = -x$ is *not* isotopic to the identity. Loosely speaking, any homotopy from f to the identity would have to exchange the endpoints, which would mean that they would have to 'pass through' each other. Moreover, f has changed the orientation of the interval, hence it cannot be isotopic to the identity. However, the maps are homotopic; one homotopy from f to the identity is $H: [-1,1] \times [0,1] \rightarrow [-1,1]$ given by $H(x,y) = 2yx - x$.

In geometric topology—for example in knot theory—the idea of isotopy is used to construct equivalence relations. For example, when should two knots be considered the same? We take two knots, K_1 and K_2 , in three-dimensional space. A knot is an embedding of a one-dimensional space, the "loop of string", into this space, and an embedding is simply a homeomorphism. The intuitive idea of *deforming* one to the other should correspond to a path of embeddings: a continuous function starting at $t=0$ with the K_1 embedding, ending at $t=1$ with the K_2 embedding, with all intermediate values being embeddings; this corresponds to the definition of isotopy. An ambient isotopy, studied in this context, is an isotopy of the larger space, considered in light of its action on the embedded submanifold.

Applications

Based on the concept of the homotopy, computation methods for algebraic and differential equations are developed. The methods for algebraic equations include the homotopy continuation method and the continuation method. The methods for differential equations include the homotopy analysis method.

See also

- Mapping class group
- Homeotopy
- Regular homotopy
- Poincaré conjecture
- Homotopy analysis method

Crossed module

In mathematics, and especially in homotopy theory, a **crossed module** consists of groups G and H , where G acts on H (which we will write on the left), and a homomorphism of groups

$$d: H \longrightarrow G,$$

that is equivariant with respect to the conjugation action of G on itself:

$$d(gh) = gd(h)g^{-1}$$

and also satisfies the so-called Peiffer identity:

$$d(h_1)h_2 = h_1h_2h_1^{-1}$$

Examples

Let N be a normal subgroup of a group G . Then, the inclusion

$$d: N \longrightarrow G$$

is a crossed module with the conjugation action of G on N .

For any group G , modules over the group ring are crossed G -modules with $d = 0$.

For any group H , the homomorphism from H to $\text{Aut}(H)$ sending any element of H to the corresponding inner automorphism can be given the structure of a crossed module. Thus we have a kind of 'automorphism structure' of a group, rather than just a group of automorphisms.

Given any central extension of groups

$$1 \rightarrow A \rightarrow H \rightarrow G \rightarrow 1$$

the onto homomorphism

$$d: H \rightarrow G$$

together with the action of G on H defines a crossed module. Thus, central extensions can be seen as special crossed modules. Conversely, a crossed module with surjective boundary defines a central extension.

If (X, A, x) is a pointed pair of topological spaces, then the homotopy boundary

$$d: \pi_2(X, A, x) \rightarrow \pi_1(A, x)$$

from the second relative homotopy group to the fundamental group, may be given the structure of crossed module. It is a remarkable fact that this functor

$$\Pi: (\text{pairs of pointed spaces}) \rightarrow (\text{crossed modules})$$

satisfies a form of the van Kampen theorem, in that it preserves certain colimits. See the article on crossed objects in algebraic topology below. The proof involves the concept of homotopy double groupoid of a pointed pair of spaces.

The result on the crossed module of a pair can also be phrased as: if

$$F \rightarrow E \rightarrow B$$

is a pointed fibration of spaces, then the induced map of fundamental groups

$$d: \pi_1(F) \rightarrow \pi_1(E)$$

may be given the structure of crossed module. This example is useful in algebraic K-theory. There are higher dimensional versions of this fact using n -cubes of spaces.

These examples suggest that crossed modules may be thought of as "2-dimensional groups". In fact, this idea can be made precise using category theory. It can be shown that a crossed module is essentially the same as a categorical group or 2-group: that is, a group object in the category of categories, or equivalently a category object in the category of groups. While this may sound intimidating, it simply means that the concept of crossed module is one version of the result of blending the concepts of "group" and "category". This equivalence is important in understanding and using even higher dimensional versions of groups.

Classifying space

Any crossed module

$$M = (d: H \longrightarrow G)$$

has a *classifying space* BM with the property that its homotopy groups are $\text{Coker } d$, in dimension 1, $\text{Ker } d$ in dimension 2, and 0 above 2. It is possible to describe conveniently the homotopy classes of maps from a CW-complex to BM . This allows one to prove that (pointed, weak) homotopy 2-types are completely described by crossed modules.

External links

- J. Baez and A. Lauda, Higher-dimensional algebra V: 2-groups ^[1]
- R. Brown, Groupoids and crossed objects in algebraic topology ^[2]
- R. Brown, Higher dimensional group theory ^[7]
- M. Forrester-Barker, Group objects and internal categories ^[3]
- Behrang Noohi, Notes on 2-groupoids, 2-groups and crossed-modules ^[4]

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- [4] <http://arxiv.org/pdf/math.CT/0512106>

Homology (mathematics)

In mathematics (especially algebraic topology and abstract algebra), **homology** (in Greek $\acute{\omicron}\mu\acute{\omicron}\varsigma$ *homos* "identical") is a certain general procedure to associate a sequence of abelian groups or modules with a given mathematical object such as a topological space or a group. See homology theory for more background, or singular homology for a concrete version for topological spaces, or group cohomology for a concrete version for groups.

For a topological space, the homology groups are generally much easier to compute than the homotopy groups, and consequently one usually will have an easier time working with homology to aid in the classification of spaces.

Construction of homology groups

The procedure works as follows. Given an object such as a topological space X , one first defines a *chain complex* $C(X)$ encoding information about X . A chain complex is a sequence of abelian groups or modules $C_0, C_1, C_2, \dots$ connected by homomorphisms $\partial_n: C_n \rightarrow C_{n-1}$, which we call **boundary operators**. That is,

$$\dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \equiv 0,$$

where 0 denotes the trivial group and $C_i \equiv 0$ for $i \leq 0$. We also require the composition of any two consecutive boundary operators to be zero. That is, for all n ,

$$\partial_n \circ \partial_{n+1} = 0,$$

i.e., the constant map to the group identity in C_{n-1} . This means $\text{im}(\partial_{n+1}) \subseteq \ker(\partial_n)$.

Now since each C_n is abelian, $\text{im}(\partial_{n+1})$ is a normal subgroup of $\ker(\partial_n)$. And we want to mod out by this subgroup, i.e., consider everything in $\text{im}(\partial_{n+1})$ equivalent and partition $\ker(\partial_n)$ using this equivalence relation. We define the **n -th homology group of X** to be the factor group (or quotient module)

$$H_n(X) = \ker(\partial_n) / \text{im}(\partial_{n+1}).$$

We also use the notation $\ker(\partial_n) = Z_n(X)$ and $\text{im}(\partial_{n+1}) = B_n(X)$, so

$$H_n(X) = Z_n(X) / B_n(X).$$

Computing these two groups is usually rather difficult since they are very large groups. On the other hand, we do have tools which make the task easier.

The *simplicial homology* groups $H_n(X)$ of a *simplicial complex* X are defined using the simplicial chain complex $C(X)$, with $C(X)_n$ the free abelian group generated by the n -simplices of X . The *singular homology* groups $H_n(X)$ are defined for any topological space X , and agree with the simplicial homology groups for a simplicial complex.

A chain complex is said to be exact if the image of the $(n+1)$ -th map is always equal to the kernel of the n th map. The homology groups of X therefore measure "how far" the chain complex associated to X is from being exact.

Cohomology groups are formally similar: one starts with a cochain complex, which is the same as a chain complex but whose arrows, now denoted d^n point in the direction of increasing n rather than decreasing n ; then the groups $\ker(d^n) = Z^n(X)$ and $\text{im}(d^{n-1}) = B^n(X)$ follow from the same description and

, as before.

Examples

The motivating example comes from algebraic topology: the **simplicial homology** of a simplicial complex X . Here A_n is the free abelian group or module whose generators are the n -dimensional oriented simplexes of X . The mappings are called the *boundary mappings* and send the simplex with vertices

$$(a[0], a[1], \dots, a[n])$$

to the sum

$$\sum_{i=0}^n (-1)^i (a[0], \dots, a[i-1], a[i+1], \dots, a[n])$$

(which is considered 0 if $n = 0$).

If we take the modules to be over a field, then the dimension of the n -th homology of X turns out to be the number of "holes" in X at dimension n .

Using this example as a model, one can define a singular homology for any topological space X . We define a chain complex for X by taking A_n to be the free abelian group (or free module) whose generators are all continuous maps from n -dimensional simplices into X . The homomorphisms ∂_n arise from the boundary maps of simplices.

In abstract algebra, one uses homology to define derived functors, for example the Tor functors. Here one starts with some covariant additive functor F and some module X . The chain complex for X is defined as follows: first find a free module F_1 and a surjective homomorphism $p_1 : F_1 \rightarrow X$. Then one finds a free module F_2 and a surjective homomorphism $p_2 : F_2 \rightarrow \ker(p_1)$. Continuing in this fashion, a sequence of free modules F_n and homomorphisms p_n can be defined. By applying the functor F to this sequence, one obtains a chain complex; the homology H_n of this complex depends only on F and X and is, by definition, the n -th derived functor of F , applied to X .

Homology functors

Chain complexes form a category: A morphism from the chain complex $(d_n : A_n \rightarrow A_{n-1})$ to the chain complex $(e_n : B_n \rightarrow B_{n-1})$ is a sequence of homomorphisms $f_n : A_n \rightarrow B_n$ such that $f_{n-1} \circ d_n = e_n \circ f_n$ for all n . The n -th homology H_n can be viewed as a covariant functor from the category of chain complexes to the category of abelian groups (or modules).

If the chain complex depends on the object X in a covariant manner (meaning that any morphism $X \rightarrow Y$ induces a morphism from the chain complex of X to the chain complex of Y), then the H_n are covariant functors from the category that X belongs to into the category of abelian groups (or modules).

The only difference between homology and cohomology is that in cohomology the chain complexes depend in a *contravariant* manner on X , and that therefore the homology groups (which are called *cohomology groups* in this context and denoted by H^n) form *contravariant* functors from the category that X belongs to into the category of abelian groups or modules.

Properties

If $(d_n : A_n \rightarrow A_{n-1})$ is a chain complex such that all but finitely many A_n are zero, and the others are finitely generated abelian groups (or finite dimensional vector spaces), then we can define the *Euler characteristic*

$$\chi = \sum (-1)^n \text{rank}(A_n)$$

(using the rank in the case of abelian groups and the Hamel dimension in the case of vector spaces). It turns out that the Euler characteristic can also be computed on the level of homology:

$$\chi = \sum (-1)^n \text{rank}(H_n)$$

and, especially in algebraic topology, this provides two ways to compute the important invariant χ for the object X which gave rise to the chain complex.

Every short exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

of chain complexes gives rise to a long exact sequence of homology groups

$$\cdots \rightarrow H_n(A) \rightarrow H_n(B) \rightarrow H_n(C) \rightarrow H_{n-1}(A) \rightarrow H_{n-1}(B) \rightarrow H_{n-1}(C) \rightarrow H_{n-2}(A) \rightarrow \cdots$$

All maps in this long exact sequence are induced by the maps between the chain complexes, except for the maps $H_n(C) \rightarrow H_{n-1}(A)$. The latter are called *connecting homomorphisms* and are provided by the snake lemma.

History

The homology group was developed by Emmy Noether^{[1] [2]} and, independently, by Leopold Vietoris and Walther Mayer, in the period 1925–28.^[3] Prior to this, topological classes in combinatorial topology were not formally considered as abelian groups. The spread of homology groups marked the change of terminology and viewpoint from "combinatorial topology" to "algebraic topology".^[4]

See also

- Simplicial homology
- Singular homology
- Cellular homology
- Homology theory
- Homological algebra

Notes

- [1] Hilton 1988, p. 284
- [2] For example *L'émergence de la notion de group d'homologie*, Nicolas Basbois (PDF) (<http://math1.unice.fr/~abbrugia/SPF/publi/expo13.pdf>), in French, note 41, explicitly names Noether as inventing the homology group.
- [3] Hirzebruch, Friedrich, "Emmy Noether and Topology" in Teicher 1999, p. 61–63.
- [4] *Bourbaki and Algebraic Topology* by John McCleary (PDF) (<http://math.vassar.edu/faculty/McCleary/BourbakiAlgTop.pdf>) gives documentation (translated into English from French originals).

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Cohomology

In mathematics, specifically in algebraic topology, **cohomology** is a general term for a sequence of abelian groups defined from a cochain complex. That is, cohomology is defined as the abstract study of **cochains**, cocycles, and coboundaries. Cohomology can be viewed as a method of assigning algebraic invariants to a topological space that has a more refined algebraic structure than does homology. Cohomology arises from the algebraic dualization of the construction of homology. In less abstract language, cochains in the fundamental sense should assign 'quantities' to the *chains* of homology theory.

From its beginning in topology, this idea became a dominant method in the mathematics of the second half of the twentieth century; from the initial idea of *homology* as a topologically invariant relation on *chains*, the range of applications of homology and cohomology theories has spread out over geometry and abstract algebra. The terminology tends to mask the fact that in many applications *cohomology*, a contravariant theory, is more natural than *homology*. At a basic level this has to do with functions and pullbacks in geometric situations: given spaces X and Y , and some kind of function F on Y , for any mapping $f: X \rightarrow Y$ composition with f gives rise to a function $F \circ f$ on X . Cohomology groups often also have a natural product, the cup product, which gives them a ring structure.

Definition

For a topological space X , the **cohomology group** $H^n(X;G)$, with coefficients in G , is defined to be the quotient $\text{Ker}(\delta)/\text{Im}(\delta)$ at $C^n(X;G)$ in the cochain complex

$$\dots \leftarrow C^{m+1}(X;G) \xleftarrow{\delta} C^m(X;G) \leftarrow \dots \leftarrow C^0(X;G) \leftarrow 0.$$

Elements in $\text{Ker}(\delta)$ are **cocycles** and elements in $\text{Im}(\delta)$ are **coboundaries**.

History

Although cohomology is fundamental to modern algebraic topology, its importance was not seen for some 40 years after the development of homology. The concept of *dual cell structure*, which Henri Poincaré used in his proof of his Poincaré duality theorem, contained the germ of the idea of cohomology, but this was not seen until later.

There were various precursors to cohomology. In the mid-1920s, J.W. Alexander and Solomon Lefschetz founded the intersection theory of cycles on manifolds. On an n -dimensional manifold M , a p -cycle and a q -cycle with nonempty intersection will, if in general position, have intersection a $(p+q-n)$ -cycle. This enables us to define a multiplication of homology classes

$$H_p(M) \times H_q(M) \rightarrow H_{p+q-n}(M).$$

Alexander had by 1930 defined a first cochain notion, based on a p -cochain on a space X having relevance to the small neighborhoods of the diagonal in X^{p+1} .

In 1931, Georges de Rham related homology and exterior differential forms, proving De Rham's theorem. This result is now understood to be more naturally interpreted in terms of cohomology.

In 1934, Lev Pontryagin proved the Pontryagin duality theorem; a result on topological groups. This (in rather special cases) provided an interpretation of Poincaré duality and Alexander duality in terms of group characters.

At a 1935 conference in Moscow, Andrey Kolmogorov and Alexander both introduced cohomology and tried to construct a cohomology product structure.

In 1936 Norman Steenrod published a paper constructing Čech cohomology by dualizing Čech homology.

From 1936 to 1938, Hassler Whitney and Eduard Čech developed the cup product (making cohomology into a graded ring) and cap product, and realized that Poincaré duality can be stated in terms of the cap product. Their theory was still limited to finite cell complexes.

In 1944, Samuel Eilenberg overcame the technical limitations, and gave the modern definition of singular homology and cohomology.

In 1945, Eilenberg and Steenrod stated the axioms defining a homology or cohomology theory. In their 1952 book, *Foundations of Algebraic Topology*, they proved that the existing homology and cohomology theories did indeed satisfy their axioms.^[1]

In 1948 Edwin Spanier, building on work of Alexander and Kolmogorov, developed Alexander-Spanier cohomology.

Cohomology theories

Eilenberg-Steenrod theories

A *cohomology theory* is a family of contravariant functors from the category of pairs of topological spaces and continuous functions (or some subcategory thereof such as the category of CW complexes) to the category of Abelian groups and group homomorphisms that satisfies the Eilenberg-Steenrod axioms.

Some cohomology theories in this sense are:

- simplicial cohomology
- singular cohomology
- de Rham cohomology
- Čech cohomology
- sheaf cohomology.

Extraordinary cohomology theories

When one axiom (*dimension axiom*) is relaxed, one obtains the idea of **extraordinary cohomology theory** or **generalized cohomology theory**; this allows theories based on K-theory and cobordism theory. There are others, coming from stable homotopy theory. In this context, singular homology is referred to as **ordinary homology**.

An extraordinary cohomology theory is "determined by its values on a point", in the sense that if one has a space given by contractible spaces (homotopy equivalent to a point), glued together along contractible spaces, as in a simplicial complex, then the cohomology of the space is determined by the cohomology of a point and the combinatorics of the patching, and effectively computable. Formally, this is computed by the excision theorem, or equivalently the Mayer-Vietoris sequence. Thus the cohomology of a point is a fundamental calculation for any extraordinary cohomology theory, though the cohomology of particular spaces is also of interest.

Other cohomology theories

Theories in a broader sense of *cohomology* include:^[2]

- Group cohomology
- Galois cohomology
- Lie algebra cohomology
- Harrison cohomology
- Γ cohomology
- Schur cohomology
- André-Quillen cohomology
- Hochschild cohomology
- Cyclic cohomology
- Topological André-Quillen cohomology
- Topological Hochschild cohomology
- Topological Cyclic cohomology
- Coherent cohomology
- Local cohomology
- Étale cohomology
- Crystalline cohomology
- Flat cohomology
- Motivic cohomology
- Deligne cohomology
- Perverse cohomology
- Intersection cohomology
- Non-abelian cohomology
- Gel'fand-Fuks cohomology
- Spencer cohomology
- Bonar-Claven cohomology
- Quantum cohomology

See also

- List of cohomology theories

Notes

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Homology theory

In mathematics, **homology theory** is the axiomatic study of the intuitive geometric idea of *homology of cycles* on topological spaces. It can be broadly defined as the study of homology theories on topological spaces.

Simple explanation

At the intuitive level *homology* is taken to be an equivalence relation, such that chains C and D are *homologous* on the space X if the chain $C - D$ is a *boundary* of a chain of one dimension higher. The simplest case is in graph theory, with C and D vertices and homology with a meaning coming from the oriented edge E from P to Q having boundary $Q - P$. A collection of edges from D to C , each one joining up to the one before, is a homology. In general, a k -chain is thought of as a formal combination

$$\sum a_i d_i$$

where the a_i are integers and the d_i are k -dimensional simplices on X . The boundary concept here is that taken over from the boundary of a simplex; it allows a high-dimensional concept which for $k = 1$ is the kind of telescopic cancellation seen in the graph theory case. This explanation is in the style of 1900, and proved somewhat naive, technically speaking.

Example of a torus surface

For example if X is a 2-torus T , a one-dimensional *cycle* on T is in intuitive terms a linear combination of curves drawn on T , which closes up on itself (cycle condition, equivalent to having no net boundary). If C and D are cycles each wrapping once round T in the same way, we can find explicitly an oriented area on T with boundary $C - D$. One can prove that the homology classes of 1-cycles with integer coefficients form a free abelian group with two generators, one generator for each of the two different ways round the 'doughnut'.

The nineteenth century

This level of understanding was common property in the mathematics of the nineteenth century, starting with the idea of Riemann surface. At the end of the century, the work of Poincaré had provided a much more general, though still intuitively-based, setting.

For example, it is considered that the general Stokes' theorem was first stated in 1899 by Poincaré: it involves necessarily both an *integrand* (we would now say, a differential form), and a region of integration (a p -chain), with two kinds of *boundary* operators, one of which in modern terms is the exterior derivative, and the other a geometric boundary operator on chains that includes *orientation* and can be used for homology theory. The two *boundaries* appear as adjoint operators, with respect to integration.

Twentieth century beginnings

Rather loose, geometric arguments with homology were only gradually replaced at the beginning of the twentieth century by rigorous techniques. To begin with, the style of the era was to use combinatorial topology (the fore-runner of today's algebraic topology). That assumes that the spaces treated are simplicial complexes, while the most interesting spaces are usually manifolds, so that artificial triangulations have to be introduced to apply the tools. Pioneers such as Solomon Lefschetz and Marston Morse still preferred a geometric approach. The combinatorial stance did allow Brouwer to prove foundational results such as the simplicial approximation theorem, at the base of the idea that homology is a functor (as it would later be put). Brouwer was able to prove the Jordan curve theorem, basic for complex analysis, and the invariance of domain, using the new tools; and remove the suspicion attached to topological arguments as handwaving.

Towards algebraic topology

The transition to *algebraic* topology is usually attributed to the influence of Emmy Noether, who insisted that homology classes lay in quotient groups — a point of view now so fundamental that it is taken as a definition.^[1] In fact Noether in the period from 1920 onwards was with her students elaborating the theory of modules for any ring, giving rise when the two ideas were combined to the concept of *homology with coefficients in a ring*. Before that, coefficients (that is, the sense in which chains are linear combinations of the basic geometric chains traced on the space) had usually been integers, real or complex numbers, or sometimes residue classes mod 2. In the new setting, there would be no reason not to take residues mod 3, for example: to be a cycle is then a more complex geometric condition, exemplified in graph theory terms by having the number of incoming edges at every vertex a multiple of 3. But in algebraic terms, the definitions present no new problem. The universal coefficient theorem explains that homology *with integer coefficients* determines all other homology theories, by use of the tensor product; it is not anodyne, in that (as we would now put it) the tensor product has derived functors that enter into a general formulation.

Cohomology, and singular homology

The 1930s were the decade of the development of cohomology theory, as several research directions grew together and the De Rham cohomology that was implicit in Poincaré's work cited earlier became the subject of definite theorems. Cohomology and homology are *dual* theories, in a sense that required detailed working out; at the same time it was realised that homology, that was, simplicial homology, was far from being at the end of its story. The definition of singular homology avoided the need for the apparatus of triangulations, at a cost of moving to infinitely-generated modules.

Axiomatics and extraordinary theories

The development of algebraic topology from 1940 to 1960 was very rapid, and the role of homology theory was often as 'baseline' theory, easy to compute and in terms of which topologists sought to calculate with other functors. The axiomatisation of homology theory by Eilenberg and Steenrod (the Eilenberg-Steenrod axioms) revealed that what various candidate homology *theories* had in common was, roughly speaking, some exact sequences

and in particular the Mayer–Vietoris sequence, and the *dimension axiom* that calculated the homology of the point. The dimension axiom was relaxed to admit the (co)homology derived from topological K-theory, and cobordism theory, in a vast extension to the **extraordinary (co)homology theories** that became standard in homotopy theory. These can be easily characterised for the category of CW complexes.

- List of cohomology theories

Current state of homology theory

For more general (i.e. less well-behaved) spaces, recourse to ideas from sheaf theory brought some extension of homology theories, particularly the Borel-Moore homology for locally compact spaces.

The basic chain complex apparatus of homology theory has long since become a separate piece of technique in homological algebra, and has been applied independently, for example to group cohomology. Therefore there is no longer one homology theory, but many homology and cohomology theories in mathematics.

Notes

[1] Hilton 1988, p. 284

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Chain complex

In mathematics, **chain complex** and **cochain complex** are constructs originally used in the field of algebraic topology. They are algebraic means of representing the relationships between the cycles and boundaries in various dimensions of some "space". Here the "space" could be a topological space or an algebraic construction such as a simplicial complex. More generally, homological algebra includes the study of chain complexes in the abstract, without any reference to an underlying space. In this case, chain complexes are studied axiomatically as algebraic structures.

Applications of chain complexes usually define and apply their homology groups (cohomology groups for cochain complexes); in more abstract settings various equivalence relations are applied to complexes (for example starting with the *chain homotopy* idea). Chain complexes are easily defined in abelian categories, also.

Formal definition

A **chain complex** $(A_\bullet, d_\bullet)$ is a sequence of abelian groups or modules $\dots A_{-2}, A_{-1}, A_0, A_1, A_2, \dots$ connected by homomorphisms (called **boundary operators**) $d_n : A_n \rightarrow A_{n-1}$, such that the composition of any two consecutive maps is zero: $d_n \circ d_{n+1} = 0$ for all n . They are usually written out as:

$$\dots \rightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} A_{n-2} \rightarrow \dots \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A_{-1} \xrightarrow{d_{-1}} A_{-2} \xrightarrow{d_{-2}} \dots$$

A variant on the concept of chain complex is that of *cochain complex*. A **cochain complex** $(A^\bullet, d^\bullet)$ is a sequence of abelian groups or modules $A_{-2}, A_{-1}, A_0, A_1, A_2, \dots$ connected by homomorphisms $d_n : A_n \rightarrow A_{n+1}$, such that the composition of any two consecutive maps is zero: $d_{n+1} \circ d_n = 0$ for all n :

$$\dots \rightarrow A_{-2} \xrightarrow{d_{-2}} A_{-1} \xrightarrow{d_{-1}} A_0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} A_2 \rightarrow \dots \rightarrow A_{n-1} \xrightarrow{d_{n-1}} A_n \xrightarrow{d_n} A_{n+1} \rightarrow \dots$$

The idea is basically the same. In either case, the index i in A_i is referred to as the **degree**.

A **bounded chain complex** is one in which almost all the A_i are 0; i.e., a finite complex extended to the left and right by 0's. An example is the complex defining the homology theory of a (finite) simplicial complex. A chain complex is **bounded above** if all degrees above some fixed degree N are 0, and is **bounded below** if all degrees below some fixed degree are 0. Clearly, a complex is bounded above and below iff the complex is bounded.

Fundamental terminology

Leaving out the indices, the basic relation on d can be thought of as

$$d^2 = 0.$$

The elements of the individual groups of a chain complex are called **chains** (or **cochains** in the case of a cochain complex.) The image of d is the group of **boundaries**, or in a cochain complex, **coboundaries**. The kernel of d (i.e., the subgroup sent to 0 by d) is the group of **cycles**, or in the case of a cochain complex, **cocycles**. From the basic relation, the (co)boundaries lie inside the (co)cycles. This phenomenon is studied in a systematic way using (co)homology groups.

Examples

Singular homology

Suppose we are given a topological space X .

Define $C_n(X)$ for natural n to be the free abelian group formally generated by singular n -simplices in X , and define the boundary map

$$\partial_n : C_n(X) \rightarrow C_{n-1}(X) : (\sigma : [v_0, \dots, v_n] \rightarrow X) \mapsto (\partial_n \sigma = \sum_{i=0}^n (-1)^i \sigma([v_0, \dots, \hat{v}_i, \dots, v_n]),$$

where the hat denotes the omission of a vertex. That is, the boundary of a singular simplex is alternating sum of restrictions to its faces. It can be shown $\partial^2 = 0$, so $(C_\bullet, \partial_\bullet)$ is a chain complex; the *singular homology* $H_\bullet(X)$ is the homology of this complex; that is,

$$H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

de Rham cohomology

The differential k -forms on any smooth manifold M form an abelian group (in fact an $\mathbf{R}$ -vector space) called $\Omega^k(M)$ under addition. The exterior derivative d_k maps $\Omega^k(M)$ to $\Omega^{k+1}(M)$, and $d^2 = 0$ follows essentially from symmetry of second derivatives, so the vector spaces of k -forms along with the exterior derivative are a cochain complex:

$$\Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \rightarrow \Omega^2(M) \rightarrow \Omega^3(M) \rightarrow \dots$$

The homology of this complex is the *de Rham cohomology*

$$H_{\text{DR}}^0(M, F) = \ker d_0 = \{\text{locally constant functions on } M \text{ with values in } F\} / \#\{\text{connected pieces of } M\} \cong F$$

$$H_{\text{DR}}^k(M) = \ker d_k / \operatorname{im} d_{k-1}.$$

Chain maps

A *chain map* f between two chain complexes $(A_\bullet, d_{A,\bullet})$ and $(B_\bullet, d_{B,\bullet})$ is a collection of module homomorphisms $f_n : A_n \rightarrow B_n$ for each n that intertwines with the differentials on the two chain complexes: $d_{B,n} \circ f_n = f_{n-1} \circ d_{A,n}$. Such a map sends cycles to cycles and boundaries to boundaries, and thus descends to a map on homology: $(f_n)_* : H_\bullet(A_\bullet, d_{A,\bullet}) \rightarrow H_\bullet(B_\bullet, d_{B,\bullet})$.

A continuous map of topological spaces induces chain maps in both the singular and de Rham chain complexes described above (and in general for the chain complex defining any homology theory of topological spaces) and thus a continuous map induces a map on homology. Because the map induced on a composition of maps is the composition of the induced maps, these homology theories are functors from the category of topological spaces with continuous maps to the category of abelian groups with group homomorphisms.

Chain homotopy

Chain homotopies give an important equivalence relation between chain maps. Chain homotopic chain maps induce the same maps on homology groups. A particular case is that homotopic maps between two spaces X and Y induce the same maps from homology of X to homology of Y . Chain homotopies have a geometric interpretation; it is described, for example, in the book of Bott and Tu. See Homotopy category of chain complexes for further information.

See also

- Homology
- Differential graded algebra

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Crystalline cohomology

In mathematics, **crystalline cohomology** is a Weil cohomology theory for schemes invented by Grothendieck (in his letter to Tate (Grothendieck 1966) and his lecture (Grothendieck 1968)) and developed by Pierre Berthelot (1974). Its values are modules over rings of Witt vectors over the base field.

Crystalline cohomology is closely related to the (algebraic) **de Rham cohomology** introduced by Grothendieck (1963). Roughly speaking, crystalline cohomology of a variety X in characteristic p is the de Rham cohomology of a smooth lift of X to characteristic 0, while de Rham cohomology of X is the crystalline cohomology reduced mod p (after taking into account higher *Tors*).

The idea of crystalline cohomology, roughly, is to replace the Zariski open sets of a scheme by infinitesimal thickenings of Zariski open sets with divided power structures. The motivation for this is that it can then be calculated by taking a local lifting of a scheme from characteristic p to characteristic 0 and employing an appropriate version of algebraic de Rham cohomology.

Applications

For schemes in characteristic p , crystalline cohomology theory can handle questions about p -torsion in cohomology groups better than p -adic étale cohomology. This makes it a natural backdrop for much of the work on p -adic L-functions.

Crystalline cohomology, from the point of view of number theory, fills a gap in the l -adic cohomology information, which occurs exactly where there are 'equal characteristic primes'. Traditionally the preserve of ramification theory, crystalline cohomology converts this situation into Dieudonné module theory, giving an important handle on arithmetic problems. Conjectures with wide scope on making this into formal statements were enunciated by Jean-Marc Fontaine, the resolution of which is called p -adic Hodge theory.

de Rham cohomology

De Rham cohomology solves the problem of finding an algebraic definition of the cohomology groups (singular cohomology)

$$H^i(X, \mathbf{C})$$

for X a smooth complex variety. These groups are the cohomology of the complex of smooth differential forms on X (with complex number coefficients), as these form a resolution of the constant sheaf $\mathbf{C}$.

The algebraic de Rham cohomology is defined to be the hypercohomology of the complex of algebraic forms (Kähler differentials) on X . The smooth i -forms form an acyclic sheaf, so the hypercohomology of the complex of smooth forms is the same as its cohomology, and the same is true for algebraic sheaves of i -forms over affine varieties, but algebraic sheaves of i -forms over non-affine varieties can have non-vanishing higher cohomology groups, so the hypercohomology can differ from the cohomology of the complex.

For smooth complex varieties Grothendieck (1963) showed that the algebraic de Rham cohomology is isomorphic to the usual smooth de Rham cohomology and therefore (by de Rham's theorem) to the cohomology with complex coefficients. This definition of algebraic de Rham cohomology is available for algebraic varieties over any field k .

Coefficients

If X is a variety over an algebraically closed field of characteristic $p > 0$, then the l -adic cohomology groups for l any prime number other than p give satisfactory cohomology groups of X , with coefficients in the ring $\mathbf{Z}_l$ of l -adic integers. It is not possible in general to find similar cohomology groups with coefficients in the p -adic numbers (or the rationals, or the integers).

The classic reason (due to Serre) is that if X is a supersingular elliptic curve, then its ring of endomorphisms generates a quaternion algebra over $\mathbf{Q}$ that is non-split at p and infinity. If X has a cohomology group over the p -adic integers with the expected dimension 2, the ring of endomorphisms would have a 2-dimensional representation; and this is not possible as it is non-split at p . (A quite subtle point is that if X is a supersingular elliptic curve over the prime field, with p elements, then its crystalline cohomology is a free rank 2 module over the p -adic integers. The argument given does not apply in this case, because some of the endomorphisms of supersingular elliptic curves are only defined over a quadratic extension of the field of order p .)

Grothendieck's crystalline cohomology theory gets around this obstruction because it takes values in the ring of Witt vectors over the ground field. So if the ground field is the algebraic closure of the field of order p , its values are modules over the p -adic completion of the maximal unramified extension of the p -adic integers, a much larger ring containing n -th roots of unity for all n not divisible by p , rather than over the p -adic integers.

Motivation

One idea for defining a Weil cohomology theory of a variety X over a field k of characteristic p is to 'lift' it to a variety X^* over the ring of Witt vectors of k (that gives back X on reduction mod p), then take the de Rham cohomology of this lift. The problem is that it is not at all obvious that this cohomology is independent of the choice of lifting.

The idea of crystalline cohomology in characteristic 0 is to find a direct definition of a cohomology theory as the cohomology of constant sheaves on a suitable site

$$\mathrm{Inf}(X)$$

over X , called the **infinitesimal site** and then show it is the same as the de Rham cohomology of any lift.

The site $\mathrm{Inf}(X)$ is a category whose objects can be thought of as some sort of generalization of the conventional open sets of X . In characteristic 0 its objects are infinitesimal thickenings $U \rightarrow T$ of Zariski open subsets U of X . This means that U is the closed subscheme of a scheme T defined by a nilpotent sheaf of ideals on T ; for example, $\mathrm{Spec}(k) \rightarrow \mathrm{Spec}(k[x]/(x^2))$.

Grothendieck showed that for smooth schemes X over $\mathbf{C}$, the cohomology of the sheaf O_X on $\mathrm{Inf}(X)$ is the same as the usual (smooth or algebraic) de Rham cohomology.

Crystalline cohomology

In characteristic p the most obvious analogue of the crystalline site defined above in characteristic 0 does not work. The reason is roughly that in order to prove exactness of the de Rham complex, one needs some sort of Poincaré lemma, whose proof in turn uses integration, and integration requires various divided powers, which exist in characteristic 0 but not always in characteristic p . Grothendieck solved this problem by defining objects of the crystalline site of X to be roughly infinitesimal thickenings of Zariski open subsets of X , together with a divided power structure giving the needed divided powers.

We will work over the ring $W_n = W/p^n W$ of Witt vectors of length n over a perfect field k of characteristic $p > 0$. For example, k could be the finite field of order p , and W_n is then the ring $\mathbf{Z}/p^n \mathbf{Z}$. (More generally one can work over a base scheme S which has a fixed sheaf of ideals I with a divided power structure.) If X is a scheme over k , then the **crystalline site of X relative to W_n** , denoted $\mathrm{Cris}(X/W_n)$, has as its objects pairs $U \rightarrow T$ consisting of a closed immersion of a Zariski open subset U of X into some W_n -scheme T defined by a sheaf of ideals J , together with a divided power structure on J compatible with the one on W_n .

Crystalline cohomology of a scheme X over k is defined to be the inverse limit

$$H^i(X/W) = \varprojlim H^i(X/W_n)$$

where

$$H^i(X/W_n) = H^i(\mathrm{Cris}(X/W_n), O)$$

is the cohomology of the crystalline site of X/W_n with values in the sheaf of rings $O = O_{X/W_n}$.

A key point of the theory is that the crystalline cohomology of a smooth scheme X over k can often be calculated in terms of the algebraic de Rham cohomology of a proper and smooth lifting of X to a scheme Z over W . There is a canonical isomorphism

$$H^i(X/W) = H_{DR}^i(Z/W) \quad (= H^i(Z, \Omega_{Z/W}^*) = \varprojlim H^i(Z, \Omega_{Z/W_n}^*))$$

of the crystalline cohomology of X with the de Rham cohomology of Z over the formal scheme of W (an inverse limit of the hypercohomology of the complexes of differential forms). Conversely the de Rham cohomology of X can be recovered as the reduction mod p of its crystalline cohomology (after taking higher *Tors* into account).

Crystals

If X is a scheme over S then the sheaf $O_{X/S}$ is defined by $O_{X/S}(T) =$ coordinate ring of T , where we write T as an abbreviation for an object $U \rightarrow T$ of $\text{Cris}(X/S)$.

A **crystal** on the site $\text{Cris}(X/S)$ is a sheaf F of $O_{X/S}$ modules that is **rigid** in the following sense:

for any map f between objects T, T' of $\text{Cris}(X/S)$, the natural map from $f^*F(T)$ to $F(T')$ is an isomorphism.

This is similar to the definition of a quasicohherent sheaf of modules in the Zariski topology.

An example of a crystal is the sheaf $O_{X/S}$.

The term *crystal* attached to the theory, explained in Grothendieck's letter to Tate (1966), was a metaphor inspired by certain properties of algebraic differential equations. These had played a role in p -adic cohomology theories (precursors of the crystalline theory, introduced in various forms by Dwork, Monsky, Washnitzer, Lubkin and Katz) particularly in Dwork's work. Such differential equations can be formulated easily enough by means of the algebraic Koszul connections, but in the p -adic theory the analogue of analytic continuation is more mysterious (since p -adic discs tend to be disjoint rather than overlap). By decree, a *crystal* would have the 'rigidity' and the 'propagation' notable in the case of the analytic continuation of complex analytic functions. (Cf. also the rigid analytic spaces introduced by Tate, in the 1960s, when these matters were actively being debated.)

See also

- Motivic cohomology
- De Rham cohomology

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Homological algebra

Homological algebra is the branch of mathematics which studies homology in a general algebraic setting. It is a relatively young discipline, whose origins can be traced to investigations in combinatorial topology (a precursor to algebraic topology) and abstract algebra (theory of modules and syzygies) at the end of the 19th century, chiefly by Henri Poincaré and David Hilbert.

The development of homological algebra was closely intertwined with the emergence of category theory. By and large, homological algebra is the study of homological functors and the intricate algebraic structures that they entail. The hidden fabric of mathematics is woven of **chain complexes**, which manifest themselves through their homology and cohomology. Homological algebra affords the means to extract information contained in these complexes and present it in the form of homological invariants of rings, modules, topological spaces, and other 'tangible' mathematical objects. A powerful tool for doing this is provided by spectral sequences.

From its very origins, homological algebra has played an enormous role in algebraic topology. Its sphere of influence has gradually expanded and presently includes commutative algebra, algebraic geometry, algebraic number theory, representation theory, mathematical physics, operator algebras, complex analysis, and the theory of partial differential equations. K-theory is an independent discipline which draws upon methods of homological algebra, as does the noncommutative geometry of Alain Connes.

Chain complexes and homology

The **chain complex** is the central notion of homological algebra. It is a sequence $(C_\bullet, d_\bullet)$ of abelian groups and group homomorphisms, with the property that the composition of any two consecutive maps is zero:

$$C_\bullet : \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots, \quad d_n \circ d_{n+1} = 0.$$

The elements of C_n are called **n -chains** and the homomorphisms d_n are called the **boundary maps** or **differentials**. The **chain groups** C_n may be endowed with extra structure; for example, they may be vector spaces or modules over a fixed ring R . The differentials must preserve the extra structure if it exists; for example, they must be linear maps or homomorphisms of R -modules. For notational convenience, restrict attention to abelian groups (more correctly, to the category **Ab** of abelian groups); a celebrated theorem by Barry Mitchell implies the results will generalize to any abelian category. Every chain complex defines two further sequences of abelian groups, the **cycles** $Z_n = \text{Ker } d_n$ and the **boundaries** $B_n = \text{Im } d_{n+1}$, where $\text{Ker } d$ and $\text{Im } d$ denote the kernel and the image of d . Since the composition of two consecutive boundary maps is zero, these groups are embedded into each other as

$$B_n \subseteq Z_n \subseteq C_n.$$

Subgroups of abelian groups are automatically normal; therefore we can define the **n th homology group** $H_n(C)$ as the factor group of the n -cycles by the n -boundaries,

$$H_n(C) = Z_n / B_n = \text{Ker } d_n / \text{Im } d_{n+1}.$$

A chain complex is called **acyclic** or an **exact sequence** if all its homology groups are zero.

Chain complexes arise in abundance in algebra and algebraic topology. For example, if X is a topological space then the singular chains $C_n(X)$ are formal linear combinations of continuous maps from the standard n -simplex into X ; if K is a simplicial complex then the simplicial chains $C_n(K)$ are formal linear combinations of the n -simplices of K ; if $A = F/R$ is a presentation of an abelian group A by generators and relations, where F is a free abelian group spanned by the generators and R is the subgroup of relations, then letting $C_1(A) = R$, $C_0(A) = F$, and $C_n(A) = 0$ for all other n defines a sequence of abelian groups. In all these cases, there are natural differentials d_n making C_n into a chain complex, whose homology reflects the structure of the topological space X , the simplicial complex K , or the abelian group A . In the case of topological spaces, we arrive at the notion of singular homology, which plays a fundamental role in investigating the properties of such spaces, for example, manifolds.

On a philosophical level, homological algebra teaches us that certain chain complexes associated with algebraic or geometric objects (topological spaces, simplicial complexes, R -modules) contain a lot of valuable algebraic information about them, with the homology being only the most readily available part. On a technical level, homological algebra provides the tools for manipulating complexes and extracting this information. Here are two general illustrations.

- Two objects X and Y are connected by a map f between them. Homological algebra studies the relation, induced by the map f , between chain complexes associated to X and Y and their homology. This is generalized to the case of several objects and maps connecting them. Phrased in the language of category theory, homological algebra studies the functorial properties of various constructions of chain complexes and of the homology of these complexes.

- An object X admits multiple descriptions (for example, as a topological space and as a simplicial complex) or the complex $C_\bullet(X)$ is constructed using some 'presentation' of X , which involves non-canonical choices. It is important to know the effect of change in the description of X on chain complexes associated to X . Typically, the complex and its homology $H_\bullet(C)$ are functorial with respect to the presentation; and the homology (although not the complex itself) is actually independent of the presentation chosen, thus it is an invariant of X .

Functoriality

A continuous map of topological spaces gives rise to a homomorphism between their n th homology groups for all n . This basic fact of algebraic topology finds a natural explanation through certain properties of chain complexes. Since it is very common to study several topological spaces simultaneously, in homological algebra one is led to simultaneous consideration of multiple chain complexes.

A **morphism** between two chain complexes, $F : C_\bullet \rightarrow D_\bullet$, is a family of homomorphisms of abelian groups $F_n : C_n \rightarrow D_n$ that commute with the differentials, in the sense that $F_{n-1} \cdot d_n^C = d_n^D \cdot F_n$ for all n . A morphism of chain complexes induces a morphism $H_\bullet(F)$ of their homology groups, consisting of the homomorphisms $H_n(F) : H_n(C) \rightarrow H_n(D)$ for all n . A morphism F is called a **quasi-isomorphism** if it induces an isomorphism on the n th homology for all n .

Many constructions of chain complexes arising in algebra and geometry, including singular homology, have the following functoriality property: if two objects X and Y are connected by a map f , then the associated chain complexes are connected by a morphism $F = C(f)$ from $C_\bullet(X)$ to $C_\bullet(Y)$, and moreover, the composition $g \cdot f$ of maps $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ induces the morphism $C(g \cdot f)$ from $C_\bullet(X)$ to $C_\bullet(Z)$ that coincides with the composition $C(g) \cdot C(f)$. It follows that the homology groups $H_\bullet(C)$ are functorial as well, so that morphisms between algebraic or topological objects give rise to compatible maps between their homology.

The following definition arises from a typical situation in algebra and topology. A triple consisting of three chain complexes $L_\bullet, M_\bullet, N_\bullet$ and two morphisms between them, $f : L_\bullet \rightarrow M_\bullet, g : M_\bullet \rightarrow N_\bullet$, is called an **exact triple**, or a **short exact sequence of complexes**, and written as

$$0 \longrightarrow L_\bullet \xrightarrow{f} M_\bullet \xrightarrow{g} N_\bullet \longrightarrow 0,$$

if for any n , the sequence

$$0 \longrightarrow L_n \xrightarrow{f_n} M_n \xrightarrow{g_n} N_n \longrightarrow 0$$

is a short exact sequence of abelian groups. By definition, this means that f_n is an injection, g_n is a surjection, and $\text{Im } f_n = \text{Ker } g_n$. One of the most basic theorems of homological algebra, sometimes known as the zig-zag lemma, states that, in this case, there is a **long exact sequence in homology**

$$\dots \longrightarrow H_n(L) \xrightarrow{H_n(f)} H_n(M) \xrightarrow{H_n(g)} H_n(N) \xrightarrow{\delta_n} H_{n-1}(L) \xrightarrow{H_{n-1}(f)} H_{n-1}(M) \longrightarrow \dots,$$

where the homology groups of L , M , and N cyclically follow each other, and δ_n are certain homomorphisms determined by f and g , called the **connecting homomorphisms**. Topological manifestations of this theorem include the Mayer-Vietoris sequence and the long exact sequence for relative homology.

Foundational aspects

Cohomology theories have been defined for many different objects such as topological spaces, sheaves, groups, rings, Lie algebras, and C^* -algebras. The study of modern algebraic geometry would be almost unthinkable without sheaf cohomology.

Central to homological algebra is the notion of exact sequence; these can be used to perform actual calculations. A classical tool of homological algebra is that of derived functor; the most basic examples are functors Ext and Tor .

With a diverse set of applications in mind, it was natural to try to put the whole subject on a uniform basis. There were several attempts before the subject settled down. An approximate history can be stated as follows:

- Cartan-Eilenberg: In their 1956 book "Homological Algebra", these authors used projective and injective module resolutions.
- 'Tohoku': The approach in a celebrated paper by Alexander Grothendieck which appeared in the Second Series of the Tohoku Mathematical Journal in 1957, using the abelian category concept (to include sheaves of abelian groups).
- The derived category of Grothendieck and Verdier. Derived categories date back to Verdier's 1967 thesis. They are examples of triangulated categories used in a number of modern theories.

These move from computability to generality.

The computational sledgehammer *par excellence* is the spectral sequence; these are essential in the Cartan-Eilenberg and Tohoku approaches where they are needed, for instance, to compute the derived functors of a composition of two functors. Spectral sequences are less essential in the derived category approach, but still play a role whenever concrete computations are necessary.

There have been attempts at 'non-commutative' theories which extend first cohomology as *torsors* (important in Galois cohomology).

See also

- Homotopical algebra

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Algebraic K-theory

In mathematics, **algebraic K-theory** is an advanced part of homological algebra concerned with defining and applying a sequence

$$K_n(R)$$

of functors from rings to abelian groups, for all integers n . For historical reasons, the **lower K-groups** K_0 and K_1 are thought of in somewhat different terms from the **higher algebraic K-groups** K_n for $n \geq 2$. Indeed, the lower groups are more accessible, and have more applications, than the higher groups. The theory of the higher K-groups is noticeably deeper, and certainly much harder to compute (even when R is the ring of integers).

The group $K_0(R)$ generalises the construction of the ideal class group of a ring, using projective modules. Its development in the 1960s and 1970s was linked to attempts to solve a conjecture of Serre on projective modules that now is the Quillen-Suslin theorem; numerous other connections with classical algebraic problems were found in this era. Similarly, $K_1(R)$ is a modification of the group of units in a ring, using elementary matrix theory. The group $K_1(R)$ is important in topology, especially when R is a group ring, because its quotient the Whitehead group contains the Whitehead torsion used to study problems in simple homotopy theory and surgery theory; the group $K_0(R)$ also contains other invariants such as the finiteness invariant. Since the 1980s, algebraic K-theory has increasingly had applications to algebraic geometry. For example, motivic cohomology is closely related to algebraic K-theory.

History

Alexander Grothendieck invented K-theory in the mid-1950s as a framework to state his far-reaching generalization of the Riemann-Roch theorem. Within a few years, its topological counterpart was considered by Atiyah and Hirzebruch and is now known as topological K-theory.

Applications of K-groups were found from 1960 onwards in surgery theory for manifolds, in particular; and numerous other connections with classical algebraic problems were brought out.

A little later a branch of the theory for operator algebras was fruitfully developed, resulting in operator K-theory and KK-theory. It also became clear that K-theory could play a role in algebraic cycle theory in algebraic geometry (Gersten's conjecture): here the *higher*

K -groups become connected with the *higher codimension* phenomena, which are exactly those that are harder to access. The problem was that the definitions were lacking (or, too many and not obviously consistent). A definition of K_2 for fields by John Milnor, for example, gave an attractive theory that was too limited in scope, constructed as a quotient of the multiplicative group of the field tensored with itself, with some explicit relations imposed; and closely connected with central extensions.

Eventually the foundational difficulties were resolved (leaving a deep and difficult theory) by Daniel Quillen, who gave several definitions of higher algebraic K -theory, via the $+$ -construction and the Q -construction.

Lower K -groups

The lower K -groups were discovered first, and given various ad hoc descriptions, which remain useful. Throughout, let A be a ring.

K_0

The 0th K -group is related to dimension and the Picard group.

The (covariant) functor K_0 goes from the category of rings to the category of groups, taking A to the Grothendieck group of the set of isomorphism classes of its finitely generated projective modules, regarded as a monoid under direct sum.

(Projective) modules over a field k are vector spaces and $K_0(k)$ is isomorphic to $\mathbf{Z}$, by dimension. For A a Dedekind ring,

$$K_0(A) = \text{Pic}A \times \mathbf{Z},$$

where $\text{Pic}(A)$ is the Picard group of A , and similarly the reduced K -theory is given by

$$\tilde{K}_0(A) = \text{Pic}A.$$

K_1

Hyman Bass provided this definition, which generalizes the group of units of a field: $K_1(A)$ is the abelianization of the infinite general linear group:

$$K_1(A) = \text{GL}(A)^{\text{ab}} = \text{GL}(A)/[\text{GL}(A), \text{GL}(A)]$$

Here

$$\text{GL}(A) = \text{colim} \text{GL}_n(A)$$

is the direct limit of the GL_n , which embeds in GL_{n+1} as the upper left block matrix, and the commutator subgroup agrees with the group generated by elementary matrices $E(A) = [\text{GL}(A), \text{GL}(A)]$, by Whitehead's lemma. Indeed, the group $\text{GL}(A)/E(A)$ was first defined and studied by Whitehead,^[1] and is called the **Whitehead group** of the ring^[2] A .

As $E(A) \triangleleft \text{SL}(A)$, one can also define the **special Whitehead group** $SK_1(A) := \text{SL}(A)/E(A)$.

Commutative rings and fields

For A a commutative ring, one can define a determinant $\det: \mathrm{GL}(A) \rightarrow A^*$ to the group of units of A , which vanishes on $E(A)$ and thus descends to a map $\det: K_1(A) \rightarrow A^*$. This map splits via the map $A^* \xrightarrow{\sim} \mathrm{GL}_1(A) \rightarrow K_1(A)$ (unit in the upper left corner), and hence is onto, and has the special Whitehead group as kernel, yielding the split short exact sequence:

$$1 \rightarrow SK_1(A) \rightarrow K_1(A) \rightarrow A^* \rightarrow 1,$$

which is a quotient of the usual split short exact sequence defining the special linear group, namely

$$1 \rightarrow \mathrm{SL}(A) \rightarrow \mathrm{GL}(A) \rightarrow A^* \rightarrow 1.$$

Thus, since the groups in question are abelian, $K_1(A)$ splits as the direct sum of the group of units and the special Whitehead group: $K_1(A) \approx A^* \oplus SK_1(A)$.

When A is a Dedekind domain (e.g. a field, or the ring of algebraic integers in an algebraic number field), $SK_1(A)$ vanishes, and the determinant map is an isomorphism. In particular, $\det: K_1(F) \xrightarrow{\sim} F^*$.

For a non-commutative ring, the determinant cannot be defined, but the map $\mathrm{GL}(A) \rightarrow K_1(A)$ generalizes the determinant.

K_2

John Milnor found the right definition of K_2 : it is the center of the Steinberg group $\mathrm{St}(A)$ of A .

It can also be defined as the kernel of the map

$$\varphi: \mathrm{St}(A) \rightarrow \mathrm{GL}(A),$$

or as the Schur multiplier of the group of elementary matrices.

For a field k one has

$$K_2(k) = k^\times \otimes_{\mathbf{Z}} k^\times / \langle a \otimes (1 - a) \mid a \neq 0, 1 \rangle.$$

Milnor K-theory

The above expression for K_2 of a field k led Milnor to the following definition of "higher" K-groups by

$$K_*^M(k) := T^* k^\times / (a \otimes (1 - a)),$$

thus as graded parts of a quotient of the tensor algebra of the multiplicative group $k^\times$ by the two-sided ideal, generated by the

$$a \otimes (1 - a)$$

for $a \neq 0, 1$. For $n = 0, 1, 2$ these coincide with those above, but for $n \geq 3$ they differ in general. For example, we have $K_n^M(\mathbb{F}_q) = 0$ for $n \geq 3$.

Higher K-theory

The master, definitive definitions of K-theory were given by Daniel Quillen, after an extended period in which uncertainty had reigned.

The +-construction

One possible definition of higher algebraic K-theory of rings was given by Quillen

$$K_n(R) = \pi_n(BGL(R)^+),$$

a very compressed piece of abstract mathematics. Here π_k is a homotopy group, $GL(R)$ is the direct limit of the general linear groups over R for the size of the matrix tending to infinity, B is the classifying space construction of homotopy theory, and the $^+$ is Quillen's plus construction.

This definition only holds for $n > 0$ so one often defines the higher algebraic K-theory via

$$K_n(R) = \pi_n(BGL(R)^+ \times K_0(R))$$

Since $BGL(R)^+$ is path connected and $K_0(R)$ discrete, this definition doesn't differ in higher degrees and also holds for $n=0$.

The Q-construction

The Q-construction gives the same results as the +-construction, but it applies in more general situations. Moreover, the definition is more direct in the sense that the K-groups, defined via the Q-construction are functorial by definition. This fact is not automatic in the +-construction.

Suppose P is an exact category; associated to P a new category QP is defined, objects of which are those of P and morphisms from M to N are isomorphism classes of exact diagrams

$$M' \longleftarrow N \longrightarrow M''.$$

where the first arrow is an admissible epimorphism and the second arrow is an admissible monomorphism.

The i -th **K-group** of P is then defined as

$$K_i(P) = \pi_{i+1}(BQP, 0)$$

with a fixed zero-object 0, where BQ is the *classifying space* of Q , which is defined to be the geometric realisation of the *nerve* of Q .

This definition coincides with the above definitions of K_0 , K_1 and K_2 .

The K-groups $K_i(A)$ of the ring A are then the K-groups $K_i(P_A)$ where P_A is the category of finitely generated projective A -modules. More generally, for a scheme X , the higher K-groups of X are by definition the K-groups of (the exact category of) locally free coherent sheaves on X .

The following variant of this is also used: instead of finitely generated projective (=locally free) modules, take finitely generated modules. The resulting K-groups are usually called *G-groups*, or *higher G-theory*. When A is a noetherian regular ring, then G - and K -theory coincide. Indeed, the global dimension of regular local rings is finite, i.e. any finitely generated module has a finite projective resolution, so the canonical map $K_0 \rightarrow G_0$ is surjective. It is also injective, as can be shown. This isomorphism extends to the higher K-groups, too.

Examples

While the Quillen algebraic K-theory has provided deep insight into various aspects of algebraic geometry and topology, the K-groups have proved particularly difficult to compute except in a few isolated but interesting cases.

Algebraic K-groups of finite fields

The first and one of the most important calculations of the higher algebraic K-groups of a ring were made by Quillen himself for the case of finite fields:

Theorem. Let F be a finite field with q elements. Then

$$K_0(F) = \mathbb{Z}, \quad K_{2i}(F) = 0$$

for $i \neq 0$, and

$$K_{2i-1}(F) = \mu_{q^i-1} \text{ for } i = 1, 2, \dots$$

where μ_r denotes the cyclic group with r elements.

Algebraic K-groups of rings of integers

Quillen proved that if A is the ring of algebraic integers in an algebraic number field F (a finite extension of the rationals), then the algebraic K-groups of A are finitely generated. Borel used this to calculate $K_i(A)$ and $K_i(F)$ modulo torsion. For example, for the integers $\mathbb{Z}$, Borel proved that (modulo torsion)

for positive i unless $i = 4k + 1$ with k positive
and (modulo torsion)
for positive k .

The torsion subgroups of $K_{2i+1}(\mathbb{Z})$, and the orders of the finite groups $K_{4k+2}(\mathbb{Z})$ have recently been determined, but whether the latter groups are cyclic, and whether the groups $K_{4k}(\mathbb{Z})$ vanish depends upon Vandiver's conjecture about the class groups of cyclotomic integers.

Applications

Algebraic K-groups are used in conjectures on special values of L-functions and the formulation of a non-commutative main conjecture of Iwasawa theory and in construction of higher regulators.

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External links

- C. Weibel "The K-book: An introduction to algebraic K-theory (<http://www.math.rutgers.edu/~weibel/Kbook.html>)"

Topological K-theory

In mathematics, **topological K-theory** is a branch of algebraic topology. It was founded to study vector bundles on general topological spaces, by means of ideas now recognised as (general) K-theory that were introduced by Alexander Grothendieck. The early work on topological K-theory is due to Michael Atiyah and Friedrich Hirzebruch.

Definitions

Topological K-theory is a generalised cohomology theory of compact Hausdorff space category by classifying vector bundles over a space to stable equivalences (vector bundles are said to be stable equivalences if and only if isomorphic vector bundles are generated by Whitney sum of the vector bundles with a trivial vector bundle^[1]). Let X be a compact Hausdorff space and $k = \mathbb{R}$ or $k = \mathbb{C}$. Then $K_k(X)$ is the Grothendieck group of the commutative monoid whose elements are the isomorphism classes of finite dimensional k -vector bundles on X with the operation

$$[E] \oplus [F] = [E \oplus F]$$

for vector bundles E, F . Usually, $K_k(X)$ is denoted $KO(X)$ in real case and $KU(X)$ in the complex case.

More explicitly, **stable equivalence**, the equivalence relation on bundles E and F on X of defining the same element in $K(X)$, occurs when there is a trivial bundle G , so that

$$E \oplus G \cong F \oplus G.$$

Under the tensor product of vector bundles $K(X)$ then becomes a commutative ring.

The rank of a vector bundle carries over to the K-group define the homomorphism

$$K(X) \rightarrow \check{H}^0(X, \mathbb{Z})$$

where $\check{H}^0(X, \mathbb{Z})$ is the 0-group de Čech cohomology which is equal to group of locally constant functions with values in $\mathbb{Z}$.

If X has a distinguished basepoint x_0 , then the reduced K-group (cf. reduced homology) satisfies

$$K(X) \cong \tilde{K}(X) \oplus K(\{x_0\})$$

and is defined as either the kernel of $K(X) \rightarrow K(\{x_0\})$ (where $\{x_0\} \rightarrow X$ is basepoint inclusion) or the cokernel of $K(\{x_0\}) \rightarrow K(X)$ (where $X \rightarrow \{x_0\}$ is the constant map).

When X is a connected space, $\tilde{K}(X) \cong \text{Ker}(K(X) \rightarrow \check{H}^0(X, \mathbb{Z}) = \mathbb{Z})$.

The definition of functor K extends to category pairs of compact spaces (an object is a pair (X, Y) , X is compact and $Y \subset X$ is closed, a morphism between (X, Y) and (X', Y') is a continuous map $f : X \rightarrow X'$ such that $f(Y) \subset Y'$)

$$K(X, Y) := \tilde{K}(X/Y).$$

The reduced K-group is given by $x_0 = \{Y\}$.

The definition

$$K_{\mathbb{C}}^n(X, Y) = \tilde{K}_{\mathbb{C}}(S^{|n|}(X/Y))$$

gives the sequence of K-groups for $n \in \mathbb{Z}$, where S denotes the reduced suspension.

Properties

- K^n is a contravariant functor.
- The classifying space of $\tilde{K}$ is BO_k (BO , in real case; BU in complex case), i.e. $\tilde{K}_k(X) \cong [X, BO_k]$.
- The classifying space of K is $\mathbb{Z} \times BO_k$ ($\mathbb{Z}$ with discrete topology), i.e. $K_k(X) \cong [X, \mathbb{Z} \times BO_k]$.
- There is a natural ring homomorphism $K^*(X) \rightarrow H^{2*}(X, \mathbb{Q})$, the Chern character, such that $K^*(X) \otimes \mathbb{Q} \rightarrow H^{2*}(X, \mathbb{Q})$ is an isomorphism.
- Topological K-theory can be generalized vastly to a functor on C*-algebras, see operator K-theory and KK-theory.

Bott periodicity

The phenomenon of periodicity named for Raoul Bott (see Bott periodicity theorem) can be formulated this way:

- $K(X \times S^2) = K(X) \otimes K(S^2)$, and $K(S^2) = \mathbb{Z}[H]/(H-1)^2$; where H is the class of the tautological bundle on the $S^2 = \mathbb{C}P^1$, i.e. the Riemann sphere as complex projective line.
- $\tilde{K}^{n+2}(X) = \tilde{K}^n(X)$.
- $\Omega^2 BU \simeq BU \times \mathbb{Z}$.

In real K-theory there is a similar periodicity, but *modulo* 8.

Notes

[1] <http://mathworld.wolfram.com/StableEquivalence.html>

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Stack (descent theory)

In mathematics a **stack** is a concept used to formalise some of the main constructions of descent theory.

Descent theory is concerned with generalisations of situations where geometrical objects (such as vector bundles on topological spaces) can be "glued together" when they are isomorphic (in a compatible way) when restricted to intersections of the sets in an open covering of a space. In more general set-up the restrictions are replaced with general pull-backs, and fibred categories form the right framework to discuss the possibility of such "glueing". The intuitive meaning of a stack is that it is a fibred category such that "all possible glueings work". The specification of glueings requires a definition of coverings with regard to which the glueings can be considered. It turns out that the general language for describing these coverings is that of a Grothendieck topology- Thus a stack is formally given as a fibred category over another *base* category, where the base has a Grothendieck topology and where the fibred category satisfies a few axioms that ensure existence and uniqueness of certain glueings with respect to the Grothendieck topology.

Archetypical examples include the stack of vector bundles on topological spaces, the stack of quasi-coherent sheaves on schemes (with respect to the fpqc-topology and weaker topologies) and the stack of affine schemes on a base scheme (again with respect to the fpqc topology or a weaker one).

Stacks are the underlying structure of algebraic stacks, which are a way to generalise schemes and algebraic spaces and which are particularly useful in studying moduli spaces. The concept of stacks has its origin in the definition of effective descent data in Grothendieck (1959). The theory was further developed by Grothendieck and Giraud (1964) and Giraud (1971); the name stack (*champ* in the original French) together with the eventual definition appears to have been introduced in the latter work.

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External links

- Stacks^[1] and descent^[2] on nLab.



^[3] This category theory-related article is a stub. You can help Wikipedia by expanding it.

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[1] <http://www.ncatlab.org/nlab/show/stack>

[2] <http://www.ncatlab.org/nlab/show/descent>

[3] http://en.wikipedia.org/w/index.php?stub&title=Stack_%28descent_theory%29&action=edit

Esquisse d'un Programme

"**Esquisse d'un Programme**" is a famous proposal for long-term mathematical research made by the German-born, French mathematician Alexander Grothendieck^[1]. He pursued the sequence of logically linked ideas in his important project proposal from 1984 until 1988, but his proposed research continues to date to be of major interest in several branches of advanced mathematics. Grothendieck's vision provides inspiration today for several developments in mathematics such as the extension and generalization of Galois theory, which is currently being extended based on his original proposal.

Brief history

Submitted in 1984, the *Esquisse d'un Programme*^[2] was a successful proposal submitted by Alexander Grothendieck for a position at the Centre National de la Recherche Scientifique, which Grothendieck held from 1984 till 1988.^[3] This proposal was however not formally published until 1997, because the author "could not be found, much less his permission requested".^[4] The dessins d'enfants, or "children's drawings" and "anabelian algebraic geometry" — non-Abelian algebraic topology and noncommutative geometry — that are contained in this long-term program, continue even today to inspire extensive mathematical studies.

Abstract of Grothendieck's programme

("Sommaire")

- 1. The Proposal and enterprise ("Envoi").
- 2. "Teichmüller's Lego-game and the Galois group of $\mathbb{Q}$ over $\mathbb{Q}$ " ("Un jeu de "Lego-Teichmüller" et le groupe de Galois de $\mathbb{Q}$ sur $\mathbb{Q}$ ").
- 3. Number fields associated with "dessin d'enfants". ("Corps de nombres associés à un dessin d'enfant").
- 4. Regular polyhedra over finite fields ("Polyèdres réguliers sur les corps finis").
- 5. General topology or a 'Moderated Topology' ("Haro sur la topologie dite 'générale', et réflexions heuristiques vers une topologie dite 'modérée').
- 6. Differentiable theories and moderated theories ("Théories différentiables" (à la Nash) et "théories modérées").
- 7. Pursuing Stacks ("À la Poursuite des Champs")^[5].
- 8. Two-dimensional geometry ("Digressions de géométrie bidimensionnelle"^[6].

- 9. Summary of proposed studies ("Bilan d'une activité enseignante").
- 10. Epilogue.
- Notes

Suggested further reading for the interested mathematical reader is provided in the *References* section.

"The Long March across the Theory of Galois"

"This manuscript, consisting of some nearly 800 hand-written double pages, dating from 1981, was left behind with Grothendieck's other unpublished manuscripts when he disappeared in 1991. Typed in Tex, it comes out to about 400 pages. It goes together with a further 1,000 pages or so of additional notes and sections which have not yet been read or typed. Many of the major themes were summarised in the 1983 manuscript *Esquisse d'un Programme*."

The Table of Contents for this important work by Alexander Grothendieck was originally compiled in French by the author and is reproduced here after the English Translation of the major parts of the Long March.

Table of Contents for the Long March across Galois Theory

1. Multi-Galois Toposes (topoi)
 2. Applications to topos coverings
 3. Pro-multi-Galois Variants Complements
 4. Introducing the arithmetic context; an 'anabelian' (non-Abelian) fundamental conjecture...
 10. Local analysis of Galois theory for reformulation of the conjecture (the necessary 'purgatorium'...)
 11. A taxonomic reflexion
 12. Tangential structure at (sections of second type extensions)
 13. Adjusting the hypotheses
 14. Conditions on the groupoid systems originating from geometric considerations (in the nonabelian case, the groupoid system can be expressed in terms of outer groups)
- Returning to the arithmetic case: the Galois-type formulation, p. 53
15. A cohomological digression, p.58
 16. Returning to the topological case: critical orbits
 17. Returning to the concept of cyclic group
 18. Application to the finite subgroups of (the discrete case, para.18)
 19. Tour of Teichmüller (spaces)
 20. Digression: the description of 2-isotopic categories of algebraic curves p.116
 21. Teichmüller spaces p.126
 22. Returning to the surfaces of (finite) groups of operators ('formulating the equations' of the problem)
 24. "Special" Teichmüller Groups
 25. The case of "two groups of operators"
-

- 26. Profinite Teichmüller Groups, connection with the modular Teichmüller topoi, conjecture
- 29. Critique of the previous approach
- 31. Digression: a finite group over a profinite cyclic group
- 32 Returning to the arithmetic aspects: a remarkable reconstruction of all of the étale topoi of a complete algebraic curve starting from an open nonabelian space...

Extensions of Galois's theory for groups: Galois groupoids, categories and functors

Galois has developed a powerful, fundamental algebraic theory in mathematics that provides very efficient computations for certain algebraic problems by utilizing the algebraic concept of groups, which is now known as the theory of Galois groups; such computations were not possible before, and also in many cases are much more effective than the 'direct' calculations without using groups^[7]. To begin with, Alexander Grothendieck stated in his proposal: *"Thus, the group of Galois is realized as the automorphism group of a concrete, pro-finite group which respects certain structures that are essential to this group."* This fundamental, Galois group theory in mathematics has been considerably expanded, at first to groupoids- as proposed in Alexander Grothendieck's *Esquisse d'un Programme (EdP)*- and now already partially carried out for groupoids; the latter are now further developed beyond groupoids to categories by several groups of mathematicians. Here, we shall focus only on the well-established and fully validated extensions of Galois' theory. Thus, EdP also proposed and anticipated, along previous Alexander Grothendieck's *IHÉS* seminars (SGA1 to SGA4) held in the 1960s, the development of even more powerful extensions of the original Galois's theory for groups by utilizing categories, functors and natural transformations, as well as further expansion of the manifold of ideas presented in Alexander Grothendieck's *Descent Theory*. The notion of motive has also been pursued actively^[8]. This was developed into the motivic Galois group, Grothendieck topology and Grothendieck category^[9]. Such developments were recently extended in algebraic topology *via* representable functors^[10] and the fundamental groupoid functor^[11].

See also

- Alexander Grothendieck
- Grothendieck's Galois theory
- Grothendieck group
- Grothendieck category^[12]
- Grothendieck inequality or Grothendieck constant
- Grothendieck-Katz p-curvature conjecture
- Grothendieck's relative point of view
- Grothendieck-Riemann-Roch theorem
- Grothendieck's Séminaire de géométrie algébrique
- Grothendieck space
- Grothendieck spectral sequence
- Grothendieck topology
- Grothendieck universe
- Tarski-Grothendieck set theory

- IHES
- IHES at Forty by Allyn Jackson ^[13]

Notes

- [1] Scharlau, Winifred (September 2008), written at Oberwolfach, Germany, "Who is Alexander Grothendieck", *Notices of the American Mathematical Society* (Providence, RI: American Mathematical Society) 55(8): 930-941, ISSN 1088-9477, OCLC 34550461, <http://www.ams.org/notices/200808/tx080800930p.pdf>
- [2] Alexander Grothendieck, 1984. "Esquisse d'un Programme", (1984 manuscript), finally published in Schneps and Lochak (1997, I), pp.5-48; English transl., *ibid.*, pp. 243-283. MR 99c:14034
- [3] Rehmeier, Julie (May 9, 2008), "Sensitivity to the Harmony of Things", *Science News*
- [4] Schneps and Lochak (1997, I) p.1
- [5] <http://www.bangor.ac.uk/r.brown/pstacks.htm>
- [6] Cartier, Pierre (2001), "A mad day's work: from Grothendieck to Connes and Kontsevich The evolution of concepts of space and symmetry", *Bull. Amer. Math. Soc.* **38**(4): 389-408, <<http://www.ams.org/bull/2001-38-04/S0273-0979-01-00913-2/S0273-0979-01-00913-2.pdf>>. An English translation of Cartier (1998)
- [7] Cartier, Pierre (1998), "La Folle Journée, de Grothendieck à Connes et Kontsevich — Évolution des Notions d'Espace et de Symétrie", *Les Relations entre les Mathématiques et la Physique Théorique — Festschrift for the 40th anniversary of the IHÉS, Institut des Hautes Études Scientifiques*, pp. 11-19
- [8] [http://en.wikipedia.org/wiki/Motive_\(algebraic_geometry\)](http://en.wikipedia.org/wiki/Motive_(algebraic_geometry))
- [9] <http://planetmath.org/encyclopedia/GrothendieckCategory.html>
- [10] <http://planetphysics.org/encyclopedia/RepresentableFunctor.html>
- [11] <http://planetphysics.org/encyclopedia/QuantumFundamentalGroupoid3.html>
- [12] <http://planetphysics.org/encyclopedia/GrothendieckCategory.html>
- [13] <http://www.ams.org/notices/199903/ihes-changes.pdf>

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- Alexander Grothendieck. 1971, Revêtements Étales et Groupe Fondamental (SGA1), chapter VI: *Catégories fibrées et descente*, Lecture Notes in Math. 224, Springer-Verlag: Berlin.
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 - Alexander Grothendieck and Jean Dieudonné.: 1960, *Éléments de géométrie algébrique.*, Publ. *Inst. des Hautes Études Scientifiques*, (IHÉS), **4**.
 - Alexander Grothendieck et al., 1971. Séminaire de Géométrie Algébrique du Bois-Marie, Vol. 1-7, Berlin: Springer-Verlag.
 - Alexander Grothendieck. 1962. *Séminaires en Géométrie Algébrique du Bois-Marie*, Vol. **2** - Cohomologie Locale des Faisceaux Cohérents et Théorèmes de Lefschetz Locaux et Globaux. , pp.287. (with an additional contributed exposé by Mme. Michele Raynaud). (Typewritten manuscript available in French; see also a brief summary in English)
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 - J. L. Verdier. 1965. Algèbre homologiques et Catégories dérivées. North Holland Publ. Cie).
 - Alexander Grothendieck. 1957, Sur quelques points d'algèbre homologique, *Tohoku Mathematics Journal*, **9**, 119-221.

- Alexander Grothendieck et al. Séminaires en Géométrie Algébrique- 4, Tome 1, Exposé 1 (or the Appendix to Exposé 1, by 'N. Bourbaki' for more detail and a large number of results. AG4 is freely available in French; also available is an extensive Abstract in English.
- Alexander Grothendieck, 1984. "Esquisse d'un Programme" (<http://people.math.jussieu.fr/~leila/grothendieckcircle/EsquisseFr.pdf>), (1984 manuscript), finally published in "Geometric Galois Actions", L. Schneps, P. Lochak, eds., *London Math. Soc. Lecture Notes* **242**, Cambridge University Press, 1997, pp.5-48; English transl., *ibid.*, pp. 243-283. MR 99c:14034 .
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- Schneps, Leila (1994), *The Grothendieck Theory of Dessins d'Enfants*, London Mathematical Society Lecture Note Series, Cambridge University Press.
- Schneps, Leila; Lochak, Pierre, eds. (1997), *Geometric Galois Actions I: Around Grothendieck's Esquisse D'un Programme*, London Mathematical Society Lecture Note Series, **242**, Cambridge University Press, ISBN 978-0-521-59642-8
- Schneps, Leila; Lochak, Pierre, eds. (1997), *Geometric Galois Actions II: The Inverse Galois Problem, Moduli Spaces and Mapping Class Groups*, London Mathematical Society Lecture Note Series, **243**, Cambridge University Press, ISBN 978-0521596411
- Harbater, David; Schneps, Leila (2000), "Fundamental groups of moduli and the Grothendieck-Teichmüller group", *Trans. Amer. Math. Soc.* **352**: 3117–3148, doi: 10.1090/S0002-9947-00-02347-3 (<http://dx.doi.org/10.1090/S0002-9947-00-02347-3>).

External links

- Fundamental Groupoid Functors (<http://planetphysics.org/encyclopedia/QuantumFundamentalGroupoid3.html>), Planet Physics.

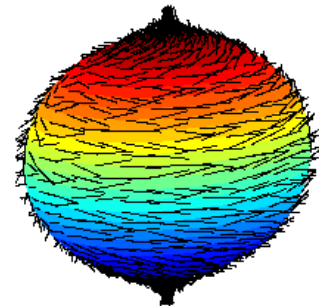
Important Theorems in Algebraic Topology

Hairy ball theorem

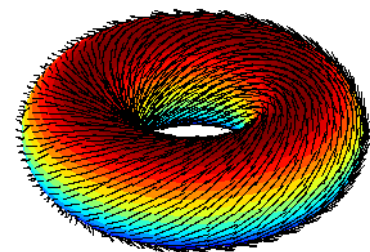
The **hairy ball theorem** of algebraic topology states that there is no nonvanishing continuous tangent vector field on the sphere. If f is a continuous function that assigns a vector in $\mathbf{R}^3$ to every point p on a sphere such that $f(p)$ is always tangent to the sphere at p , then there is at least one p such that $f(p) = \mathbf{0}$. The theorem was first stated by Henri Poincaré in the late 19th century.

This is famously stated as "*you can't comb a hairy ball flat without creating a cowlick*", or sometimes, "*you can't comb the hair on a coconut*". It was first proved in 1912 by Brouwer^[1].

In fact from a more advanced point of view it can be shown that the sum at the zeros of such a vector field of a certain 'index' must be 2, the Euler characteristic of the 2-sphere; and that therefore there must be at least some zero. This is a consequence of the Poincaré-Hopf theorem. In the case of the torus, the Euler characteristic is 0; and it *is* possible to 'comb a hairy doughnut flat'. In this regard, it follows that for any compact regular 2-dimensional manifold with non-zero Euler characteristic, any continuous tangent vector field has at least one zero.



A failed attempt to comb a hairy ball flat, leaving an uncombable tuft at each pole.



A hairy doughnut, on the other hand, is quite easily combable.

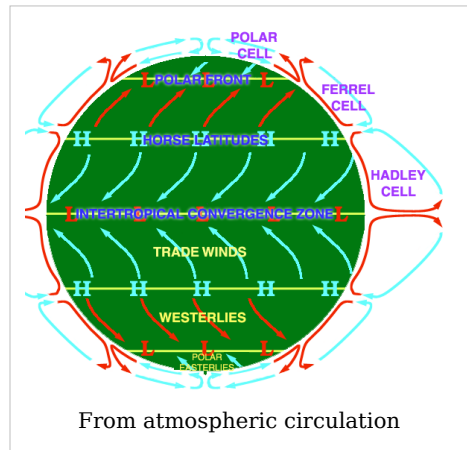
Cyclone consequences

A curious meteorological application of this theorem involves considering the wind as a vector defined at every point continuously over the surface of a planet with an atmosphere. As an idealisation, take wind to be a two-dimensional vector: suppose that relative to the planetary diameter of the Earth, its vertical (i.e., non-tangential) motion is negligible.

One scenario, in which there is absolutely no wind (air movement), corresponds to a field of zero-vectors. This scenario is uninteresting from the point of view of this theorem, and physically unrealistic (there will always be wind). In the case where there is at least some wind, the Hairy Ball Theorem dictates that there must be at least one point on a planet at all times with no wind at all and therefore a tuft. This corresponds to the above statement that there will always be p such that $f(p) = \mathbf{0}$.

In a physical sense, this zero-wind point will be the eye of a cyclone or anticyclone. (Like the swirled hairs on the tennis ball, the wind will spiral around this zero-wind point - under our assumptions it cannot flow into or out of the point.) In brief, then, the Hairy Ball Theorem dictates that, given at least some wind on Earth, there must at all times be a cyclone somewhere. Note that the eye can be arbitrarily large or small and the magnitude of the wind surrounding it is irrelevant.

This is not strictly true as the air above the earth has multiple layers, but for each layer there must be a point with zero horizontal windspeed.



Application to computer graphics

A common problem in computer graphics is to generate a non-zero vector that is orthogonal to a given one. There is no continuous function that can do this. If the given vector is considered to be the radius vector of a sphere, then it follows that that is a corollary of the hairy ball theorem. To see this, note that finding a non-zero vector orthogonal to the given one is equivalent to finding a non-zero vector that is tangent to the surface of the sphere produced by the given vector. However, the hairy ball theorem says there exists no *continuous* function that can do this for every point on the sphere (ie. every given vector).

Lefschetz connection

There is a closely-related argument from algebraic topology, using the Lefschetz fixed point theorem. Since the Betti numbers of a 2-sphere are 1, 0, 1, 0, 0, ... the *Lefschetz number* (total trace on homology) of the identity mapping is 2. By integrating a vector field we get (at least a small part of) a one-parameter group of diffeomorphisms on the sphere; and all of the mappings in it are homotopic to the identity. Therefore they all have Lefschetz number 2, also. Hence they have fixed points (since the Lefschetz number is nonzero). Some more work would be needed to show that this implies there must actually be a zero of the vector field. It does suggest the correct statement of the more general Poincaré-Hopf index theorem.

Corollary

A consequence of the hairy ball theorem is that any continuous function that maps a sphere into itself has either a fixed point or a point that maps onto its own antipodal point. This can be seen by transforming the function into a tangential vector field as follows.

Let s be the function mapping the sphere to itself, and let v be the tangential vector function to be constructed. For each point p , construct the stereographic projection of $s(p)$ with p as the point of tangency. Then $v(p)$ is the displacement vector of this projected point relative to p . According to the hairy ball theorem, there is a p such that $v(p) = \mathbf{0}$, so that $s(p) = p$.

This argument breaks down only if there exists a point p for which $s(p)$ is the antipodal point of p , since such a point is the only one that cannot be stereographically projected onto the tangent plane of p .

Higher dimensions

The connection with the Euler characteristic χ suggests the correct generalisation: the $2n$ -sphere has no non-vanishing vector field for $n \geq 1$. The difference in even and odd dimension is that the Betti numbers of the m -sphere are 0 except in dimensions 0 and m . Therefore their alternating sum χ is 2 for m even, and 0 for m odd.

References

- Murray Eisenberg, Robert Guy, *A Proof of the Hairy Ball Theorem*, The American Mathematical Monthly, Vol. 86, No. 7 (Aug. - Sep., 1979), pp. 571-574

[1] <http://dz-srv1.sub.uni-goettingen.de/sub/digbib/loader?ht=VIEW&did=D28661>

See also

- Vector fields on spheres
- Tokamak

Universal coefficient theorem

In mathematics, the **universal coefficient theorem** in algebraic topology establishes the relationship in homology theory between the *integral homology* of a topological space X , and its *homology with coefficients* in any abelian group A . It shows that the integral homology groups

$$H_i(X, \mathbf{Z})$$

do in a certain, definite sense determine the groups

$$H_i(X, A).$$

Here $H_\square$ might be the simplicial homology or more general singular homology theory: the result itself is a pure piece of homological algebra about chain complexes of free abelian groups. The form of the result is that other coefficients A may be used, at the cost of using a Tor functor.

For example it is common to take A to be $\mathbf{Z}/2\mathbf{Z}$, so that coefficients are modulo 2. This becomes straightforward in the absence of 2-torsion in the homology. Quite generally, the result indicates the relationship that holds between the Betti numbers b_i of X and the Betti numbers $b_{i,F}$ with coefficients in a field F . These can differ, but only when the characteristic of F is a prime number p for which there is some p -torsion in the homology.

The statement of the universal coefficient theorem runs as follows. Consider

$$H_i \otimes A$$

where H_i means $H_i(X, \mathbf{Z})$. Then there is an injective group homomorphism ι from it to $H_i(X, A)$. The theorem describes the cokernel of ι as

$$\text{Tor}(H_{i-1}, A).$$

This Tor group can therefore be described as the obstruction to ι being an isomorphism, which could be thought of as the 'expected' result.

This can be summarized saying there is a natural short exact sequence

$$0 \rightarrow H_i \otimes A \rightarrow H_i(X, A) \rightarrow \text{Tor}(H_{i-1}, A) \rightarrow 0$$

Furthermore, this is a split sequence (but the splitting is *not* natural).

There is also a **universal coefficient theorem for cohomology** involving the Ext functor, stating that there is a natural short exact sequence

$$0 \rightarrow \text{Ext}(H_{i-1}, A) \rightarrow H^i(X, A) \rightarrow \text{Hom}(H_i, A) \rightarrow 0$$

As in the homological case, the sequence splits, though not naturally.

References

- Allen Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002. ISBN 0-521-79540-0. A modern, geometrically flavored introduction to algebraic topology. The book is available free in PDF and PostScript formats on the author's homepage^[1].

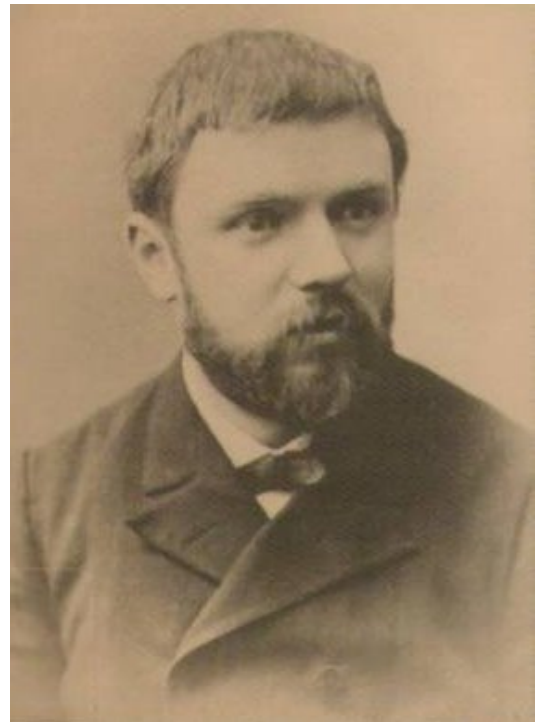
Borsuk-Ulam theorem

1. redirect Borsuk-Ulam theorem

Brouwer fixed point theorem

In mathematics, **Brouwer's fixed point theorem** is a theorem in topology. It is one of many fixed point theorems, which state that for any continuous function f with certain properties there is a point x_0 such that $f(x_0) = x_0$. The simplest form of Brouwer's theorem is for continuous functions f from a disk D to itself. A more general form is for continuous functions from a convex compact subset K of Euclidean space to itself.

Among hundreds of fixed point theorems^[1], Brouwer's is particularly well known, due in part to the fact that it is used across numerous fields of mathematics. In its original field, this result is one of the key theorems characterizing the topology of Euclidean spaces, along with the Jordan curve theorem, the hairy ball theorem or the Borsuk-Ulam theorem.^[2] This gives it a place among the fundamental theorems of topology^[3]. The theorem is also used for proving deep results about differential equations and is covered in introductory course on differential geometry. It appears in unlikely fields such as game theory, where John Nash used it to prove the existence of a winning strategy for the game Hex.



In 1886, Henri Poincaré proved a result that is equivalent to Brouwer's fixed point theorem.

The three-dimensional case of the exact statement was proved in 1904 by Piers Bohl, and the general case in 1910 by Jacques Hadamard. Luitzen Brouwer proposed a new proof in 1912.

The theorem was first studied in view of work on differential equations by the French mathematicians around Poincaré and Picard. Proving results such as the Poincaré-Bendixson theorem requires the use of topological methods. This work at the end of the 19th century opened into several successive versions of the theorem. The general case was first proved in 1910 by Hadamard, and then in 1912 by Luitzen Egbertus Jan Brouwer.

Statement

The theorem has several formulations, depending on the context in which it is used. The simplest is sometimes given as follows:

In the plane

Every continuous function f from a closed disk to itself has at least one fixed point.^[4]

This can be generalized to an arbitrary finite dimension:

In Euclidean space

Every continuous function from a closed ball of a Euclidean space to itself has a fixed point.^[5]

A slightly more general version is as follows:^[6]

Convex compact set

Every continuous function f from a convex compact subset K of a Euclidean space to K itself has a fixed point.^[7]

An even more general form is better known under a different name:

Schauder fixed point theorem

Every continuous function from a convex compact subset K of a Banach space to K itself has a fixed point.^[8]

Notes

The function f in this theorem is not required to be bijective or even surjective. Since any closed ball in Euclidean n -space is homeomorphic to the closed unit ball D^n , the theorem also has equivalent formulations that only state it for D^n .

Because the properties involved (continuity, being a fixed point) are invariant under homeomorphisms, the theorem is equivalent to forms in which the domain is required to be a closed unit ball D^n . For the same reason it holds for every set that is homeomorphic to a closed ball (and therefore also closed, bounded, connected, without holes, etc.).

The statement of the theorem is false if formulated for the *open* unit disk, the set of points with distance strictly less than 1 from the origin. Consider for example the function

$$f(x, y) = \left(\frac{1}{2}(x + \sqrt{1 - y^2}), y\right)$$

which maps every point of the open unit disk in $\mathbf{R}^2$ to another point of the open unit disk slightly to the right of the given one.

Illustrations

The theorem has several "real world" illustrations. For example: take two sheets of graph paper of equal size with coordinate systems on them, lay one flat on the table and crumple up (without ripping or tearing) the other one and place it any fashion on top of the first so that the crumpled paper does not reach outside the flat one. There will then be at least one point of the crumpled sheet that lies exactly on top of its corresponding point (i.e. the point with the same coordinates) of the flat sheet. This is a consequence of the $n = 2$ case of Brouwer's theorem applied to the continuous map that assigns to the coordinates of every point of the crumpled sheet the coordinates of the point of the flat sheet immediately beneath it.

In three dimensions the consequence of the Brouwer fixed point theorem is that no matter how much you stir or shake a cocktail in a glass some point in the liquid will remain in the exact same place in the glass as before you took any action, assuming that the final position of each point is a continuous function of its original position, and that the liquid after stirring or shaking is contained within the space originally taken up by it.

Another consequence of the case $n = 3$ is given by an informational display of a map in, for example, an airport terminal. The function that sends points of the terminal to their image on the map is continuous and therefore has a fixed point, usually indicated by a cross or arrow with the text *You are here*. A similar display outside the terminal would violate the condition that the function is "to itself" and fail to have a fixed point. For this example, the existence of a fixed point is also a consequence of the Banach fixed point theorem, since the function mapping points in space to the display is a contraction mapping.

Intuitive approach

Explanations attributed to Brouwer

The theorem is supposed to have originated from Brouwer's observation of a cup of coffee.^[9] If one stirs to dissolve a lump of sugar, it appears there is always a point without motion. He drew the conclusion that at any moment, there is a point on the surface that is not moving.^[10] The fixed point is not necessarily the point that seems to be motionless, since the centre of the turbulence moves a little bit. The result is not intuitive, since the original fixed point may become mobile when another fixed point appears.

Brouwer is said to have added: "I can formulate this splendid result differently, I take a horizontal sheet, and another identical one which I crumple, flatten and place on the other. Then a point of the crumpled sheet is in the same place as on the other sheet."^[10] Brouwer "flattens" his sheet as with a flat iron, without removing the folds and wrinkles.

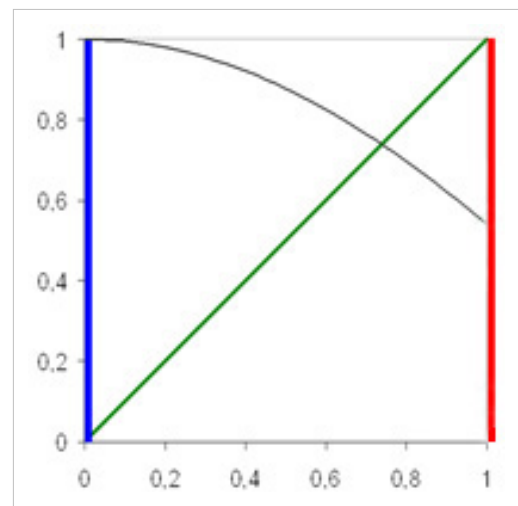
One-dimensional case

In one dimension, the result is intuitive and easy to prove. The continuous function f is defined on a closed interval $[a, b]$ and takes values in the same interval. Saying that this function has a fixed point amounts to saying that its graph (black in the figure on the right) intersects that of the function defined on the same interval $[a, b]$ which maps x to x (green).

Intuitively, any continuous line from the left edge of the square to the right edge must necessarily intersect the green diagonal.

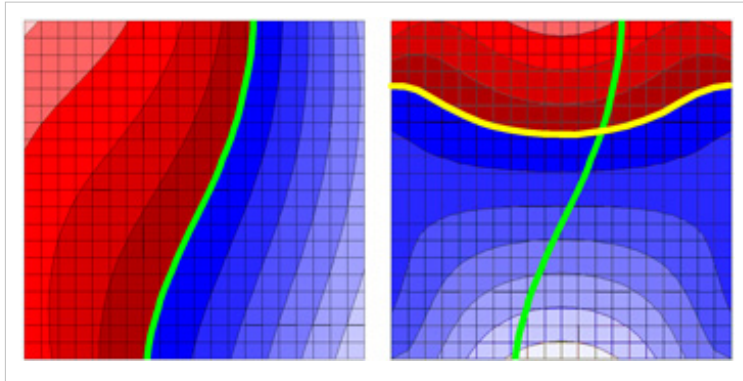
It is not hard to give a formal proof. It suffices to consider the function g which maps x to $f(x) - x$. It is ≥ 0 on a and ≤ 0 on b . By the intermediate value theorem, g has a zero in $[a, b]$; this zero is a fixed point.

Brouwer is said to have expressed this as follows: "Instead of examining a surface, we will prove the theorem about a piece of string. Let us begin with the string in an unfolded state,



then refold it. Let us flatten the refolded string. Again a point of the string has not changed its position with respect to its original position on the unfolded string."^[10]

Two-dimensional case



In other words, one can assume that K is a closed unit ball. The norm can moreover be chosen arbitrarily. Choosing the maximum norm (the maximum of the absolute values of the coordinates) shows that without loss of generality K can be assumed to be the set $[-1, 1] \times [-1, 1]$.

Denoting by g the function that maps x to $h(x) - x$, the goal is to prove that the zero vector is in the image of g on $[-1, 1] \times [-1, 1]$. If g_k , for k equal to 1 or 2, are the two coordinate functions of g , this amounts to proving the existence of a point x_0 such that g_1 and g_2 both have a zero at x_0 .

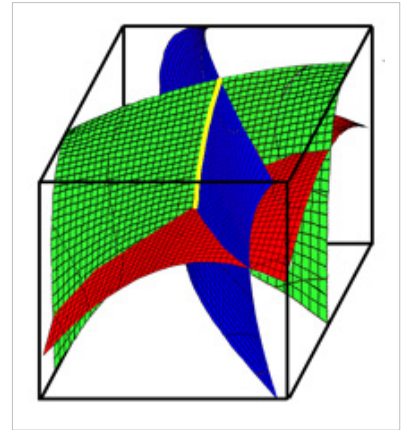
The function g_1 goes from $[-1, 1] \times [-1, 1]$ to $[-1, 1]$. It can be interpreted as a map of a region which indicates the altitude of every point (see first figure on the right). In the area $\{-1\} \times [-1, 1]$, this altitude is ≥ 0 (red in the figure), while on $\{1\} \times [-1, 1]$ it is ≤ 0 (blue). This suggests that the contour line 0 is a line (green in the figure) from a point in $[-1, 1] \times \{1\}$ to a point in $[-1, 1] \times \{-1\}$. The same reasoning applied to g_2 suggests that the contour line 0 is this time a line from somewhere in $\{-1\} \times [-1, 1]$ to somewhere in $\{1\} \times [-1, 1]$ (see yellow line in the second figure).

Intuitively, it seems obvious that these two contour lines (green and yellow) must necessarily intersect, and the point of intersection is a fixed point of $f \circ \varphi$. Verifying this intuition is not as easy as it appears. The green zone is not necessarily a connected line, not even necessarily a line, even though it always contains one. For example it can be a strip.

Finite-dimensional case

The intuitive approach of the previous section generalizes to any finite dimension. To understand how, it is sufficient to look at dimension 3.

The goal is again to prove that the function g , which now has three coordinates, has a zero. The first coordinate is ≥ 0 on the left face of the cube and ≤ 0 on the right face. There is every reason to think that the set of zeros contains a sheet, shown in blue in the figure on the right. This sheet *cuts* the cube into two connected components, one containing part of the right face and one containing part of the left face.



Assuming that the y axis is in *front-back* direction, the same reasoning suggests the existence of a sheet, shown in green in the figure, which also *cuts* the cube into at least two connected components. The intersection of the two sheets probably contains a *line*, depicted in yellow, going from the upper face to the lower face.

Now the third component of g describes a face shown in red. It appears that this sheet must intersect the yellow line. The point of intersection is the desired fixed point.

History

The Brouwer fixed point theorem was one of the early achievements of algebraic topology, and is the basis of more general fixed point theorems which are important in functional analysis. The case $n = 3$ first was proved by Piers Bohl in 1904 (published in *Journal für die reine und angewandte Mathematik*). It was later proved by L. E. J. Brouwer in 1909. Jacques Hadamard proved the general case in 1910, and Brouwer found a different proof in 1912. Since these early proofs were all non-constructive indirect proofs, they ran contrary to Brouwer's intuitionist ideals. Methods to construct (approximations to) fixed points guaranteed by Brouwer's theorem are now known, however; see for example (Karamadian 1977) and (Istrăţescu 1981).

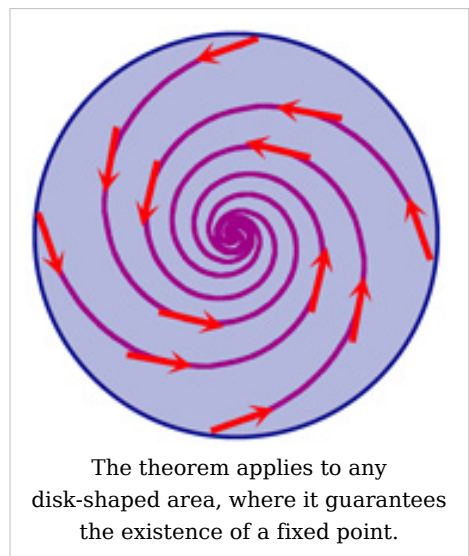
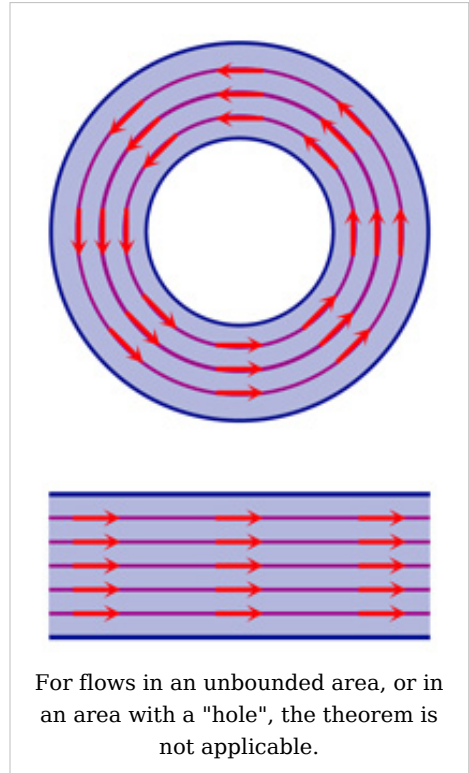
Prehistory

To understand the prehistory of Brouwer's fixed point theorem one needs to pass through differential equations. At the end of the 19th century, the old problem^[11] of the stability of the solar system returned into the focus of the mathematical community.^[12] Its solution required new methods. As noted by Henri Poincaré, who worked on the three-body problem, there is no hope to find an exact solution: "Nothing is more proper to give us an idea of the hardness of the three-body problem, and generally of all problems of Dynamics where there is no uniform integral and the Bohlin series diverge."^[13] He also noted that the search for an approximate solution is no more efficient: "the more we seek to obtain precise approximations, the more the result will diverge towards an increasing imprecision."^[14]

He studied a question analogous to that of the surface movement in cup of coffee. What can we say, in general, about the trajectories on a surface animated by a constant flow?^[15] Poincaré discovered that the answer can be found in what we now call the topological properties in the area containing the trajectory. If this area is compact, i.e. both closed and bounded, then the trajectory either becomes stationary, or it approaches a limit cycle.^[16] Poincaré went further; if the area is of the same kind as a disk, as is the case for the cup of coffee, there must necessarily be a fixed point. This fixed point is invariant under all functions which associate to each point of the original surface its position after a short time interval t . If the area is a circular band, or if it is not closed^[17], then this is not necessarily the case.

To understand differential equations better, a new branch of mathematics was born. Poincaré called it *analysis situs*. The French Encyclopædia Universalis defines it as the branch which "treats the properties of an object that are invariant if it is deformed in any continuous way, without tearing"^[18]. In 1886, Poincaré proved a result that is equivalent to Brouwer's fixed point theorem^[19], although the connection with the subject of this article was not yet apparent^[20]. A little later, he developed one of the fundamental tools for better understanding the analysis situs, now known as the fundamental group or sometimes the Poincaré group.^[21] This method can be used for a very compact proof of the theorem under discussion.

Poincaré's method was analogous to that of Emile Picard, a contemporary mathematician who generalized the Cauchy-Lipschitz theorem.^[22] Picard's approach is based on a result



that would later be formalised by another fixed point theorem, named after Banach. Instead of the topological properties of the domain, this theorem uses the fact that the function in question is a contraction.

First proofs

At the dawn of the 20th century, the interest in analysis situs did not stay unnoticed. However, the necessity of a theorem equivalent to the one discussed in this article was not yet evident. Piers Bohl, a Latvian mathematician, applied topological methods to the study of differential equations.^[23] In 1904 he proved the three-dimensional case of our theorem, but his publication was not noticed.^[24]

It was Brouwer, finally, who gave the theorem its first patent of nobility. His goals were different from those of Poincaré. This mathematician was inspired by the foundations of mathematics, especially mathematical logic and topology. His initial interest lay in an attempt to solve Hilbert's fifth problem.^[25] In 1909, during a voyage to Paris, he met Poincaré, Hadamard and Borel. The ensuing discussions convinced Brouwer of the importance of a better understanding of Euclidean spaces, and were the origin of a fruitful exchange of letters with Hadamard. For the next four years, he concentrated on the proof of certain great theorems on this question. In 1912 he proved the hairy ball theorem for the two-dimensional sphere, as well as the fact that every continuous map from the two-dimensional ball to itself has a fixed point.^[26] These two results in themselves were not really new. As Hadamard observed, Poincaré had shown a theorem equivalent to the hairy ball theorem^[27]. The revolutionary aspect of Brouwer's approach was his systematic use of recently developed tools such as homotopy, the underlying concept of the Poincaré group. In the following year, Hadamard generalised the theorem under discussion to an arbitrary finite dimension, but he employed different methods. H. Freudenthal comments on the respective roles as follows: "Compared to Brouwer's revolutionary methods, those of Hadamard were very traditional, but Hadamard's participation in the birth of Brouwer's ideas resembles that of a midwife more than that of a mere spectator."^[28]

Brouwer's approach yielded its fruits, and in 1912 he also found a proof that was valid for any finite dimension.^[29], as well as other key theorems such as the invariance of dimension^[30]. In the context of this work, Brouwer also generalized the Jordan curve theorem to arbitrary dimension and established the properties connected with the degree of a continuous mapping.^[31] This branch of mathematics, originally envisioned by Poincaré and developed by Brouwer, changed its name. In the 1930s, analysis situs became algebraic topology.^[32]



Hadamard played the role of a midwife, helping Brouwer to formalize his ideas.

Brouwer's celebrity is not exclusively due to his topological work. He was also the originator and zealous defender of a way of formalising mathematics that is known as intuitionism, which at the time made a stand against set theory.^[33] While Brouwer preferred constructive proofs, ironically, the original proofs of his great topological theorems were not constructive^[34], and it took until 1967 for constructive proofs to be found.^[35]

Reception



John Nash used the theorem in game theory to prove the existence of a winning strategy.

The theorem proved its worth in more than one way. During the 20th century numerous fixed point theorems were developed, and even a branch of mathematics called fixed point theory.^[36] Brouwer's theorem is probably the most important.^[37] It is also among the foundational theorems on the topology of topological manifolds and is often used to prove other important results such as the Jordan curve theorem.^[38]

Besides the fixed point theorems for more or less contracting functions, there are many that are derived directly or indirectly from the result under discussion. A continuous map from a

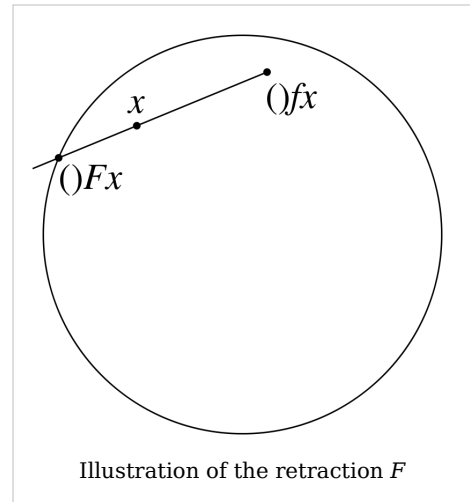
closed ball of Euclidean space to its boundary cannot be the identity on the boundary. Similarly, the Borsuk-Ulam theorem says that a continuous map from the n -dimensional sphere to $\mathbf{R}^n$ has a pair of antipodal points that are mapped to the same point. In the finite-dimensional case, the Lefschetz fixed point theorem provided from 1926 a method for counting fixed points. In 1930, Brouwer's fixed point theorem was generalized to Banach spaces.^[39] This generalization is known as Schauder's fixed point theorem, a result generalized further by S. Kakutani to multivalued functions.^[40] One also meets the theorem and its variants outside topology. It can be used to prove the Hartman-Grobman theorem, which describes the qualitative behaviour of certain differential equations near certain equilibria. Similarly, Brouwer's theorem is used for the proof of the *théorème de la variété centrale*. The theorem can also be found in existence proofs for the solutions of certain partial differential equations.^[41]

Other areas are also touched. In game theory, John Nash used the theorem to prove that in the game of Hex there is a winning strategy for white.^[42] In economy, P. Bich explains that certain generalizations of the theorem show that its use is helpful for certain classical problems in game theory and generally for equilibria (Hotelling's law), financial equilibria and incomplete markets.^[43]

Proof outlines

A full proof of the theorem would be too long to reproduce here, but the following outlines a proof omitting the most difficult part. The proof uses the observation that the boundary of D^n is S^{n-1} , the $(n-1)$ -sphere.

The argument proceeds by contradiction, supposing that a continuous function $f: D^n \rightarrow D^n$ has *no* fixed point, and then attempting to derive an inconsistency, which proves that the function must in fact have a fixed point. For each x in D^n , there is only one straight line that passes through $f(x)$ and x , because it must be the case that $f(x)$ and x are distinct by hypothesis (recall that f having no fixed points means that $f(x) \neq x$). Following this line from $f(x)$ through x leads to a point on S^{n-1} , denoted by $F(x)$. This defines a continuous function $F: D^n \rightarrow S^{n-1}$, which is a special type of continuous function known as a retraction: every point of the codomain (in this case S^{n-1}) is a fixed point of the function.



Intuitively it seems unlikely that there could be a retraction of D^n onto S^{n-1} , and in the case $n = 1$ it is obviously impossible because S^0 (i.e., the endpoints of the closed interval D^1) is not even connected. The case $n = 2$ is less obvious, but can be proven by using basic arguments involving the fundamental groups of the respective spaces: the retraction would induce an injective group homomorphism from the fundamental group of S^1 to that of D^2 , but the first group is isomorphic to $\mathbf{Z}$ while the latter group is trivial, so this is impossible. The case $n = 2$ can also be proven by contradiction based on a theorem about non-vanishing vector fields.

For $n > 2$, however, proving the impossibility of the retraction is more difficult. One way is to make use of homology groups: it can be shown that the homology $H_{n-1}(D^n)$ is trivial, while $H_{n-1}(S^{n-1})$ is infinite cyclic. This shows that the retraction is impossible, because again the retraction would induce an injective group homomorphism from the latter to the former group.

There is also a more elementary combinatorial proof, whose main step consists in establishing Sperner's lemma in n dimensions.

There is also a quick proof, by Morris Hirsch, based on the impossibility of a differentiable retraction. The indirect proof starts by noting that the map f can be approximated by a smooth map retaining the property of not fixing a point; this can be done by using the Weierstrass approximation theorem, for example. One then defines a retraction as above which must now be differentiable. Such a retraction must have a non-singular value, by Sard's theorem, which is also non-singular for the restriction to the boundary (which is just the identity). Thus the inverse image would be a 1-manifold with boundary. The boundary would have to contain at least two end points, both of which would have to lie on the boundary of the original ball—which is impossible in a retraction.

A quite different proof given by David Gale is based on the game of Hex. The basic theorem about Hex is that no game can end in a draw. This is equivalent to the Brouwer fixed point theorem for dimension 2. By considering n -dimensional versions of Hex, one can prove in

general that Brouwer's theorem is equivalent to the determinacy theorem for Hex. ^[44]

Generalizations

The Brouwer fixed point theorem forms the starting point of a number of more general fixed point theorems.

The straightforward generalization to infinite dimensions, i.e. using the unit ball of an arbitrary Hilbert space instead of Euclidean space, is not true. The main problem here is that the unit balls of infinite-dimensional Hilbert spaces are not compact. For example, in the Hilbert space ℓ^2 of square-summable real (or complex) sequences, consider the map $f : \ell^2 \rightarrow \ell^2$ which sends a sequence (x_n) from the closed unit ball of ℓ^2 to the sequence (y_n) defined by

$$y_0 = \sqrt{1 - \|x\|_2^2} \quad \text{and} \quad y_n = x_{n-1} \quad \text{for} \quad n \geq 1.$$

It is not difficult to check that this map is continuous, has its image in the unit sphere of ℓ^2 , but does not have a fixed point.

The generalizations of the Brouwer fixed point theorem to infinite dimensional spaces therefore all include a compactness assumption of some sort, and in addition also often an assumption of convexity. See fixed point theorems in infinite-dimensional spaces for a discussion of these theorems.

The Kakutani fixed point theorem generalizes the Brouwer fixed point theorem in a different direction: it stays in $\mathbf{R}^n$, but considers upper semi-continuous correspondences (functions that assign to each point of the set a subset of the set). It also requires compactness and convexity of the set.

By using forcing to collapse cardinals, the Brouwer fixed point theorem can be generalized to arbitrary cardinality: if L is a compact, connected order topology, then any continuous function from L^n to itself has a fixed point. Note that if we require L to be separable, this is precisely the Brouwer fixed point theorem.

The Lefschetz fixed-point theorem applies to (almost) arbitrary compact topological spaces, and gives a condition in terms of singular homology that guarantees the existence of fixed points; this condition is trivially satisfied for any map in the case of D^n .

See also

- Sperner's lemma
- Borsuk-Ulam theorem
- Tucker's lemma
- Topological combinatorics

Notes

- [1] E.g. F & V Bayart *Théorèmes du point fixe* (<http://www.bibmath.net/dico/index.php3?action=affiche&quoi=.p/pointfixe.html>) on Bibm@th.net
- [2] See page 15 of: D. Leborgne *Calcul différentiel et géométrie* Puf (1982) ISBN 2130374956
- [3] More exactly, according to Encyclopédie Universalis: *Il en a démontré l'un des plus beaux théorèmes, le théorème du point fixe, dont les applications et généralisations, de la théorie des jeux aux équations différentielles, se sont révélées fondamentales.* Luizen Brouwer (http://www.universalis.fr/encyclopedie/T705705/BROUWER_L.htm) by G. Sabbagh
- [4] D. Violette *Applications du lemme de Sperner pour les triangles* (<http://newton.mat.ulaval.ca/amq/bulletins/dec06/sperner.pdf>) Bulletin AMQ, V. XLVI N° 4, (2006) p 17.

- [5] Page 15 of: D. Leborgne *Calcul différentiel et géométrie* Puf (1982) ISBN 2130374956.
- [6] This version follows directly from the previous one because every convex compact subset of a Euclidean space is homeomorphic to a closed ball. For details see convex set.
- [7] V. & F. Bayart *Point fixe, et théorèmes du point fixe* (<http://www.bibmath.net/dico/index.php3?action=affiche&quoi=.p/pointfixe.html>) on Bibmath.net.
- [8] C. Minazzo K. Rider *Théorèmes du Point Fixe et Applications aux Equations Différentielles* (<http://math1.unice.fr/~eaubry/Enseignement/M1/memoire.pdf>) Université de Nice-Sophia Antipolis.
- [9] The interest of this anecdote rests in its intuitive and didactic character, but its accuracy is dubious. As the history section shows, the origin of the theorem is not Brouwer's work. More than 20 years earlier Henri Poincaré had proved an equivalent results, and 5 years before Brouwer P. Bohl had proved the three-dimensional case.
- [10] Cette citation provient d'une émission de télévision : *Archimède* (<http://archives.arte.tv/hebdo/archimed/19990921/ftext/sujet5.html>), Arte, 21 septembre 1999
- [11] See F. Brechenmacher *L'identité algébrique d'une pratique portée par la discussion sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des planètes* (<http://arxiv.org/ftp/arxiv/papers/0704/0704.2931.pdf>) CNRS Fédération de Recherche Mathématique du Nord-Pas-de-Calais
- [12] Henri Poincaré won the King of Sweden's mathematical competition in 1889 for his work on the related three-body problem: J. Tits *Célébrations nationales 2004* (<http://www.culture.gouv.fr/culture/actualites/celebrations2004/poincare.htm>) Site du Ministère Culture et Communication
- [13] H. Poincaré *Les méthodes nouvelles de la mécanique céleste* T Gauthier-Villars, Vol 3 p 389 (1892) new edition Paris: Blanchard, 1987.
- [14] Quotation from H. Poincaré taken from: P. A. Miquel *La catégorie de désordre* (<http://www.arches.ro/revue/no03/no3art03.htm>), on the website of l'Association roumaine des chercheurs francophones en sciences humaines
- [15] This question was studied in: H. Poincaré *Sur les courbes définies par les équations différentielles* J. de Math. V 2 (1886)
- [16] This follows from the Poincaré-Bendixson theorem.
- [17] Multiplication by $1 \square_2$ on $]0, 1[$ has no fixed point.
- [18] "concerne les propriétés invariantes d'une figure lorsqu'on la déforme de manière continue quelconque, sans déchirure (par exemple, dans le cas de la déformation de la sphère, les propriétés corrélatives des objets tracés sur sa surface". From C. Houzel M. Paty *Poincaré, Henri (1854-1912)* (http://www.scientiaestudia.org.br/associac/paty/pdf/Paty,M_1997g-PoincareEU.pdf) Encyclopædia Universalis Albin Michel, Paris, 1999, p. 696-706
- [19] Poincaré's theorem is stated in: V. I. Istratescu *Fixed Point Theory an Introduction* Kluwer Academic Publishers (réédition de 2001) p 113 ISBN 1402003013
- [20] M.I. Voitsekhovskii *Brouwer theorem* (<http://eom.springer.de/b/b017670.htm>) Encyclopaedia of Mathematics ISBN 1402006098
- [21] J. Dieudonné, *A History of Algebraic and differential Topology, 1900-1960*, pages 17-24
- [22] See for example: E Picard *Sur l'application des méthodes d'approximations successives à l'étude de certaines équations différentielles ordinaires* (http://portail.mathdoc.fr/JMPA/PDF/JMPA_1893_4_9_A4_0.pdf) Journal de Mathématiques p 217 (1893)
- [23] J. J. O'Connor E. F. Robertson *Piers Bohl* (<http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Bohl.html>)
- [24] A. D. Myskis I. M. Rabinovic *The first proof of a fixed-point theorem for a continuous mapping of a sphere into itself, given by the Latvian mathematician P G Bohl (Russian)* Uspekhi matematicheskikh nauk (NS) Vol 10 (N° 3) (65) (1955) pp 188-192
- [25] J. J. O'Connor E. F. Robertson *Luitzen Egbertus Jan Brouwer* (<http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Brouwer.html>)
- [26] H. Freudenthal *The cradle of modern topology, according to Brouwer's inedita* Hist. Math. 2 p 495 (1975)
- [27] Freudenthal explains: "... cette dernière propriété, bien que sous des hypothèses plus grossières, ait été démontré par H. Poincaré" H. Freudenthal *The cradle of modern topology, according to Brouwer's inedita* Hist. Math. 2 p 495 (1975)
- [28] H. Freudenthal *The cradle of modern topology, according to Brouwer's inedita* Hist. Math. 2 p 501 (1975)
- [29] L. Brouwer *Über Abbildungen von Mannigfaltigkeiten* Mathematische Annalen 38(71) pp 97 115 (1912)
- [30] If an open subset of a n -manifold is homeomorphic to an open subset of a Euclidean space of dimension n , and if p is a positive integer other than n , then the open set is never homeomorphic to an open subset of a Euclidean space of dimension p .
- [31] J. J. O'Connor E. F. Robertson *Luitzen Egbertus Jan Brouwer* (<http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Brouwer.html>).

- [32] The term *algebraic topology* first appeared 1931 under the pen of David van Dantzig: J. Miller *Topological algebra* (<http://jeff560.tripod.com/t.html>) on the site Earliest Known Uses of Some of the Words of Mathematics (2007)
- [33] Later it would be shown that the formalism that was combatted by Brouwer can also serve to formalise intuitionism. For further details see intuitionistic logic.
- [34] For a long explanation, see: J.P. Dubucs *L.J.E. Brouwer : Topologie et constructivisme* (http://www.persee.fr/web/revues/home/prescript/article/rhs_0151-4105_1988_num_41_2_4094) Revue d'histoire des sciences V. 41 N°41-2 pp 133-155 (1988)
- [35] H. Scarf found the first algorithmic proof: M.I. Voitsekhovskii *Brouwer theorem* (<http://eom.springer.de/b/b017670.htm>) Encyclopaedia of Mathematics ISBN 1402006098.
- [36] V. I. Istratescu *Fixed Point Theory. An Introduction* Kluwer Academic Publishers (new edition 2001) ISBN 1402003013.
- [37] "... Brouwer's fixed point theorem, perhaps the most important fixed point theorem." p xiii V. I. Istratescu *Fixed Point Theory an Introduction* Kluwer Academic Publishers (new edition 2001) ISBN 1402003013.
- [38] E.g.: S. Greenwood J. Cao *Brouwer's Fixed Point Theorem and the Jordan Curve Theorem* (<http://www.math.auckland.ac.nz/class750/section5.pdf>) University of Auckland, New Zealand.
- [39] J. Schauder *Der Fixpunktsatz in Funktionalraumen* Studia. Math. 2 (1930) pp 171-180
- [40] S. Kakutani *A generalization of Brouwer's Fixed Point Theorem* Duke Math. Journal 8 (1941) pp 457-459
- [41] These examples are taken from: F. Boyer *Théorèmes de point fixe et applications* (http://www.cmi.univ-mrs.fr/~fboyer/ter_fboyer2.pdf) CMI Université Paul Cézanne (2008-2009)
- [42] For context and references see the article Hex (board game).
- [43] P. Bich *Une extension discontinue du théorème du point fixe de Schauder, et quelques applications en économie* (<http://www.ann.jussieu.fr/~plc/code2007/bich.pdf>) Institut Henri Poincaré, Paris (2007)
- [44] David Gale (1979). "The Game of Hex and Brouwer Fixed-Point Theorem". *The American Mathematical Monthly* **86**: 818-827. doi: 10.2307/2320146 (<http://dx.doi.org/10.2307/2320146>).

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- Gale, D. (1979). "The Game of Hex and Brouwer Fixed-Point Theorem". *The American Mathematical Monthly* **86**: 818-827. doi: 10.2307/2320146 (<http://dx.doi.org/10.2307/2320146>).
- Morris W. Hirsch, "Differential Topology", Springer, 1980 (see p. 72-73 for Hirsch's proof utilizing non-existence of a differentiable retraction)
- S. Karamadian (ed.), *Fixed points. Algorithms and applications*, Academic Press, 1977
- V.I. Istrăţescu, *Fixed point theory*, Reidel, 1981

External links

- Brouwer's Fixed Point Theorem for Triangles (http://www.cut-the-knot.org/do_you_know/poincare.shtml#brouwertheorem) at cut-the-knot
- Brouwer theorem (<http://planetmath.org/encyclopedia/BrouwerFixedPointTheorem.html>), from PlanetMath with attached proof.
- Reconstructing Brouwer (<http://www.mathpages.com/home/kmath262/kmath262.htm>) at MathPages

Cellular approximation theorem

In algebraic topology, in the **cellular approximation theorem**, a map between CW-complexes can always be taken to be of a specific type. Concretely, if X and Y are CW-complexes, and $f : X \rightarrow Y$ is a continuous map, then f is said to be *cellular*, if f takes the n -skeleton of X to the n -skeleton of Y for all n , i.e. if $f(X^n) \subseteq Y^n$ for all n . The content of the cellular approximation theorem is then that any continuous map $f : X \rightarrow Y$ between CW-complexes X and Y is homotopic to a cellular map, and if f is already cellular on a subcomplex A of X , then we can furthermore choose the homotopy to be stationary on A . From an algebraic topological viewpoint, any map between CW-complexes can thus be taken to be cellular.

Idea of proof

The proof can be given by induction after n , with the statement that f is cellular on the skeleton X^n . For the base case $n = 0$, notice that every path-component of Y must contain a 0-cell. The image under f of a 0-cell of X can thus be connected to a 0-cell of Y by a path, but this gives a homotopy from f to a map, which is cellular on the 0-skeleton of X .

Assume inductively that f is cellular on the $(n - 1)$ -skeleton of X , and let e^n be an n -cell of X . The closure of e^n is compact in X , being the image of the characteristic map of the cell, and hence the image of the closure of e^n under f is also compact in Y . Then it is a general result of CW-complexes that any compact subspace of a CW-complex meets (that is, intersects non-trivially) only finitely many cells of the complex. Thus $f(e^n)$ meets at most finitely many cells of Y , so we can take $e^k \subseteq Y$ to be a cell of highest dimension meeting $f(e^n)$. If $k \leq n$, the map f is already cellular on e^n , since in this case only cells of the n -skeleton of Y meets $f(e^n)$, so we may assume that $k > n$. It is then a technical, non-trivial result (see Hatcher) that the restriction of f to $X^{n-1} \cup e^n$ can be homotoped relative to X^{n-1} to a map missing a point $p \in e^k$. Since $Y^k - \{p\}$ deformation retracts onto the subspace $Y^k - e^k$, we can further homotope the restriction of f to $X^{n-1} \cup e^n$ to a map, say, g , with the property that $g(e^n)$ misses the cell e^k of Y , still relative to X^{n-1} . Since $f(e^n)$ met only finitely many cells of Y to begin with, we can repeat this process finitely many times to make $f(e^n)$ miss all cells of Y of dimension larger than n .

We repeat this process for every n -cell of X , fixing cells of the subcomplex A on which f is already cellular, and we thus obtain a homotopy (relative to the $(n - 1)$ -skeleton of X and the n -cells of A) of the restriction of f to X^n to a map cellular on all cells of X of dimension at most n . Using then the homotopy extension property to extend this to a homotopy on all of X , and patching these homotopies together, will finish the proof. For details, consult Hatcher.

Applications

Some homotopy groups

The cellular approximation theorem can be used to immediately calculate some homotopy groups. In particular, if $n < k$, then $\pi_n(S^k) = 0$: Give S^n and S^k their canonical CW-structure, with one 0-cell each, and with one n -cell for S^n and one k -cell for S^k . Any base-point preserving map $f: S^n \rightarrow S^k$ is by the cellular approximation theorem homotopic

to a constant map, whence $\pi_n(S^k) = 0$.

Cellular approximation for pairs

Let $f:(X,A) \rightarrow (Y,B)$ be a map of CW-pairs, that is, f is a map from X to Y , and the image of $A \subseteq X$ under f sits inside B . Then f is homotopic to a cellular map $(X,A) \rightarrow (Y,B)$. To see this, restrict f to A and use cellular approximation to obtain a homotopy of f to a cellular map on A . Use the homotopy extension property to extend this homotopy to all of X , and apply cellular approximation again to obtain a map cellular on X , but without violating the cellular property on A .

As a consequence, we have that a CW-pair (X,A) is n -connected, if all cells of $X - A$ have dimension strictly greater than n : If $i \leq n$, then any map $(D^i, \partial D^i) \rightarrow (X,A)$ is homotopic to a cellular map of pairs, and since the n -skeleton of X sits inside A , any such map is homotopic to a map whose image is in A , and hence it is 0 in the relative homotopy group $\pi_i(X, A)$.

We have in particular that (X, X^n) is n -connected, so it follows from the long exact sequence of homotopy groups for the pair (X, X^n) that we have isomorphisms $\pi_i(X^n) \rightarrow \pi_i(X)$ for all $i < n$ and a surjection $\pi_n(X^n) \rightarrow \pi_n(X)$.

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Eilenberg-Zilber theorem

In mathematics, specifically in algebraic topology, the **Eilenberg-Zilber theorem** is an important result in establishing the link between the homology groups of a product space $X \times Y$ and those of the spaces X and Y . The theorem first appeared in a 1953 paper in the American Journal of Mathematics.

Statement of the theorem

The theorem can be formulated as follows. Suppose X and Y are topological spaces, Then we have the three chain complexes $C_*(X)$, $C_*(Y)$, and $C_*(X \times Y)$. (The argument applies equally to the simplicial or singular chain complexes.) We also have the *tensor product complex* $C_*(X) \otimes C_*(Y)$, whose differential is, by definition,

$$\delta(\sigma \otimes \tau) = \delta_X \sigma \otimes \tau + (-1)^p \sigma \otimes \delta_Y \tau$$

for $\sigma \in C_p(X)$ and δ_X, δ_Y the differentials on $C_*(X), C_*(Y)$.

Then the theorem says that we have a chain maps

$$F : C_*(X \times Y) \rightarrow C_*(X) \otimes C_*(Y), \quad G : C_*(X) \otimes C_*(Y) \rightarrow C_*(X \times Y)$$

such that FG is the identity and GF is chain-homotopic to the identity. Moreover, the maps are natural in X and Y . Consequently the two complexes must have the same homology:

$$H_*(C_*(X \times Y)) \cong H_*(C_*(X) \otimes C_*(Y)).$$

Consequences

The Eilenberg–Zilber theorem is a key ingredient in establishing the Künneth theorem, which expresses the homology groups $H_*(X \times Y)$ in terms of $H_*(X)$ and $H_*(Y)$. In light of the Eilenberg–Zilber theorem, the content of the Künneth theorem consists in analysing how the homology of the tensor product complex relates to the homologies of the factors; the answer is somewhat subtle.

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Hurewicz theorem

In mathematics, the **Hurewicz theorem** is a basic result of algebraic topology, connecting homotopy theory with homology theory via a map known as the **Hurewicz homomorphism**. The theorem is named after Witold Hurewicz, and generalizes earlier results from Henri Poincaré.

Statement of the theorems

The Hurewicz theorems are a key link between homotopy groups and homology groups.

Absolute version

For any space X and positive integer k there exists a group homomorphism

$$h_*: \pi_k(X) \rightarrow H_k(X)$$

called the Hurewicz homomorphism from the k -th homotopy group to the k -th homology group (with integer coefficients), which for $k = 1$ is equivalent to the canonical abelianization map

$$h_*: \pi_1(X) \rightarrow \pi_1(X)/[\pi_1(X), \pi_1(X)].$$

The Hurewicz theorem states that if X is $(n-1)$ -connected, the Hurewicz map is an isomorphism for all $k \leq n$. In particular, this theorem says that the abelianization of the first homotopy group (the fundamental group) is isomorphic to the first homology group:

$$H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)].$$

The first homology group therefore vanishes if X is path-connected and $\pi_1(X)$ is a perfect group.

In addition, the Hurewicz homomorphism is an epimorphism from $\pi_{n+1}(X) \rightarrow H_{n+1}(X)$ whenever X is $(n-1)$ connected, for $n \geq 2$.

The group homomorphism is given in the following way. Choose canonical generators $u_n \in H_n(S^n)$. Then a homotopy class of maps $f \in \pi_n(X)$ is taken to $f_*(u_n) \in H_n(X)$.

Relative version

For any pair of spaces (X, A) and integer $k > 1$ there exists a homomorphism

$$h_*: \pi_k(X, A) \rightarrow H_k(X, A)$$

from relative homotopy groups to relative homology groups. The Relative Hurewicz Theorem states that if each of X, A are connected and the pair (X, A) is $(n-1)$ -connected then $H_k(X, A) = 0$ for $k < n$ and $H_n(X, A)$ is obtained from $\pi_n(X, A)$ by factoring out the action of $\pi_1(A)$. This is proved in, for example, Whitehead (1978) by induction, proving in turn the absolute version and the Homotopy Addition Lemma.

This relative Hurewicz theorem is reformulated by Brown & Higgins (1981) as a statement about the morphism

$$\pi_n(X, A) \rightarrow \pi_n(X \cup CA).$$

This statement is a special case of a homotopical excision theorem, involving induced modules for $n > 2$ (crossed modules if $n = 2$), which itself is deduced from a higher homotopy van Kampen theorem for relative homotopy groups, whose proof requires development of techniques of a cubical higher homotopy groupoid of a filtered space.

Triadic version

For any triad of spaces $(X; A, B)$ (i.e. space X and subspaces A, B) and integer $k > 2$ there exists a homomorphism

$$h_*: \pi_k(X; A, B) \rightarrow H_k(X; A, B)$$

from triad homotopy groups to triad homology groups. Note that $H_k(X; A, B) \cong H_k(X \cup (C(A \cup B)))$. The Triadic Hurewicz Theorem states that if X, A, B , and $C = A \cap B$ are connected, the pairs $(A, C), (B, C)$ are respectively $(p-1)$ -, $(q-1)$ -connected, and the triad $(X; A, B)$ is $p+q-2$ connected, then $H_k(X; A, B) = 0$ for $k < p+q-2$ and $H_{p+q-1}(X; A)$ is obtained from $\pi_{p+q-1}(X; A, B)$ by factoring out the action of $\pi_1(A \cap B)$ and the generalised Whitehead products. The proof of this theorem uses a higher homotopy van Kampen type theorem for triadic homotopy groups, which requires a notion of the fundamental cat^n -group of an n -cube of spaces.

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Künneth theorem

In mathematics, especially in homological algebra and algebraic topology, a **Künneth theorem** is a statement relating the homology of two objects to the homology of their product. The classical statement of the Künneth theorem relates the singular homology of two topological spaces X and Y and their product space $X \times Y$. In the simplest possible case the relationship is that of a tensor product, but for applications it is very often necessary to apply certain tools of homological algebra to express the answer.

A Künneth theorem or Künneth formula is true in many different homology and cohomology theories, and the name has become generic. These many results are named for the German mathematician Otto Hermann Künneth (1892–1975).

Singular homology with coefficients in a field

Let X and Y be two topological spaces, and let F be a field. In this situation, the Künneth theorem for singular homology states that for any integer k ,

$$\bigoplus_{i+j=k} H_i(X, F) \otimes H_j(Y, F) \cong H_k(X \times Y, F).$$

Furthermore, the isomorphism is natural isomorphism. The map from the sum to the homology group of the product is called the *cross product*. More precisely, there is a cross product operation showing how an i -cycle on X and a j -cycle on Y can be combined to create an $(i + j)$ -cycle on $X \times Y$; so that there is an explicit linear mapping defined from the direct sum to $H_k(X \times Y)$.

A consequence of this result is that the Betti numbers, the dimensions of the homology with $\mathbf{Q}$ coefficients, of $X \times Y$ can be determined from those of X and Y . If $p_Z(t)$ is the generating function of the sequence of Betti numbers $b_k(Z)$ of a space Z , then

$$p_{X \times Y}(t) = p_X(t)p_Y(t).$$

Here when there are finitely many Betti numbers of X and Y , each of which is a natural number rather than ∞ , this reads as an identity on Poincaré polynomials. In the general case these are formal power series with possibly infinite coefficients, and have to be interpreted accordingly. Furthermore, the above statement holds not only for the Betti numbers but also for the generating functions of the dimensions of the homology over any field. (If the integer homology is not torsion-free then these numbers may be different.)

Singular homology with coefficients in a PID

The above formula is simple because vector spaces over a field have very restricted behavior. As the coefficient ring becomes more general, the relationship becomes more complicated. The next simplest case is the case when the coefficient ring is a principal ideal domain. This case is particularly important because the integers are a PID.

In this case the equation above is no longer true. Instead a correction factor appears to account for the possibility of torsion phenomena. For example, if $H_1(X, \mathbf{Z}) = \mathbf{Z}/(2)$ and $H_1(Y, \mathbf{Z}) = \mathbf{Z}/(3)$, then the tensor product of these homology groups will be zero. But the second homology of $X \times Y$ will *always* contain a correction factor to account for the vanishing of this product. This correction factor is expressed in terms of the Tor functor, the first derived functor of the tensor product.

When X and Y are CW complexes and R is a PID, then the correct statement of the Künneth theorem is that there are natural short exact sequences

$$0 \rightarrow \bigoplus_{i+j=k} H_i(X, R) \otimes_R H_j(Y, R) \rightarrow H_k(X \times Y, R) \rightarrow \bigoplus_{i+j=k-1} \text{Tor}_1^R(H_i(X, R), H_j(Y, R)) \rightarrow 0.$$

Furthermore these sequences split, but not canonically.

The Künneth spectral sequence

For general R , the homology of X and Y is related to the homology of their product by a spectral sequence. In the cases described above, this spectral sequence collapses to give an isomorphism or a short exact sequence. The Künneth spectral sequence is

$$E_{pq}^2 = \bigoplus_{q_1+q_2=q} \text{Tor}_p^R(H_{q_1}(X, R), H_{q_2}(Y, R)) \Rightarrow H_{p+q}(X \times Y, R).$$

The Künneth formula in the derived category

A much cleaner statement of the Künneth formula becomes possible in the derived category. In this case, the formula becomes a natural isomorphism between quasi-isomorphism classes of singular chain complexes:

$$C_*(X) \otimes^{\mathbf{L}} C_*(Y) \cong C_*(X \times Y).$$

Here $\otimes^{\mathbf{L}}$ denotes the derived tensor product.

Künneth theorems in other homology and cohomology theories

The above statements are also true for singular cohomology and sheaf cohomology. For sheaf cohomology on an algebraic variety, Grothendieck found six spectral sequences relating the possible hyperhomology groups of two chain complexes of sheaves and the hyperhomology groups of their tensor product. (See EGA III₂, Théorème 6.7.3) Künneth theorems are also true for K-theory, cobordism, and l-adic cohomology.

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Poincaré duality theorem

In mathematics, the **Poincaré duality** theorem, named after Henri Poincaré, is a basic result on the structure of the homology and cohomology groups of manifolds. It states that if M is an n -dimensional compact oriented manifold, then the k th cohomology group of M is isomorphic to the $(n - k)$ th homology group of M , for all integers k . It further states that if mod 2 homology and cohomology is used, then the assumption of orientability can be dropped.

History

A form of Poincaré duality was first stated, without proof, by Henri Poincaré in 1893. It was stated in terms of Betti numbers: The k th and $(n - k)$ th Betti numbers of a closed (i.e. compact and without boundary) orientable n -manifold are equal. The *cohomology* concept was at that time about 40 years from being clarified. In his 1895 paper *Analysis Situs*, Poincaré tried to prove the theorem using topological intersection theory, which he had invented. Criticism of his work by Poul Heegaard led him to realize that his proof was seriously flawed. In the first two complements to *Analysis Situs*, Poincaré gave a new proof in terms of dual triangulations.

Poincaré duality did not take on its modern form until the advent of cohomology in the 1930s, when Eduard Čech and Hassler Whitney invented the cup and cap products and formulated Poincaré duality in these new terms.

Dual cell structures

Poincaré duality was classically thought of in terms of dual triangulations, which are generalizations of dual polyhedra. Given a triangulation X of an n -dimensional manifold M , one replaces each k -simplex with an $(n - k)$ -cell to produce a new decomposition Y of M . If each $(n - k)$ -cell is indeed a simplex then one says that Y is the dual triangulation of X , and the chain groups are related by isomorphisms

$$C_k(Y) \rightarrow \operatorname{Hom}_Z(C_{n-k}(X), Z).$$

Considering the tetrahedron as a triangulation of the 2-sphere, the dual triangulation of the tetrahedron is another tetrahedron. This construction does not necessarily yield another

triangulation, as the examples of the octahedron and icosahedron demonstrate. Poincaré used a (not entirely correct) method involving barycentric subdivision to show that we may always obtain a dual triangulation for compact oriented manifolds.

In more precise terms, one may describe the dual of a triangulation X as a triangulation Y such that given a k -simplex α in X , there is one $(n - k)$ -simplex in Y whose intersection number with α is 1, and such that the intersection number of α with any other $(n - k)$ -simplex of Y is 0.

The boundary operator in a chain complex can be viewed as a matrix. Let M be a closed n -manifold, X a triangulation of M , and Y the dual triangulation of X . Then one can show that the boundary operator

$$C_k(X) \rightarrow C_{k-1}(X)$$

is the transpose of the boundary operator

$$C_{n-k+1}(Y) = \text{Hom}_Z(C_{k-1}(X), Z) \rightarrow C_{n-k}(Y) = \text{Hom}_Z(C_k(X), Z).$$

Using the fact that the homology groups of a manifold are independent of the triangulation used to compute them, one can easily show that the k th and $(n - k)$ th Betti numbers of M are equal.

Modern formulation

The modern statement of the Poincaré duality theorem is in terms of homology and cohomology: if M is a closed oriented n -manifold, and k is an integer, then there is a canonically defined isomorphism from the k -th homology group $H_k(M)$ to the $(n - k)$ th cohomology group $H^{n-k}(M)$. (Here, homology and cohomology is taken with coefficients in the ring of integers, but the isomorphism holds for any coefficient ring.) Specifically, one maps an element of $H^k(M)$ to its cap product with a fundamental class of M , which will exist for oriented M .

For non-compact oriented manifolds, one has to replace cohomology by cohomology with compact support.

Homology and cohomology groups are defined to be zero for negative degrees, so Poincaré duality in particular implies that the homology and cohomology groups of orientable closed n -manifolds are zero for degrees bigger than n .

Naturality

Note that H^k is a contravariant functor while H_{n-k} is covariant. The family of isomorphisms

$$D_M : H^k(M) \rightarrow H_{n-k}(M)$$

is natural in the following sense: if

$$f : M \rightarrow N$$

is a continuous map between two oriented n -manifolds which is compatible with orientation, i.e. which maps the fundamental class of M to the fundamental class of N , then

$$D_N = f_{\square} D_M f^{\square},$$

where $f_{\square}$ and $f^{\square}$ are the maps induced by f in homology and cohomology, respectively.

Bilinear pairings formulation

Assuming M is compact boundaryless and orientable, let $\tau H_i M$ denote the torsion subgroup of $H_i M$ and let $f H_i M = H_i M / \tau H_i M$ be the free part. Then there are bilinear maps which are duality pairings

$$f H_i M \otimes f H_{n-i} M \rightarrow \mathbb{Z}$$

and

$$\tau H_i M \otimes \tau H_{n-i-1} M \rightarrow \mathbb{Q}/\mathbb{Z}.$$

$\mathbb{Q}/\mathbb{Z}$ is the quotient of the rationals by the integers (taken as an additive quotient group).

The first form is typically called the *intersection product* and the 2nd the *torsion linking form*. Assuming the manifold M is smooth, the intersection product is computed by perturbing the homology classes to be transverse and computing their oriented intersection number. For the torsion linking form, one computes the pairing of x and y by realizing nx as the boundary of some class z . The form is the fraction with numerator the transverse intersection number of z with y and denominator n .

The statement that the pairings are duality pairings means that the adjoint maps

$$f H_i M \rightarrow \text{Hom}_{\mathbb{Z}}(f H_{n-i} M, \mathbb{Z})$$

and

$$\tau H_i M \rightarrow \text{Hom}_{\mathbb{Z}}(\tau H_{n-i-1} M, \mathbb{Q}/\mathbb{Z})$$

are isomorphisms of groups.

Thus, Poincaré duality says that $f H_i M$ and $f H_{n-i} M$ are isomorphic, although there is no natural map giving the isomorphism, and similarly $\tau H_i M$ and $\tau H_{n-i-1} M$ are also isomorphic.

This approach to Poincaré duality was used by Przytycki and Yasuhara to give an elementary homotopy and diffeomorphism classification of 3-dimensional lens spaces. ^[1]

Generalizations and related results

The Poincaré-Lefschetz duality theorem is a generalisation for manifolds with boundary. In the non-orientable case, taking into account the sheaf of local orientations, one can give a statement that is independent of orientability.

Blanchfield duality is a version of Poincaré duality which provides an isomorphism between the homology of an abelian covering space of a manifold and the corresponding cohomology with compact supports. It is used to get basic structural results about the Alexander module and can be used to define the signatures of a knot.

With the development of homology theory to include K-theory and other *extraordinary* theories from about 1955, it was realised that the homology H_* could be replaced by other theories, once the products on manifolds were constructed; and there are now textbook treatments in generality.

Verdier duality is the appropriate generalization to (possibly singular) geometric objects, such as analytic spaces or schemes, while intersection homology was developed R. MacPherson and M. Goresky for stratified spaces, such as real or complex algebraic varieties, precisely so as to generalise Poincaré duality to such stratified spaces.

There are many other forms of geometric duality in algebraic topology, including Lefschetz duality, Alexander duality, Hodge duality, and S-duality (homotopy theory).

See also

- Bruhat decomposition
- Fundamental class
- Weyl group

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Tychonoff's theorem

In mathematics, **Tychonoff's theorem** states that the product of any collection of compact topological spaces is compact. The theorem is named after Andrey Nikolayevich Tychonoff, who proved it first in 1930 for powers of the closed unit interval and in 1935 stated the full theorem along with the remark that its proof was the same as for the special case. The earliest known published proof is contained in a 1937 paper of Eduard Čech.

Several texts identify Tychonoff's theorem as the single most important result in general topology [e.g. Willard, p. 120]; others allow it to share this honor with Urysohn's lemma.

Definition

The theorem crucially depends upon the precise definitions of compactness and of the product topology; in fact, Tychonoff's 1935 paper defines the product topology for the first time. Conversely, part of its importance is to give confidence that these particular definitions are the correct (i.e., most useful) ones.

Indeed, the Heine-Borel definition of compactness — that every covering of a space by open sets admits a finite subcovering — is relatively recent. More popular in the 19th and early 20th centuries was the Bolzano-Weierstrass criterion that every sequence admits a convergent subsequence, now called sequential compactness. These conditions are equivalent for metrizable spaces, but neither implies the other on the class of all topological spaces.

It is almost trivial to prove that the product of two sequentially compact spaces is sequentially compact -- one passes to a subsequence for the first component and then a subsubsequence for the second component. An only slightly more elaborate "diagonalization" argument establishes the sequential compactness of a countable product of sequentially compact spaces. However, the product of an uncountable number of copies of the closed unit interval fails to be sequentially compact.

This is a critical failure: if X is a completely regular Hausdorff space, there is a natural embedding from X into $[0,1]^{C(X,[0,1])}$, where $C(X,[0,1])$ is the set of continuous maps from X to $[0,1]$. The compactness of $[0,1]^{C(X,[0,1])}$ thus shows that every completely regular Hausdorff space embeds in a compact Hausdorff space (or, can be "compactified".) This construction is none other than the Stone-Čech compactification. Conversely, all subspaces of compact Hausdorff spaces are completely regular Hausdorff, so this characterizes the completely regular Hausdorff spaces as those which can be compactified. Such spaces are now called Tychonoff spaces.

Applications

Tychonoff's theorem is also used in the proof of the Theorem of Banach-Alaoglu and in that of Ascoli's Theorem. As a rule of thumb, any sort of construction that takes as input a fairly general object (often of an algebraic, or topological-algebraic nature) and outputs a compact space is likely to use Tychonoff: e.g., the Gelfand space of maximal ideals of a commutative C^* algebra, the Stone space of maximal ideals of a Boolean algebra, the Berkovich spectrum of a commutative Banach ring.

Proofs of Tychonoff's theorem

- 1) Tychonoff's 1930 proof used the concept of a complete accumulation point.
- 2) The theorem is a quick corollary of the Alexander subbase theorem.

More modern proofs have been motivated by the following considerations: the approach to compactness via convergence of subsequences leads to a simple and transparent proof in the case of countable index sets. However, the approach to convergence in a topological space using sequences is sufficient when the space satisfies the first axiom of countability (as metrizable spaces do), but generally not otherwise. However, the product of uncountably many metrizable spaces, each with at least two points, fails to be first countable. So it is natural to hope that a suitable notion of convergence in arbitrary spaces will lead to a compactness criterion generalizing sequential compactness in metrizable spaces that will be as easily applied to deduce the compactness of products. This has turned out to be the case.

- 3) The theory of convergence via filters, due to Henri Cartan and developed by Bourbaki in 1937, leads to the following criterion: assuming the ultrafilter lemma, a space is compact if and only if each ultrafilter on the space converges. With this in hand, the proof becomes easy: the (filter generated by the) image of an ultrafilter on the product space under any projection map is an ultrafilter on the factor space, which therefore converges, to at least one x_i . One then shows that the original ultrafilter converges to $x = (x_i)$.

Munkres gives in his textbook a reworking of the Cartan-Bourbaki proof which does not explicitly use any filter-theoretic language or preliminaries.

- 4) Similarly, the Moore-Smith theory of convergence via nets, as supplemented by Kelley's notion of a universal net, leads to the criterion that a space is compact if and only if each universal net on the space converges. This criterion leads to a proof (Kelley, 1950) of Tychonoff's theorem which is word for word identical to the Cartan/Bourbaki proof using filters, save for the repeated substitution of "universal net" for "ultrafilter base".
- 5) A proof using nets but not universal nets was given in 1992 by Paul Chernoff.

Tychonoff's theorem and the axiom of choice

All of the above proofs use the axiom of choice (AC) in some way. For instance, the second proof uses that every filter is contained in an ultrafilter (i.e., a maximal filter), and this is seen by invoking Zorn's Lemma. Zorn's Lemma is also used to prove Kelley's theorem, that every net has a universal subnet. In fact these uses of AC are essential: in 1950 Kelley proved that Tychonoff's theorem implies the axiom of choice. Note that one formulation of AC is that the Cartesian product of a family of nonempty sets is nonempty; but since the empty set is most certainly compact, the proof cannot proceed along such straightforward lines. Thus Tychonoff's theorem joins several other basic theorems (e.g. that every nonzero vector space has a basis) in being *equivalent* to AC.

On the other hand, the statement that every filter is contained in an ultrafilter does not imply AC. Indeed, it is not hard to see that it is equivalent to the Boolean prime ideal theorem (BPIT), a well-known intermediate point between the axioms of Zermelo-Fraenkel set theory (ZF) and the ZF theory augmented by the axiom of choice (ZFC). A first glance at the second proof of Tychonoff may suggest that the proof uses no more than (BPIT), in contradiction to the above. However, the spaces in which every convergent filter has a unique limit are precisely the Hausdorff spaces. In general we must select, for each element of the index set, an element of the nonempty set of limits of the projected ultrafilter base, and of course this uses AC. However, it also shows that the compactness of the product of compact Hausdorff spaces can be proved using (BPIT), and in fact the converse also holds. Studying the *strength* of Tychonoff's theorem for various restricted classes of spaces is an active area in set-theoretic topology.

The analogue of Tychonoff's theorem in pointless topology does not require any form of the axiom of choice.

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Tietze extension theorem

In topology, the **Tietze extension theorem** states that, if X is a normal topological space and

$$f : A \rightarrow \mathbf{R}$$

is a continuous map from a closed subset A of X into the real numbers carrying the standard topology, then there exists a continuous map

$$F : X \rightarrow \mathbf{R}$$

with $F(a) = f(a)$ for all a in A . F is called a *continuous extension* of f .

The theorem generalizes Urysohn's lemma and is widely applicable, since all metric spaces and all compact Hausdorff spaces are normal. It can be generalized by replacing $\mathbf{R}$ with $\mathbf{R}^J$ for some indexing set J , any retract of $\mathbf{R}^J$, or any normal absolute retract whatsoever.

The theorem is due to Heinrich Franz Friedrich Tietze.

References

- Tietze extension theorem ^[1] on PlanetMath
- Proof of Tietze extension theorem ^[2] on PlanetMath

References

[1] <http://planetmath.org/?op=getobj&from=objects&id=4215>

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Universal coefficient theorem

In mathematics, the **universal coefficient theorem** in algebraic topology establishes the relationship in homology theory between the *integral homology* of a topological space X , and its *homology with coefficients* in any abelian group A . It shows that the integral homology groups

$$H_i(X, \mathbf{Z})$$

do in a certain, definite sense determine the groups

$$H_i(X, A).$$

Here $H_\square$ might be the simplicial homology or more general singular homology theory: the result itself is a pure piece of homological algebra about chain complexes of free abelian groups. The form of the result is that other coefficients A may be used, at the cost of using a Tor functor.

For example it is common to take A to be $\mathbf{Z}/2\mathbf{Z}$, so that coefficients are modulo 2. This becomes straightforward in the absence of 2-torsion in the homology. Quite generally, the result indicates the relationship that holds between the Betti numbers b_i of X and the Betti numbers $b_{i,F}$ with coefficients in a field F . These can differ, but only when the characteristic of F is a prime number p for which there is some p -torsion in the homology.

The statement of the universal coefficient theorem runs as follows. Consider

$$H_i \otimes A$$

where H_i means $H_i(X, \mathbf{Z})$. Then there is an injective group homomorphism ι from it to $H_i(X, A)$. The theorem describes the cokernel of ι as

$$\text{Tor}(H_{i-1}, A).$$

This Tor group can therefore be described as the obstruction to ι being an isomorphism, which could be thought of as the 'expected' result.

This can be summarized saying there is a natural short exact sequence

$$0 \rightarrow H_i \otimes A \rightarrow H_i(X, A) \rightarrow \text{Tor}(H_{i-1}, A) \rightarrow 0$$

Furthermore, this is a split sequence (but the splitting is *not* natural).

There is also a **universal coefficient theorem for cohomology** involving the Ext functor, stating that there is a natural short exact sequence

$$0 \rightarrow \text{Ext}(H_{i-1}, A) \rightarrow H^i(X, A) \rightarrow \text{Hom}(H_i, A) \rightarrow 0$$

As in the homological case, the sequence splits, though not naturally.

References

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Seifert-van Kampen theorem

In mathematics, the **Seifert-van Kampen theorem** of algebraic topology, sometimes just called **van Kampen's theorem**, expresses the structure of the fundamental group of a topological space X , in terms of the fundamental groups of two open, path-connected subspaces U and V that cover X . It can therefore be used for computations of the fundamental group of spaces that are constructed out of simpler ones.

The underlying idea is that paths in X can be partitioned: into journeys through the intersection W of U and V , through U but outside V , and through V outside U . In order to move segments of paths around, by homotopy to form loops returning to a base point w in W , we should assume U , V and W are path-connected; and that W isn't empty. We assume also that U and V are open subspaces with union X .

Under these conditions, $\pi_1(U, w)$, $\pi_1(V, w)$, and $\pi_1(W, w)$, together with the inclusion homomorphisms (induced by the inclusion map):

$$I: \pi_1(W, w) \rightarrow \pi_1(U, w)$$

and

$$J: \pi_1(W, w) \rightarrow \pi_1(V, w)$$

are sufficient data to determine $\pi_1(X, w)$. The maps I and J extend to an epimorphism

$$\Phi: \pi_1(U, w) * \pi_1(V, w) \rightarrow \pi_1(X, w)$$

where $\pi_1(U, w) * \pi_1(V, w)$ is the free product of $\pi_1(U, w)$ and $\pi_1(V, w)$. The kernel of the map Φ are the loops in W that, when viewed in X , are homotopic to the trivial one at w .

The group $\pi_1(X, w)$ is therefore isomorphic to $\pi_1(U, w) * \pi_1(V, w)$ modulo such elements.

In particular, when W is simply connected (so that its fundamental group is the trivial group), the theorem says that $\pi_1(X, w)$ is isomorphic to the free product $\pi_1(U, w) * \pi_1(V, w)$.

Equivalent formulations

In the language of combinatorial group theory, $\pi_1(X, w)$ is the free product with amalgamation of those of U and V , with respect to the homomorphisms I and J (which might not be injective): given group presentations

$$\pi_1(U, w) = \langle u_1, \dots, u_k \mid \alpha_1, \dots, \alpha_l \rangle$$

$$\pi_1(V, w) = \langle v_1, \dots, v_m \mid \beta_1, \dots, \beta_n \rangle$$

$$\pi_1(W, w) = \langle w_1, \dots, w_p \mid \gamma_1, \dots, \gamma_q \rangle$$

the amalgamation can be written in terms of generators and relations as $\pi_1(X, w) = \langle u, v \mid \alpha, \beta, \gamma, I(w_r) \cdot J(w_r)^{-1} \rangle$ where each letter $u, v, w, \alpha, \beta, \gamma$ stands for the respective set of generators or relators, and the final relator means that the images of each generator w_r under the inclusions I, J are equivalent in the fundamental group of X .

In category theory, the fundamental group of X is a colimit of the diagram of those of U, V and W . More precisely, $\pi_1(X, w)$ is the pushout of the diagram.

Van Kampen's theorem for fundamental groups

Van Kampen's theorem for fundamental groups^[1]:

Let X be a topological space which is the union of the interiors of two path connected subspaces X_1, X_2 . Suppose $X_0 := X_1 \cap X_2$ is path connected. Let also $*$ $\in X_0$ and $i_k : \pi_1(X_0, *) \rightarrow \pi_1(X_k, *)$, $j_k : \pi_1(X_k, *) \rightarrow \pi_1(X, *)$ be induced by the inclusions for $k = 1, 2$. Then X is path connected and the natural morphism $\pi_1(X_1, *) * \pi_1(X_0, *) \pi_1(X_2, *) \rightarrow \pi_1(X, *)$ is an isomorphism, that is, the fundamental group of X is the free product of the fundamental groups of X_1 and X_2 with amalgamation of $\pi_1(X_0, *)$.

Usually the morphisms induced by inclusion in this theorem are not themselves injective, and the more precise version of the statement is in terms of pushouts of groups. The notion of pushout in the category of groupoids allows for a version of the theorem for the non path connected case, using the fundamental groupoid $\pi_1(X, A)$ on a set A of base points,^[2]. This groupoid consists of homotopy classes relative to the end points of paths in X joining points of $A \cap X$. In particular, if X is a contractible space, and A consists of two distinct points of X , then $\pi_1(X, A)$ is easily seen to be isomorphic to the groupoid often written $\mathcal{I}$ with two vertices and exactly one morphism between any two vertices. This groupoid plays a role in the theory of groupoids analogous to that of the group of integers in the theory of groups^[3].

Theorem: Let the topological space X be covered by the interiors of two subspaces X_1, X_2 and let A be a set which meets each path component of X_1, X_2 and $X_0 := X_1 \cap X_2$. Then A meets each path component of X and the diagram **P** of morphisms induced by inclusion

$$\begin{array}{ccc} P & \xrightarrow{p_2} & Y \\ p_1 \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

--with $P = \pi_1(X_0, A)$, $X = \pi_1(X_1, A)$, $p_2 = \pi_1(i_2)$, $p_1 = \pi_1(i_1)$, $f = \pi_1(j_1)$, $g = \pi_1(j_2)$, and $Y = Z = \pi_1(X, A)$ -- is a pushout diagram in the category of groupoids.^[4]

The interpretation of this theorem as a calculational tool for fundamental groups needs some development of 'combinatorial groupoid theory',^{[5] [6]}. This theorem implies the calculation of the fundamental group of the circle as the group of integers, since the group of integers is obtained from the groupoid $\mathcal{I}$ by identifying, in the category of groupoids, its two vertices.

There is a version of the last theorem when X is covered by the union of the interiors of a family $\{U_\lambda : \lambda \in \Lambda\}$ of subsets^{[7] [8]}. The conclusion is that if A meets each path component of all 1,2,3-fold intersections of the sets U_λ , then A meets all path components

of X and the diagram $\bigsqcup_{(\lambda, \mu) \in \Lambda^2} \pi_1(U_\lambda \cap U_\mu, A) \rightrightarrows \bigsqcup_{\lambda \in \Lambda} \pi_1(U_\lambda, A) \rightarrow \pi_1(X, A)$ of morphisms induced by inclusions is a coequaliser in the category of groupoids.

Examples

One can use Van Kampen's theorem to calculate fundamental groups for topological spaces that can be decomposed into simpler spaces. For example, consider the sphere S^2 . Pick open sets $A = S^2 - n$ and $B = S^2 - s$ where n and s denote the north and south poles respectively. Then we have the property that A , B and $A \cap B$ are open path connected sets. Thus we can see that there is a commutative diagram including $A \cap B$ into A and B and then another inclusion from A and B into S^2 and that there is a corresponding diagram of homomorphisms between the fundamental groups of each subspace. Applying Van Kampen's theorem gives the result $\pi_1(S^2) = \pi_1(A) * \pi_1(B) / \ker(\Phi)$. However A and B are both homeomorphic to $\mathbb{R}^2$ which is simply connected, so both A and B have trivial fundamental groups. It is clear from this that the fundamental group of S^2 is trivial.

A more complicated example is the calculation of the fundamental group of a genus n orientable surface S , otherwise known as the *genus n surface group*. One can construct S using its standard fundamental polygon. For the first open set A , pick a disk within the center of the polygon. Pick B to be the complement in S of the center point of A . Then the intersection of A and B is an annulus, which is known to be homotopy equivalent to (and so has the same fundamental group as) a circle. Then $\pi_1(A \cap B) = \pi_1(S^1)$, which is the integers, and $\pi_1(A) = \pi_1(D^2) = 1$. Thus the inclusion of $\pi_1(A \cap B)$ into $\pi_1(A)$ sends any generator to the trivial element. However, the inclusion of $\pi_1(A \cap B)$ into $\pi_1(B)$ is not trivial. In order to understand this, first one must calculate $\pi_1(B)$. This is easily done as one can deformation retract B (which is S with one point deleted) onto the edges labeled by $A_1 B_1 A_1^{-1} B_1^{-1} A_2 B_2 A_2^{-1} B_2^{-1} \dots A_n B_n A_n^{-1} B_n^{-1}$. This space is known to be the wedge sum of $2n$ circles (also called a bouquet of circles), which further is known to have fundamental group isomorphic to the free group with $2n$ generators, which in this case can be represented by the edges themselves: $\{A_1, B_1, \dots, A_n, B_n\}$. We now have enough information to apply Van Kampen's theorem. The generators are the loops $\{A_1, B_1, \dots, A_n, B_n\}$ (A is simply connected, so it contributes no generators) and there is exactly one relation: $A_1 B_1 A_1^{-1} B_1^{-1} A_2 B_2 A_2^{-1} B_2^{-1} \dots A_n B_n A_n^{-1} B_n^{-1} = 1$. Using generators and relations, this group is denoted

$$\langle A_1, B_1, \dots, A_n, B_n \mid A_1 B_1 A_1^{-1} B_1^{-1} \dots A_n B_n A_n^{-1} B_n^{-1} \rangle.$$

Generalizations

This theorem has been extended to the non-connected case by using the fundamental groupoid $\pi_1(X, A)$ on a set A of base points, which consists of homotopy classes of paths in X joining points of X which lie in A . The connectivity conditions for the theorem then become that A meets each path-component of U, V, W . The pushout is now in the category of groupoids. This extended theorem allows the determination of the fundamental group of the circle, and many other useful cases. For example, if the intersection W has two path components, it is convenient to let A consist of one point in each of these components. A theorem for arbitrary covers, with the restriction that A meets all three fold intersections of the sets of the cover, is given in the paper by Brown and Razak cited below. Applications of the fundamental groupoid on a set of base points to the Jordan curve theorem, Covering space, and orbit space are given in Ronald Brown's book cited below.

In the case of orbit spaces, it is convenient to take A to include all the fixed points of the action. An example here is the conjugation action on the circle.

The version that allows more than two overlapping sets but with A a singleton is also given in Allen Hatcher's book below, theorem 1.20.

In fact, we can extend van Kampen's theorem significantly further by considering the fundamental groupoid $\Pi(X)$, an element of the category of small categories whose objects are points of X and whose arrows are paths between points modulo homotopy equivalence. In this case, to determine the fundamental groupoid of a space, we need only know the fundamental groupoids of the open sets covering X as follows: create a new category in which the objects are fundamental groupoids of the open sets, with an arrow between groupoids if the domain space is a subspace of the codomain. Then van Kampen's theorem is the assertion that the fundamental groupoid of X is the colimit of the diagram. For details, see Peter May's book, chapter 2.

References to higher dimensional versions of the theorem which yield some information on homotopy types are given in an article on higher dimensional group theories and groupoids^[9].

See also

- Higher dimensional algebra
- Higher category theory
- Alexander Grothendieck
- Van Kampen
- Ronald Brown (mathematician)

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Notes

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Whitehead theorem

In homotopy theory (a branch of mathematics), the **Whitehead theorem** states that if a continuous mapping f between topological spaces X and Y induces isomorphisms on all homotopy groups, then f is a homotopy equivalence provided X and Y are connected CW complexes. This result was proved by J. H. C. Whitehead in two landmark papers from 1949, and provides a justification for working with the CW complex concept that he introduced there.

Stating it more accurately, we suppose given CW complexes X and Y , with respective base points x and y . Given a continuous mapping

$$f: X \rightarrow Y$$

such that $f(x) = y$, we consider for $n \geq 0$ the induced homomorphisms

$$f_*: \pi_n(X, x) \rightarrow \pi_n(Y, y),$$

where π_n denotes for $n \geq 1$ the n -th homotopy group. For $n = 0$ this means the mapping of the path-connected components; if we assume both X and Y are connected we can ignore this as containing no information. We say that f is a **weak homotopy equivalence** if the homomorphisms f_* are all bijective. The Whitehead theorem then states that a weak homotopy equivalence, for connected CW complexes, is an actual homotopy equivalence.

A word of caution: it is not enough to assume $\pi_n(X)$ is isomorphic to $\pi_n(Y)$ for each $n \geq 1$ in order to conclude that X and Y are homotopy equivalent. One really needs a map $f: X \rightarrow Y$ inducing such isomorphisms in homotopy. For instance, take $X = S^2 \times \mathbf{RP}^3$ and $Y = \mathbf{RP}^2 \times S^3$. Then X and Y have the same fundamental group, namely $\mathbf{Z}_2$, and the same universal cover, namely $S^2 \times S^3$; thus, they have isomorphic homotopy groups. On the other hand their homology groups are different (as can be seen from the Künneth formula); thus, X and Y are not homotopy equivalent.

The Whitehead theorem does not hold for general topological spaces or even for all subspaces of $\mathbf{R}^n$. For example, the Warsaw circle, a subset of the plane, has all homotopy groups zero, but the map from the Warsaw circle to a single point is not a homotopy equivalence. The study of possible generalizations of Whitehead's theorem to more general spaces is part of the subject of shape theory.

Generalization to model categories

In any model category, a weak equivalence between cofibrant-fibrant objects is a homotopy equivalence.

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List of important publications in mathematics

This is a list of **important publications** in mathematics, organized by field.

Some reasons why a particular publication might be regarded as important:

- **Topic creator** – A publication that created a new topic
- **Breakthrough** – A publication that changed scientific knowledge significantly
- **Introduction** – A publication that is a good introduction or survey of a topic
- **Influence** – A publication which has significantly influenced the world
- **Latest and greatest** – The current most advanced result in a topic

Algebra

Theory of equations

Hisab al-Jabr w'al-muqabala, Kitab al-Jabr wa-l-Muqabala

- Muhammad ibn Mūsā al-Khwārizmī (820)

Description: The first book on the systematic algebraic solutions of linear and quadratic equations. The book is considered to be the foundation of modern algebra and Islamic mathematics. The word "algebra" itself is derived from the *al-Jabr* in the title of the book.

Ars Magna

- Gerolamo Cardano (1545)

Description: Provided the first published methods for solving cubic and quartic equations (due to Scipione del Ferro, Niccolò Fontana Tartaglia, and Lodovico Ferrari), and exhibited the first published calculations involving non-real complex numbers.^[1]

Vollständige Anleitung zur Algebra

- Leonhard Euler (1770)

Description: Also known as Elements of Algebra, Euler's textbook on elementary algebra is one of the first to set out algebra in the modern form we would recognize today. The first volume deals with determinate equations, while the second part deals with Diophantine equations. The last section contains a proof of Fermat's Last Theorem for the case $n = 3$, making some valid assumptions regarding $\mathbb{Q}(\sqrt{-3})$ that Euler did not prove.^[2]

Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse

- Carl Friedrich Gauss (1799)

Description: Gauss' doctoral dissertation^[3], which contained a widely accepted (at the time) but incomplete proof^[4] of the fundamental theorem of algebra.

Abstract algebra

Group theory

Réflexions sur la résolution algébrique des équations

- Joseph Louis Lagrange (1770)

Description: Made the prescient observation that the roots of the Lagrange resolvent of a polynomial equation are tied to permutations of the roots of the original equation, laying a more general foundation for what had previously been an ad hoc analysis and helping motivate the later development of the theory of permutation groups, group theory, and Galois theory. The Lagrange resolvent also introduced the discrete Fourier transform of order 3.

Articles Publiés par Galois dans les Annales de Mathématiques

- Journal de Mathématiques pures et Appliquées, II (1846)

Description: Posthumous publication of the mathematical manuscripts of Évariste Galois by Joseph Liouville. Included are Galois' papers *Mémoire sur les conditions de résolubilité des équations par radicaux* and *Des équations primitives qui sont solubles par radicaux*.

Traité des substitutions et des équations algébriques

- Camille Jordan (1870)

Description: The first book on group theory, giving a then-comprehensive study of permutation groups and Galois theory. In this book, Jordan introduced the notion of a simple group and epimorphism (which he called *l'isomorphisme méridrique*)^[5], proved part of the Jordan-Hölder theorem, and discussed matrix groups over finite fields as well as the Jordan normal form.^[6]

Theorie der Transformationsgruppen

- Sophus Lie, Friedrich Engel (1888-1893).

Publication data: 3 volumes, B.G. Teubner, Verlagsgesellschaft, mbH, Leipzig, 1888-1893. Volume 1^[7], Volume 2^[8], Volume 3^[8].

Description: The first comprehensive work on transformation groups, serving as the foundation for the modern theory of Lie groups.

Solvability of groups of odd order

- Walter Feit and John Thompson (1960)

Description: Gave a complete proof of the solvability of finite groups of odd order, establishing the long-standing Burnside conjecture that all finite non-abelian simple groups are of even order. Many of the original techniques used in this paper were used in the eventual classification of finite simple groups.

Homological algebra**Homological Algebra**

- Henri Cartan and Samuel Eilenberg (1956)

Description: Provided the first fully-worked out treatment of abstract homological algebra, unifying previously disparate presentations of homology and cohomology for associative algebras, Lie algebras, and groups into a single theory.

Sur Quelques Points d'Algèbre Homologique

- Alexander Grothendieck (1957)

Description: Revolutionized homological algebra by introducing abelian categories and providing a general framework for Cartan and Eilenberg's notion of derived functors.

Algebraic geometry**Theorie der Abelschen Functionen**

- Bernhard Riemann (1857)

Publication data: *Journal für die Reine und Angewandte Mathematik*

Description: Developed the concept of Riemann surfaces and their topological properties beyond Riemann's 1851 thesis work, proved an index theorem for the genus (the original formulation of the Riemann-Hurwitz formula), proved the Riemann inequality for the dimension of the space of meromorphic functions with prescribed poles (the original formulation of the Riemann-Roch theorem), discussed birational transformations of a given curve and the dimension of the corresponding moduli space of inequivalent curves of a given genus, and solved more general inversion problems than those investigated by Abel and Jacobi. André Weil once wrote that this paper "*is one of the greatest pieces of mathematics that has ever been written; there is not a single word in it that is not of consequence.*" [9]

Faisceaux Algébriques Cohérents

- Jean-Pierre Serre

Publication data: *Annals of Mathematics*, 1955

Description: *FAC*, as it is usually called, first introduced the use of sheaves into algebraic geometry. Serre introduced Čech cohomology of sheaves in this paper, and, despite its technical deficiencies, revolutionized algebraic geometry. For example, the long exact sequence in sheaf cohomology allows one to show that some surjective maps of sheaves induce surjective maps on sections; specifically, these are the maps whose kernel (as a sheaf) has a vanishing first cohomology group. Before *FAC*, this was next to impossible.

While Grothendieck's derived functor cohomology has replaced Čech cohomology for technical reasons, actual calculations, such as of the cohomology of projective space, are usually carried out by Čech techniques, and for this reason Serre's paper remains important even today.

Géométrie Algébrique et Géométrie Analytique

- Jean-Pierre Serre (1956)

Description: In mathematics, algebraic geometry and analytic geometry are closely related subjects, where *analytic geometry* is the theory of complex manifolds and the more general analytic spaces defined locally by the vanishing of analytic functions of several complex variables. A (mathematical) theory of the relationship between the two was put in place during the early part of the 1950s, as part of the business of laying the foundations of algebraic geometry to include, for example, techniques from Hodge theory. (NB While analytic geometry as use of Cartesian coordinates is also in a sense included in the scope of algebraic geometry, that is not the topic being discussed in this article.) The major paper consolidating the theory was *Géométrie Algébrique et Géométrie Analytique* by Serre, now usually referred to as *GAGA*. A *GAGA-style result* would now mean any theorem of comparison, allowing passage between a category of objects from algebraic geometry, and their morphisms, and a well-defined subcategory of analytic geometry objects and holomorphic mappings.

Le théorème de Riemann-Roch, d'après A. Grothendieck

- Armand Borel, Jean-Pierre Serre (1958)

Description: Borel and Serre's exposition of Grothendieck's version of the Riemann Roch theorem, published after Grothendieck made it clear that he was not interested in writing up his own result. Grothendieck reinterpreted both sides of the formula that Hirzebruch proved in 1953 in the framework of morphisms between varieties, resulting in a sweeping generalization.^[10] In his proof, Grothendieck broke new ground with his concept of Grothendieck groups, which led to the development of K-theory.^[11]

Éléments de géométrie algébrique

- Alexander Grothendieck (1960-1967)

Description: Written with the assistance of Jean Dieudonné, this is Grothendieck's exposition of his reworking of the foundations of algebraic geometry. It has become the most important foundational work in modern algebraic geometry. The approach expounded in EGA, as these books are known, transformed the field and led to monumental advances.

Séminaire de géométrie algébrique

- Alexander Grothendieck et al.

Description: These seminar notes on Grothendieck's reworking of the foundations of algebraic geometry report on work done at IHÉS starting in the 1960s. SGA 1 dates from the seminars of 1960-1961, and the last in the series, SGA 7, dates from 1967-1969. In contrast to EGA, which is intended to set foundations, SGA describes ongoing research as it unfolded in Grothendieck's seminar; as a result, it is quite difficult to read, since many of the more elementary and foundational results were relegated to EGA. One of the major

results building on the results in SGA is Pierre Deligne's proof of the last of the open Weil conjectures in the early 1970s. Other authors who worked on one or several volumes of SGA include Michel Raynaud, Michael Artin, Jean-Pierre Serre, Jean-Louis Verdier, Pierre Deligne, and Nicholas Katz.

Number theory

De fractionibus continuis dissertatio

- Leonhard Euler (1744)

Description: First presented in 1737, this paper^[12] provided the first then-comprehensive account of the properties of continued fractions. It also contains the first proof that the number e is irrational.^[13]

Recherches d'Arithmétique

- Joseph Louis Lagrange (1775)

Description: Developed a general theory of binary quadratic forms to handle the general problem of when an integer is representable by the form $ax^2 + by^2 + cxy$. This included a reduction theory for binary quadratic forms, where he proved that every form is equivalent to a certain canonically chosen reduced form.^[14] ^[15]

Disquisitiones Arithmeticae

- Carl Friedrich Gauss (1801)

Description: The *Disquisitiones Arithmeticae* is a profound and masterful book on number theory written by German mathematician Carl Friedrich Gauss and first published in 1801 when Gauss was 24. In this book Gauss brings together results in number theory obtained by mathematicians such as Fermat, Euler, Lagrange and Legendre and adds many important new results of his own. Among his contributions was the first complete proof known of the Fundamental theorem of arithmetic, the first two published proofs of the law of quadratic reciprocity, a deep investigation of binary quadratic forms going beyond Lagrange's work in *Recherches d'Arithmétique*, a first appearance of Gauss sums, cyclotomy, and the theory of constructible polygons with a particular application to the constructibility of the regular 17-gon. Of note, in section V, article 303 of *Disquisitiones*, Gauss summarized his calculations of class numbers of imaginary quadratic number fields, and in fact found all imaginary quadratic number fields of class numbers 1, 2, and 3 (confirmed in 1986) as he had conjectured.^[16] In section V, article 358, Gauss proved what can be interpreted as the first non-trivial case of the Riemann Hypothesis for curves over finite fields (the Hasse-Weil theorem).^[17]

Beweis des Satzes, daß jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält

- Johann Peter Gustav Lejeune Dirichlet (1837)

Description: Pioneering paper in analytic number theory, which introduced Dirichlet characters and their L-functions to establish Dirichlet's theorem on arithmetic progressions.^[18] In subsequent publications, Dirichlet used these tools to determine, among other things, the class number for quadratic forms.

Über die Anzahl der Primzahlen unter einer gegebenen Grösse

- Bernhard Riemann (1859)

Description: *Über die Anzahl der Primzahlen unter einer gegebenen Grösse* (or *On the Number of Primes Less Than a Given Magnitude*) is a seminal 8-page paper by Bernhard Riemann published in the November 1859 edition of the *Monthly Reports of the Berlin Academy*. Although it is the only paper he ever published on number theory, it contains ideas which influenced dozens of researchers during the late 19th century and up to the present day. The paper consists primarily of definitions, heuristic arguments, sketches of proofs, and the application of powerful analytic methods; all of these have become essential concepts and tools of modern analytic number theory. It also contains the famous Riemann Hypothesis, one of the most important open problems in mathematics.

Vorlesungen über Zahlentheorie

- P.G.L. Dirichlet and Richard Dedekind

Description: *Vorlesungen über Zahlentheorie* (*Lectures on Number Theory*) is a textbook of number theory written by German mathematicians P.G.L. Dirichlet and Richard Dedekind, and published in 1863. The *Vorlesungen* can be seen as a watershed between the classical number theory of Fermat, Jacobi and Gauss, and the modern number theory of Dedekind, Riemann and Hilbert. Dirichlet does not explicitly recognise the concept of the group that is central to modern algebra, but many of his proofs show an implicit understanding of group theory

Zahlbericht

- David Hilbert (1897)

Description: Unified and made accessible many of the developments in algebraic number theory made during the nineteenth century. Although criticized by André Weil (who stated "more than half of his famous *Zahlbericht* is little more than an account of Kummer's number-theoretical work, with inessential improvements")^[19] and Emmy Noether^[20], it was highly influential for many years following its publication.

Fourier Analysis in Number Fields and Hecke's Zeta-Functions

- John Tate (1950)

Description: Generally referred to simply as *Tate's Thesis*, Tate's Princeton Ph.D. thesis, under Emil Artin, is a reworking of Erich Hecke's theory of zeta- and L -functions in terms of Fourier analysis on the adèles. The introduction of these methods into number theory made it possible to formulate extensions of Hecke's results to more general L -functions such as those arising from automorphic forms.

Automorphic Forms on $GL(2)$

- Hervé Jacquet and Robert Langlands (1970)

Description: This publication offers evidence towards Langlands' conjectures by reworking and expanding the classical theory of modular forms and their L -functions through the introduction of representation theory.

La conjecture de Weil. I.

- Pierre Deligne (1974)

Description: Proved the Riemann hypothesis for varieties over finite fields, settling the last of the open Weil conjectures.

Endlichkeitssätze für abelsche Varietäten über Zahlkörpern

- Gerd Faltings (1983)

Description: Faltings proves a collection of important results in this paper, the most famous of which is the first proof of the Mordell conjecture (a conjecture dating back to 1922). Other theorems proved in this paper include an instance of the Tate conjecture (relating the homomorphisms between two abelian varieties over a number field to the homomorphisms between their Tate modules) and some finiteness results concerning abelian varieties over number fields with certain properties.

Modular Elliptic Curves and Fermat's Last Theorem

- Andrew Wiles (1995)

Description: This article proceeds to prove a special case of the Shimura-Taniyama conjecture through the study of the deformation theory of Galois representations. This in turn implies the famed Fermat's Last Theorem. The proof's method of identification of a deformation ring with a Hecke algebra (now referred to as an $R=T$ theorem) to prove modularity lifting theorems has been an influential development in algebraic number theory.

The geometry and cohomology of some simple Shimura varieties

- Michael Harris and Richard Taylor (2001)

Description: Harris and Taylor provide the first proof of the local Langlands conjecture for $GL(n)$. As part of the proof, this monograph also makes an in depth study of the geometry and cohomology of certain Shimura varieties at primes of bad reduction.

Analysis

Introductio in analysin infinitorum

- Leonhard Euler (1748)

Description: The eminent historian of mathematics Carl Boyer once called Euler's *Introductio in analysin infinitorum* the greatest modern textbook in mathematics.^[21] Published in two volumes^{[22] [23]}, this book more than any other work succeeded in establishing analysis as a major branch of mathematics, with a focus and approach distinct from that used in geometry and algebra.^[24] Notably, Euler identified functions rather than curves to be the central focus in his book.^[25] Logarithmic, exponential, trigonometric, and transcendental functions were covered, as were expansions into partial fractions, evaluations of $\zeta(2k)$ for k a positive integer between 1 and 13, infinite series-infinite product formulas^[21], continued fractions, and partitions of integers.^[26] In this work, Euler proved that every rational number can be written as a finite continued fraction, that the continued fraction of an irrational number is infinite, and derived continued fraction expansions for e and $\sqrt{e}$.^[22] This work also contains a statement of Euler's formula and a statement of the pentagonal number theorem, which he had discovered earlier and would publish a proof for in 1751.

Calculus

Yuktibhasa

- Jyeshtadeva (1501)

Description: Written in India in 1501, this was the world's first calculus text. "This work laid the foundation for a complete system of fluxions" (Charles Whish, 1835) and served as a summary of the Kerala School's achievements in calculus, trigonometry and mathematical analysis, most of which were earlier discovered by the 14th century mathematician Madhava. It's possible that this text influenced the later development of calculus in Europe. Some of its important developments in calculus include: the fundamental ideas of differentiation and integration, the derivative, differential equations, term by term integration, numerical integration by means of infinite series, the relationship between the area of a curve and its integral, and the mean value theorem.

Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illi calculi genus

- Gottfried Leibniz (1684)

Description: Leibniz's first publication on differential calculus, containing the now familiar notation for differentials as well as rules for computing the derivatives of powers, products and quotients.

Philosophiae Naturalis Principia Mathematica

- Isaac Newton

Description: The *Philosophiae Naturalis Principia Mathematica* (Latin: "mathematical principles of natural philosophy", often *Principia* or *Principia Mathematica* for short) is a three-volume work by Isaac Newton published on July 5, 1687. Perhaps the most influential scientific book ever published, it contains the statement of Newton's laws of motion forming the foundation of classical mechanics as well as his law of universal gravitation, and derives Kepler's laws for the motion of the planets (which were first obtained empirically). Here was born the practice, now so standard we identify it with science, of explaining nature by postulating mathematical axioms and demonstrating that their conclusion are observable phenomena. In formulating his physical theories, Newton freely used his unpublished work on calculus. When he submitted *Principia* for publication, however, Newton chose to recast the majority of his proofs as geometric arguments.^[27]

Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum

- Leonhard Euler (1755)

Description: Published in two books^[28], Euler's textbook on differential calculus presented the subject in terms of the function concept, which he had introduced in his 1748 *Introductio in analysin infinitorum*. This work opens with a study of the calculus of finite differences and makes a thorough investigation of how differentiation behaves under substitutions.^[1] Also included is a systematic study of Bernoulli polynomials and the Bernoulli numbers (naming them as such), a demonstration of how the Bernoulli numbers are related to the coefficients in the Euler-Maclaurin formula and the values of $\zeta(2n)$ ^[29], a further study of Euler's constant (including its connection to the gamma function), and an application of partial fractions to differentiation.^[30]

Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe

- Bernhard Riemann (1867)

Description: Written in 1853, Riemann's work on trigonometric series was published posthumously. In it, he extended Cauchy's definition of the integral to that of the Riemann integral, allowing some functions with dense subsets of discontinuities on an interval to be integrated (which he demonstrated by an example)^[31]. He also stated the Riemann series theorem^[31], proved the Riemann-Lebesgue lemma for the case of bounded Riemann integrable functions^[32], and developed the Riemann localization principle^[33].

Intégrale, longueur, aire

- Henri Lebesgue (1901)

Description: Lebesgue's doctoral dissertation, summarizing and extending his research to date regarding his development of measure theory and the Lebesgue integral.

Complex analysis**Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse**

- Bernhard Riemann (1851)

Description: Riemann's doctoral dissertation introduced the notion of a Riemann surface, conformal mapping, simple connectivity, the Riemann sphere, the Laurent series expansion for functions having poles and branch points, and the Riemann mapping theorem.

Functional analysis**Théorie des opérations linéaires**

- Stefan Banach (1932; originally published 1931 in Polish under the title *Teorja operacyj*.)

Description: The first mathematical monograph on the subject of linear metric spaces, bringing the abstract study of functional analysis to the wider mathematical community. The book introduced the ideas of a normed space and the notion of a so-called B -space, a complete normed space. The B -spaces are now called Banach spaces and are one of the basic objects of study in all areas of modern mathematical analysis. Banach also gave proofs of versions of the open mapping theorem, closed graph theorem, and Hahn-Banach theorem.

Fourier analysis**Mémoire sur la propagation de la chaleur dans les corps solides**

- Joseph Fourier (1807)^[34]

Description: Introduced Fourier analysis, specifically Fourier series. Key contribution was to not simply use trigonometric series, but to model *all* functions by trigonometric series.

$$\varphi(y) = a \cos \frac{\pi y}{2} + a' \cos 3 \frac{\pi y}{2} + a'' \cos 5 \frac{\pi y}{2} + \dots$$

Multiplying both sides by $\cos(2i+1)\frac{\pi y}{2}$, and then integrating from $y = -1$ to $y = +1$ yields:

$$a_i = \int_{-1}^1 \varphi(y) \cos(2i+1) \frac{\pi y}{2} dy.$$

When Fourier submitted his paper in 1807, the committee (which included Lagrange, Laplace, Malus and Legendre, among others) concluded: *...the manner in which the author arrives at these equations is not exempt of difficulties and [...] his analysis to integrate them still leaves something to be desired on the score of generality and even rigour*. Making Fourier series rigorous, which in detail took over a century, lead directly to a number of developments in analysis, notably the rigorous statement of the integral via the Dirichlet integral and later the Lebesgue integral.

Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données

- Johann Peter Gustav Lejeune Dirichlet (1829, expanded German edition in 1837)

Description: In his habilitation thesis on Fourier series, Riemann characterized this work of Dirichlet as the "*the first profound paper about the subject*"^[35]. This paper gave the first rigorous proof of the convergence of Fourier series under fairly general conditions (piecewise continuity and monotonicity) by considering partial sums, which Dirichlet transformed into a particular Dirichlet integral involving what is now called the Dirichlet kernel. This paper introduced the nowhere continuous Dirichlet function and an early version of the Riemann-Lebesgue lemma.^[36]

On convergence and growth of partial sums of Fourier series

- Lennart Carleson (1966)

Description: Settled Luzin's conjecture that the Fourier expansion of any L^2 function converges almost everywhere.

Geometry

Baudhayana Sulba Sutra

- Baudhayana

Description: Written around the 8th century BC, this is one of the oldest geometrical texts. It laid the foundations of Indian mathematics and was influential in South Asia and its surrounding regions, and perhaps even Greece. Among the important geometrical discoveries included in this text are: the earliest list of Pythagorean triples discovered algebraically, the earliest statement of the Pythagorean theorem, geometric solutions of linear equations, several approximations of π , the first use of irrational numbers, and an accurate computation of the square root of 2, correct to a remarkable five decimal places. Though this was primarily a geometrical text, it also contained some important algebraic developments, including the earliest use of quadratic equations of the forms $ax^2 = c$ and $ax^2 + bx = c$, and integral solutions of simultaneous Diophantine equations with up to four unknowns.

Euclid's Elements

- Euclid

Publication data: c. 300 BC

Online version: Interactive Java version ^[37]

Description: This is often regarded as not only the most important work in geometry but one of the most important works in mathematics. It contains many important results in geometry, number theory and the first algorithm as well. More than any specific result in the publication, it seems that the major achievement of this publication is the popularization of logic and mathematical proof as a method of solving problems.

The Nine Chapters on the Mathematical Art

- Unknown author

Description: This was a Chinese mathematics book, mostly geometric, composed during the Han Dynasty, perhaps as early as 200 BC. It remained the most important textbook in China and East Asia for over a thousand years, similar to the position of Euclid's *Elements* in Europe. Among its contents: Linear problems solved using the principle known later in the West as the *rule of false position*. Problems with several unknowns, solved by a principle similar to Gaussian elimination. Problems involving the principle known in the West as the Pythagorean theorem. The earliest solution of a matrix using a method equivalent to the modern method.

The Conics

- Apollonius of Perga

Description: The Conics was written by Apollonius of Perga, a Greek mathematician. His innovative methodology and terminology, especially in the field of conics, influenced many later scholars including Ptolemy, Francesco Maurolico, Isaac Newton, and René Descartes. It was Apollonius who gave the ellipse, the parabola, and the hyperbola the names by which we know them.

La Géométrie

- René Descartes

Description: La Géométrie was published in 1637 and written by René Descartes. The book was influential in developing the Cartesian coordinate system and specifically discussed the representation of points of a plane, via real numbers; and the representation of curves, via equations.

Grundlagen der Geometrie

- David Hilbert

Publication data: Hilbert, David (1899). *Grundlagen der Geometrie*. Teubner-Verlag Leipzig.

Description: Hilbert's axiomatization of geometry, whose primary influence was in its pioneering approach to metamathematical questions including the use of models to prove axiom independence and the importance of establishing the consistency and completeness of an axiomatic system.

Regular Polytopes

- H.S.M. Coxeter

Description: *Regular Polytopes* is a comprehensive survey of the geometry of regular polytopes, the generalisation of regular polygons and regular polyhedra to higher dimensions. Originating with an essay entitled *Dimensional Analogy* written in 1923, the first edition of the book took Coxeter 24 years to complete. Originally written in 1947, the book was updated and republished in 1963 and 1973.

Differential geometry

Recherches sur la courbure des surfaces

- Leonard Euler (1760)

Publication data: Mémoires de l'académie des sciences de Berlin **16** (1760) pp. 119–143; published 1767. (Full text ^[38] and an English translation available from the Dartmouth Euler archive.)

Description: Established the theory of surfaces, and introduced the idea of principal curvatures, laying the foundation for subsequent developments in the differential geometry of surfaces.

Disquisitiones generales circa superficies curvas

- Carl Friedrich Gauss (1827)

Publication data: "Disquisitiones generales circa superficies curvas" ^[39], *Commentationes Societatis Regiae Scientiarum Gottingensis Recentiores* Vol. **VI** (1827), pp. 99–146; "General Investigations of Curved Surfaces" ^[40] (published 1965) Raven Press, New York, translated by A.M.Hiltebeitel and J.C.Morehead.

Description: Groundbreaking work in differential geometry, introducing the notion of Gaussian curvature and Gauss' celebrated Theorema Egregium.

Über die Hypothesen, welche der Geometrie zu Grunde Liegen

- Bernhard Riemann (1854)

Publication data: "Über die Hypothesen, welche der Geometrie zu Grunde Liegen" ^[41], *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, Vol. 13, 1867.

Description: Riemann's famous Habilitationsvortrag, in which he introduced the notions of a manifold, Riemannian metric, and curvature tensor.

Leçons sur la théorie générale des surfaces

- Gaston Darboux

Publication data: Darboux, Gaston (1887,1889,1896). *Leçons sur la théorie générale des surfaces: Volume I* ^[42], *Volume II* ^[43], *Volume III* ^[44], *Volume IV* ^[44]. Gauthier-Villars.

Description: A treatise covering virtually every aspect of the 19th century differential geometry of surfaces.

Topology

Analysis situs

- Henri Poincaré (1895, 1899-1905)

Description: Poincaré's *Analysis situs* and his *Compléments à l'Analysis Situs* laid the general foundations for algebraic topology. In these papers, Poincaré introduced the notions of homology and the fundamental group, provided an early formulation of Poincaré duality, gave the Euler-Poincaré characteristic for chain complexes, and mentioned several important conjectures including the Poincaré conjecture.

Grundzüge der Mengenlehre

- Felix Hausdorff (1914)
- Reprinted and commented in: *Gesammelte Werke Bd. 2* / Herausgegeben von E. Brieskorn, S.D.Chatterji *et al.* – Berlin, 2002

Description: This book founded (general) topology by giving the axioms for a (Hausdorff) topological space.

L'anneau d'homologie d'une représentation, Structure de l'anneau d'homologie d'une représentation

- Jean Leray (1946)

Description: These two Comptes Rendus notes of Leray from 1946 introduced the novel concepts of sheafs, sheaf cohomology, and spectral sequences, which he had developed during his years of captivity as a prisoner of war. Leray's announcements and applications (published in other Comptes Rendus notes from 1946) drew immediate attention from other mathematicians. Subsequent clarification, development, and generalization by Henri Cartan, Jean-Louis Koszul, Armand Borel, Jean-Pierre Serre, and Leray himself allowed these concepts to be understood and applied to many other areas of mathematics.^[45] Dieudonné would later write that these notions created by Leray "*undoubtedly rank at the same level in the history of mathematics as the methods invented by Poincaré and Brouwer*".^[11]

Quelques propriétés globales des variétés différentiables

- René Thom (1954)

Description: In this paper, Thom proved the Thom transversality theorem, introduced the notions of oriented and unoriented cobordism, and demonstrated that cobordism groups could be computed as the homotopy groups of certain Thom spaces. Thom completely characterized the unoriented cobordism ring and achieved strong results for several problems, including Steenrod's problem on the realization of cycles.^[11] [46]

Category theory

General theory of natural equivalences

- Samuel Eilenberg and Saunders Mac Lane (1945)

Description: The first paper on category theory. Mac Lane later wrote in *Categories for the Working Mathematician* that he and Eilenberg introduced categories so that they could introduce functors, and they introduced functors so that they could introduce natural equivalences. Prior to this paper, "natural" was used in an informal and imprecise way to designate constructions that could be made without making any choices. Afterwards, "natural" had a precise meaning which occurred in a wide variety of contexts and had powerful and important consequences.

Categories for the Working Mathematician

- Saunders Mac Lane (1971, second edition 1998)

Description: Saunders Mac Lane, one of the founders of category theory, wrote this exposition to bring categories to the masses. Mac Lane does not get lost in pointless abstraction, but instead brings to the fore the important concepts that make category theory useful, such as adjoint functors and universal properties. His text is more comprehensive than most mathematicians will ever need, and consequently is also an excellent reference.

Set theory

Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen

- Georg Cantor (1874)

Description: Contains the first proof that the set of all real numbers is uncountable; also contains a proof that the set of algebraic numbers is denumerable.

Grundzüge der Mengenlehre

- Felix Hausdorff

Description: First published in 1914, this was the first comprehensive introduction to set theory. Besides the systematic treatment of known results in set theory, the book also contains chapters on measure theory and topology, which were then still considered parts of set theory. Here Hausdorff presents and develops highly original material which was later to become the basis for those areas.

The consistency of the axiom of choice and of the generalized continuum-hypothesis with the axioms of set theory

- Kurt Gödel (1938)

Description: Gödel proves the results of the title. Also, in the process, introduces the class L of constructible sets, a major influence in the development of axiomatic set theory.

The Independence of the Continuum Hypothesis

- Paul J. Cohen (1963, 1964)

Description: Cohen's breakthrough work proved the independence of the continuum hypothesis and axiom of choice with respect to Zermelo-Fraenkel set theory. In proving this Cohen introduced the concept of *forcing* which led to many other major results in axiomatic set theory.

Logic

Begriffsschrift

- Gottlob Frege (1879)

Description: Published in 1879, the title ***Begriffsschrift*** is usually translated as *concept writing* or *concept notation*; the full title of the book identifies it as "*a formula language, modelled on that of arithmetic, of pure thought*". Frege's motivation for developing his formal logical system was similar to Leibniz's desire for a *calculus ratiocinator*. Frege defines a logical calculus to support his research in the foundations of mathematics. ***Begriffsschrift*** is both the name of the book and the calculus defined therein. It was arguably the most significant publication in logic since Aristotle.

Formulario mathematico

- Giuseppe Peano (1895)

Description: First published in 1895, the **Formulario mathematico** was the first mathematical book written entirely in a formalized language. It contained a description of mathematical logic and many important theorems in other branches of mathematics. Many of the notations introduced in the book are now in common use.

Principia Mathematica

- Bertrand Russell and Alfred North Whitehead (1910-1913)

Description: The ***Principia Mathematica*** is a three-volume work on the foundations of mathematics, written by Bertrand Russell and Alfred North Whitehead and published in 1910-1913. It is an attempt to derive all mathematical truths from a well-defined set of axioms and inference rules in symbolic logic. The questions remained whether a contradiction could be derived from the Principia's axioms, and whether there exists a mathematical statement which could neither be proven nor disproven in the system. These questions were settled, in a rather surprising way, by Gödel's incompleteness theorem in 1931.

Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I

- Kurt Gödel (1931)

Online version: Online version ^[47]

Description: In mathematical logic, **Gödel's incompleteness theorems** are two celebrated theorems proved by Kurt Gödel in 1931. The first incompleteness theorem states:

For any formal system such that (1) it is ω -consistent (omega-consistent), (2) it has a recursively definable set of axioms and rules of derivation, and (3) every recursive relation of natural numbers is definable in it, there exists a formula of the system such that, according to the intended interpretation of the system, it expresses a truth about natural numbers and yet it is not a theorem of the system.

Combinatorics

On sets of integers containing no k elements in arithmetic progression

- Endre Szemerédi (1975)

Description: Settled a conjecture of Paul Erdős and Paul Turán that if a sequence of natural numbers has positive upper density then it contains arbitrarily long arithmetic progressions. Szemerédi's solution has been described as a "masterpiece of combinatorics"^[48] and it introduced new ideas and tools to the field including the Szemerédi regularity lemma.

Graph theory

Solutio problematis ad geometriam situs pertinentis

- Leonhard Euler (1741)
- Euler's original publication ^[49] (in Latin)

Description: Euler's solution of the Königsberg bridge problem in *Solutio problematis ad geometriam situs pertinentis* (*The solution of a problem relating to the geometry of position*) is considered to be the first theorem of graph theory.

On the evolution of random graphs

- Paul Erdős and Alfréd Rényi (1960)

Description: Provides a detailed discussion of sparse random graphs, including distribution of components, occurrence of small subgraphs, and phase transitions.^[50]

Network Flows and General Matchings

- Ford, L., & Fulkerson, D.
- Flows in Networks. Prentice-Hall, 1962.

Description: Ford and Fulkerson paper on Network Flows. The algorithm along with many ideas on flow-based models can be found in their book.

Complexity theory

See List of important publications in computer science.

Probability theory

See list of important publications in statistics.

Game theory

Zur Theorie der Gesellschaftsspiele

- John von Neumann (1928)

Description: Went well beyond Émile Borel's initial investigations into strategic two-person game theory by proving the minimax theorem for two-person, zero-sum games.

Theory of Games and Economic Behavior

- Oskar Morgenstern, John von Neumann (1944)

Description: This book led to the investigation of modern game theory as a prominent branch of mathematics. This profound work contained the method for finding optimal solutions for two-person zero-sum games.

Equilibrium Points in N-person Games

- John Forbes Nash
- *Proceedings of the National Academy of Sciences* 36 (1950), 48–49. MR0031701 ^[51]
- "Equilibrium Points in N-person Games" ^[52]

Description: Nash equilibrium

On Numbers and Games

- John Horton Conway

Description: The book is in two, $\{0,1\}$, parts. The zeroth part is about numbers, the first part about games - both the values of games and also some real games that can be played such as Nim, Hackenbush, Col and Snort amongst the many described.

Winning Ways for your Mathematical Plays

- Elwyn Berlekamp, John Conway and Richard K. Guy

Description: A compendium of information on mathematical games. It was first published in 1982 in two volumes, one focusing on Combinatorial game theory and surreal numbers, and the other concentrating on a number of specific games.

Fractals

How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension

- Benoît Mandelbrot

Description: A discussion of self-similar curves that have fractional dimensions between 1 and 2. These curves are examples of fractals, although Mandelbrot does not use this term in the paper, as he did not coin it until 1975. Shows Mandelbrot's early thinking on fractals, and is an example of the linking of mathematical objects with natural forms that was a theme of much of his later work.

Numerical analysis

Numerical linear algebra

The Algebraic Eigenvalue Problem

- James H. Wilkinson (1965)

Description:

Optimization

Method of Fluxions

- Isaac Newton

Description: *Method of Fluxions* was a book written by Isaac Newton. The book was completed in 1671, and published in 1736. Within this book, Newton describes a method (the Newton-Raphson method) for finding the real zeroes of a function.

Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies

- Joseph Louis Lagrange (1761)

Description: Major early work on the calculus of variations, building upon some of Lagrange's prior investigations as well as those of Euler. Contains investigations of minimal surface determination as well as the initial appearance of Lagrange multipliers.

The New Variational Method

- Leonid Kantorovich
- This work was published in 1930's in the erstwhile Soviet Republic.

Description: Kantorovich wrote the first paper on production planning, which used Linear Programs as the model. He proposed the simplex algorithm as a systematic procedure to solve these Linear Programs. He received the Nobel prize for this work in 1975.

Decomposition Principle for Linear Programs

- George Dantzig and P. Wolfe
- Operations Research 8:101-111, 1960.

Description: Dantzig's is considered the father of Linear Programming in the western world. He independently invented the simplex algorithm. Dantzig and Wolfe worked on decomposition algorithms for large scale linear programs in factory and production planning.

How good is the simplex algorithm?

- Victor Klee and George J. Minty
- In: O. Shisha (ed.) *Inequalities III*, Academic Press (1972) 159–175.

Description: Klee and Minty gave an example showing that the simplex method can take exponentially many steps to solve a linear program if it chooses the greedy ascent rule.

Linear Programming and Polynomial time algorithms

- L. Khachiyan
- Doklady Akademii Nauk SSSR 244 (1979) pp. 1093–1096 (Russian).

Description: Khachiyan's work on Ellipsoid method. This was the first polynomial time algorithm for Linear programming.

New polynomial-time algorithm for linear programming

- Karmarkar, N.
- *Combinatorica* 4, 373–395, 1984.

Description: Karmarkar's path-breaking work on Interior-Point algorithms for Linear Programming.

Interior Point Polynomial Algorithms in Convex Programming

- Yurii Nesterov and A. Nemirovski
- Philadelphia : Society for Industrial and Applied Mathematics, 1994. (SIAM Studies in Applied Mathematics).

Description: Nesterov and Nemirovski's work on self-concordant barriers and interior-point methods for general convex programming. All their series of papers (both individual and combined) is compiled more coherently in the following "bible" of convex optimization.

Early manuscripts

These are publications that are not necessarily relevant to a mathematician nowadays, but are nonetheless important publications in the history of mathematics.

Rhind Mathematical Papyrus

- Ahmes (scribe)

Description: It is one of the oldest mathematical texts, dating to the Second Intermediate Period of ancient Egypt. It was copied by the scribe Ahmes (properly *Ahmose*) from an older Middle Kingdom papyrus. It laid the foundations of Egyptian mathematics and in turn, later influenced Greek and Hellenistic mathematics. Besides describing how to obtain an approximation of π only missing the mark by less than one per cent, it describes one of the earliest attempts at squaring the circle and in the process provides persuasive evidence against the theory that the Egyptians deliberately built their pyramids to enshrine the value of π in the proportions. Even though it would be a strong overstatement to suggest that the papyrus represents even rudimentary attempts at analytical geometry, Ahmes did make use of a kind of an analogue of the cotangent.

Archimedes Palimpsest

- Archimedes of Syracuse

Description: Although the only mathematical tools at its author's disposal were what we might now consider secondary-school geometry, he used those methods with rare brilliance, explicitly using infinitesimals to solve problems that would now be treated by integral calculus. Among those problems were that of the center of gravity of a solid hemisphere, that of the center of gravity of a frustum of a circular paraboloid, and that of the area of a region bounded by a parabola and one of its secant lines. For explicit details of the method used, see Archimedes' use of infinitesimals.

The Sand Reckoner

- Archimedes of Syracuse

Online version: Online version ^[53]

Description: The first known (European) system of number-naming that can be expanded beyond the needs of everyday life.

Textbooks

Synopsis of Pure Mathematics

- G. S. Carr

Description: Contains over 6000 theorems of mathematics, assembled by George Shoobridge Carr for the purpose of training students in the art of mathematics, studied extensively by Ramanujan. (first half here) ^[54] It was one of the few books that attempts to summarize the entirety of known mathematics.

Arithmetick: or, The Grounde of Arts

- Robert Recorde

Description: Written in 1542, it was the first really popular arithmetic book written in the English Language.

Cocker's Arithmetick

- Edward Cocker (authorship disputed)

Description: Textbook of arithmetic published in 1678 by John Hawkins, who claimed to have edited manuscripts left by Edward Cocker, who had died in 1676. This influential mathematics textbook used to teach arithmetic in schools in the United Kingdom for over 150 years.

The Schoolmaster's Assistant, Being a Compendium of Arithmetic both Practical and Theoretical

- Thomas Dilworth

Description: An early and popular English arithmetic textbook published in America in the eighteenth century. The book reached from the introductory topics to the advanced in five sections.

Course of Pure Mathematics

- G. H. Hardy

Online version: Online version ^[55]

Description: A classic textbook in introductory mathematical analysis, written by G. H. Hardy. It was first published in 1908, and went through many editions. It was intended to help reform mathematics teaching in the UK, and more specifically in the University of Cambridge, and in schools preparing pupils to study mathematics at Cambridge. As such, it was aimed directly at "scholarship level" students — the top 10% to 20% by ability. The book contains a large number of difficult problems. The content covers introductory calculus and the theory of infinite series.

Moderne Algebra

- B. L. van der Waerden

Description: The first introductory textbook (graduate level) expounding the abstract approach to algebra developed by Emil Artin and Emmy Noether. First published in German in 1931 by Springer Verlag. A later English translation was published in 1949 by Frederick Ungar Publishing Company.

Algebra

- Saunders Mac Lane and Garrett Birkhoff

Description: A definitive introductory text for abstract algebra using a category theoretic approach. Both a rigorous introduction from first principles, and a reasonably comprehensive survey of the field.

Algebraic Geometry

- Robin Hartshorne

Description: The first comprehensive introductory (graduate level) text in algebraic geometry that used the language of schemes and cohomology. Published in 1977, it lacks aspects of the scheme language which are nowadays considered central, like the functor of points.

Naive Set Theory

- Paul Halmos

Description: An undergraduate introduction to not-very-naive set theory which has lasted for decades. It is still considered by many to be the best introduction to set theory for beginners. While the title states that it is naive, which is usually taken to mean without axioms, the book does introduce all the axioms of Zermelo-Fraenkel set theory and gives correct and rigorous definitions for basic objects. Where it differs from a "true" axiomatic set theory book is its character: There are no long-winded discussions of axiomatic minutiae, and there is next to nothing about advanced topics like large cardinals. Instead it tried, and succeeds, in being intelligible to someone who has never thought about set theory before.

Cardinal and Ordinal Numbers

- Waclaw Sierpinski

Description: The *nec plus ultra* reference for basic facts about cardinal and ordinal numbers. If you have a question about the cardinality of sets occurring in everyday mathematics, the first place to look is this book, first published in the early 1950s but based on the author's lectures on the subject over the preceding 40 years.

Set Theory: An Introduction to Independence Proofs

- Kenneth Kunen

Description: This book is not really for beginners, but graduate students with some minimal experience in set theory and formal logic will find it a valuable self-teaching tool, particularly in regard to forcing. It is far easier to read than a true reference work such as Jech, *Set Theory*. It may be the best textbook from which to learn forcing, though it has the disadvantage that the exposition of forcing relies somewhat on the earlier presentation of Martin's axiom.

Topologie

- Pavel Sergeevich Alexandrov
- Heinz Hopf

Description: First published round 1935, this text was a pioneering "reference" text book in topology, already incorporating many modern concepts from set-theoretic topology, homological algebra and homotopy theory.

General Topology

- John L. Kelley

Description: First published in the mid-1950s, for many years the only introductory graduate level textbook in the U.S.A. teaching the basics of point set, as opposed to algebraic, topology. Prior to this the material, essential for advanced study in many fields, was only available in bits and pieces from texts on other topics or journal articles.

Topology from the Differentiable Viewpoint

- John Milnor

Description: This short book introduces the main concepts of differential topology in Milnor's lucid and concise style. While the book does not cover very much, its topics are explained beautifully in a way that illuminates all their details.

Number Theory, An approach through history from Hammurapi to Legendre

- André Weil

Description: An historical study of number theory, written by one of the 20th century's greatest researchers in the field. The book covers some thirty six centuries of arithmetical work but the bulk of it is devoted to a detailed study and exposition of the work of Fermat, Euler, Lagrange, and Legendre. The author wishes to take the reader into the workshop of his subjects to share their successes and failures. A rare opportunity to see the historical development of a subject through the mind of one of its greatest practitioners.

An Introduction to the Theory of Numbers

- G. H. Hardy and E. M. Wright

Description: This book was first published in 1938, and is still in print, with the latest edition being the 6th (2008). It is likely that almost every serious student and researcher into number theory has consulted this book, and probably has it on their bookshelf. It was not intended to be a textbook, and is rather an introduction to a wide range of differing areas of number theory which would now almost certainly be covered in separate volumes. The writing style has long been regarded as exemplary, and the approach gives insight into a variety of areas without requiring much more than a good grounding in algebra, calculus and complex numbers.

Popular writing

Gödel, Escher, Bach

- Douglas Hofstadter

Description: Gödel, Escher, Bach: an Eternal Golden Braid is a Pulitzer Prize-winning book, first published in 1979 by Basic Books. It is a book about how the creative achievements of logician Kurt Gödel, artist M. C. Escher and composer Johann Sebastian Bach interweave. As the author states: "I realized that to me, Gödel and Escher and Bach were only shadows cast in different directions by some central solid essence. I tried to reconstruct the central object, and came up with this book."

The World of Mathematics

- James R. Newman

Description: The World of Mathematics was specially designed to make mathematics more accessible to the inexperienced. It comprises nontechnical essays on every aspect of the vast subject, including articles by and about scores of eminent mathematicians, as well as literary figures, economists, biologists, and many other eminent thinkers. Includes the work of Archimedes, Galileo, Descartes, Newton, Gregor Mendel, Edmund Halley, Jonathan

Swift, John Maynard Keynes, Henri Poincaré, Lewis Carroll, George Boole, Bertrand Russell, Alfred North Whitehead, John von Neumann, and many others. In addition, an informative commentary by distinguished scholar James R. Newman precedes each essay or group of essays, explaining their relevance and context in the history and development of mathematics. Originally published in 1956, it does not include many of the exciting discoveries of the later years of the 20th century but it has no equal as a general historical survey of important topics and applications.

See also

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Higher Dimensional Algebra

Higher-dimensional algebra

This article is about **higher-dimensional algebra and supercategories** in generalized category theory, super-category theory, and also its extensions in metamathematics^[1].

In **higher-dimensional algebra**, a double groupoid is a generalisation of a one-dimensional groupoid to two dimensions^[2], and the latter groupoid can be considered as a special case of a category with all invertible arrows, or morphisms.

Double groupoids are often used to capture information about geometrical objects such as higher-dimensional manifolds (or n-dimensional manifolds)^[3]. In general, an n-dimensional manifold is a space that locally looks like an n-dimensional Euclidean space, but whose global structure may be non-Euclidean. A first step towards defining higher dimensional algebras is the concept of 2-category, followed by the more 'geometric' concept of double category^{[4][5][6]}.

A higher level concept is that of a category of categories, or **super-category** which generalises to higher dimensions the notion of category -regarded as any structure which is an interpretation of Lawvere's axioms of the *elementary theory of abstract categories* (ETAC)^{[7][8][9][10]}. Thus, a supercategory and also a super-category, can be regarded as natural extensions of the concepts of meta-category,^[11] multicategory, and multi-graph, k-partite graph, or colored graph (see a color figure, and also its definition in graph theory).

Double groupoids were first introduced by Ronald Brown in 1976, in ref.^[12] and were further developed towards applications in nonabelian algebraic topology^{[13][14][15][16]}.

Supercategories were first introduced in 1970,^[17] and were subsequently developed for applications in Theoretical Physics (especially Quantum Field Theory and Topological quantum field theory) and Mathematical Biology or Mathematical Biophysics.^[18]

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See also

- Higher category theory
 - Category theory
 - Algebraic topology
 - Seifert-van Kampen theorem
 - Abstract algebra
 - Categorical algebra
 - Esquisse d'un Programme
 - Grothendieck's Galois theory
 - Metatheory
 - Metalogic
 - Metamathematics
 - Colored graphs
 - Multicategory
 - Enriched category
-

Higher category theory

Higher category theory is the part of category theory at a *higher order*, which means that some equalities are replaced by explicit arrows in order to be able to explicitly study the structure behind those equalities.

Strict higher categories

N-categories are defined inductively using the enriched category theory: 0-categories are sets and (n+1)-categories are categories enriched over the monoidal category of n-categories (with the monoidal structure given by finite products).^[1] This construction is well defined, as shown in the article on n-categories. This concept introduces higher arrows, higher compositions and higher identities, which must well behave together. For example, the category of small categories is in fact a 2-category, with natural transformations as second degree arrows. However this concept is too strict for some purposes (for example, homotopy theory), where "weak" structures arise in the form of higher categories.^[2]

Weak higher categories

In weak n-categories, the associativity and identity conditions are not strict anymore (that is, they are not given by equalities) but they are satisfied up to an isomorphism of the next level. An example in topology is the composition of paths which is associative only up to homotopy. These isomorphisms must well behave between them and expressing this is the difficulty in the definition of weak n-categories. Weak 2-categories, also called bicategories were the first to be defined explicitly. A particularity of these is that a bicategory with one object is exactly a monoidal category, which makes say that bicategories are "monoidal categories with many objects". Weak 3-categories, also called tricategories, are harder to define explicitly, and so on. Several definitions have been given, and telling when they are equivalent, and in what sense, has become a new object of study in category theory.

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See also

- Network Science
- Polytely
- Higher-dimensional algebra

External links

- John Baez Tale of n -Categories (<http://math.ucr.edu/home/baez/week73.html>)

Enriched category

In category theory and its applications to mathematics, an **enriched category** is a category whose hom-sets are replaced by objects from some other category, in a well-behaved manner.

Definition

We define here what it means for $\mathbf{C}$ to be an **enriched category** over a monoidal category $(\mathbf{M}, \otimes, I)$.

We require the following structures:

- Let $\text{Ob}(\mathbf{C})$ be a set (or proper class, if you prefer). An element of $\text{Ob}(\mathbf{C})$ is called an *object* of $\mathbf{C}$.
- For each pair (A, B) of objects of $\mathbf{C}$, let $\text{Hom}(A, B)$ be an object of $\mathbf{M}$, called the *hom-object* of A and B .
- For each object A of $\mathbf{C}$, let id_A be a morphism in $\mathbf{M}$ from I to $\text{Hom}(A, A)$, called the *identity morphism* of A .
- For each triple (A, B, C) of objects of $\mathbf{C}$, let

$$\circ : \text{Hom}(B, C) \otimes \text{Hom}(A, B) \rightarrow \text{Hom}(A, C)$$

be a morphism in $\mathbf{M}$ called the *composition* morphism of A , B , and C .

We require the following axioms:

- **Associativity:** Given objects A , B , C , and D of $\mathbf{C}$, we can go from $\text{Hom}(C, D) \square \text{Hom}(B, C) \square \text{Hom}(A, B)$ to $\text{Hom}(A, D)$ in two ways, depending on which composition we do first. These must give the same result.

$$\begin{array}{ccccc}
 (\text{Hom}(C, D) \otimes \text{Hom}(B, C)) \otimes \text{Hom}(A, B) & \xrightarrow{\alpha} & \text{Hom}(C, D) \otimes (\text{Hom}(B, C) \otimes \text{Hom}(A, B)) & \xrightarrow{1 \otimes \circ} & \text{Hom}(C, D) \otimes \text{Hom}(A, C) \\
 \downarrow \circ \otimes 1 & & & & \downarrow \circ \\
 \text{Hom}(B, D) \otimes \text{Hom}(A, B) & \xrightarrow{\quad \quad \quad \circ \quad \quad \quad} & & & \text{Hom}(A, D)
 \end{array}$$

- **Left identity:** Given objects A and B of $\mathbf{C}$, we can go from $I \square \text{Hom}(A, B)$ to just $\text{Hom}(A, B)$ in two ways, either by using id_A on I and then using composition, or by simply using the fact that I is an identity for $\square$ in $\mathbf{M}$. These must give the same result.
- **Right identity:** Given objects A and B of $\mathbf{C}$, we can go from $\text{Hom}(A, B) \square I$ to just $\text{Hom}(A, B)$ in two ways, either by using id_B on I and then using composition, or by simply using the fact that I is an identity for $\square$ in $\mathbf{M}$. These must give the same result.

Then $\mathbf{C}$ (consisting of all the structures listed above) is a category enriched over $\mathbf{M}$.

Examples

The most straightforward example is to take $\mathbf{M}$ to be a category of sets, with the Cartesian product for the monoidal operation. Then $\mathbf{C}$ is nothing but an ordinary category. If $\mathbf{M}$ is the category of small sets, then $\mathbf{C}$ is a locally small category, because the hom-sets will all be small. Similarly, if $\mathbf{M}$ is the category of finite sets, then $\mathbf{C}$ is a locally finite category.

If $\mathbf{M}$ is the category $\mathbf{2}$ with $\text{Ob}(\mathbf{2}) = \{0, 1\}$, a single nonidentity morphism (from 0 to 1), and ordinary multiplication of numbers as the monoidal operation, then $\mathbf{C}$ can be interpreted as a preordered set. Specifically, $A \leq B$ iff $\text{Hom}(A, B) = 1$.

If $\mathbf{M}$ is a category of pointed sets with smash product for the monoidal operation, then $\mathbf{C}$ is a category with zero morphisms. Specifically, the zero morphism from A to B is the special point in the pointed set $\text{Hom}(A, B)$.

If $\mathbf{M}$ is a category of abelian groups with tensor product as the monoidal operation, then $\mathbf{C}$ is a preadditive category.

A property

If there is a monoidal functor from a monoidal category $\mathbf{M}$ to a monoidal category $\mathbf{N}$, then any category enriched over $\mathbf{M}$ can be reinterpreted as a category enriched over $\mathbf{N}$. Every monoidal category $\mathbf{M}$ has a monoidal functor $\mathbf{M}(I, -)$ to the category of sets, so any enriched category has an underlying ordinary category. In many examples (such as those above) this functor is faithful, so a category enriched over $\mathbf{M}$ can be described as an ordinary category with certain additional structure or properties.

Enriched functors

An **enriched functor** is the appropriate generalization of the notion of a functor to enriched categories. Enriched functors are then maps between enriched categories which respect the enriched structure.

If C and D are $\mathbf{M}$ -categories (that is, categories enriched over monoidal category $\mathbf{M}$), an $\mathbf{M}$ -enriched functor $T: C \rightarrow D$ is a map which assigns to each object of C an object of D and for each pair of objects a and b in C provides a morphism in $\mathbf{M}$ $T_{ab}: C(a, b) \rightarrow D(T(a), T(b))$ between the hom-objects of C and D (which are objects in $\mathbf{M}$), satisfying enriched versions of the axioms of a functor, viz preservation of identity and composition.

Because the hom-objects need not be sets in an enriched category, one cannot speak of a particular morphism. There is no longer any notion of an identity morphism, nor of a particular composition of two morphisms. Instead, morphisms from the unit to a hom-object should be thought of as selecting an identity and morphisms from the monoidal product should be thought of as composition. The usual functorial axioms are replaced with corresponding commutative diagrams involving these morphisms.

In detail, one has that the diagram

$$\begin{array}{ccc}
 & I & \\
 \text{id}_a \swarrow & & \searrow \text{id}_b \\
 C(a, a) & \xrightarrow{T_{aa}} & D(T(a), T(a))
 \end{array}$$

commutes, which amounts to the equation

$$T_{aa} \circ \text{id}_a = \text{id}_{T(a)},$$

where I is the unit object of $\mathbf{M}$. This is analogous to the rule $F(\text{id}_a) = \text{id}_{F(a)}$ for ordinary functors. Additionally, one demands that the diagram

$$\begin{array}{ccc} C(b, c) \otimes C(a, b) & \xrightarrow{\circ} & C(a, c) \\ \downarrow T_{bc} \otimes T_{ab} & & \downarrow T_{ac} \\ D(T(b), T(c)) \otimes D(T(a), T(b)) & \xrightarrow{\circ} & D(T(a), T(c)) \end{array}$$

commute, which is analogous to the rule $F(fg) = F(f)F(g)$ for ordinary functors.

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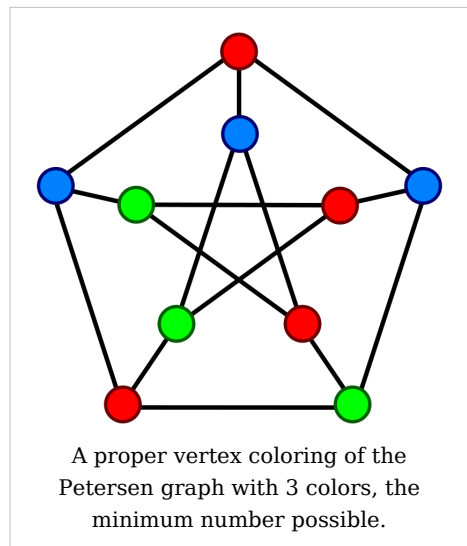
- [1] <http://www.tac.mta.ca/tac/reprints/articles/10/tr10.pdf>

Graph coloring

In graph theory, **graph coloring** is a special case of graph labeling; it is an assignment of labels traditionally called "colors" to elements of a graph subject to certain constraints. In its simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color; this is called a **vertex coloring**. Similarly, an **edge coloring** assigns a color to each edge so that no two adjacent edges share the same color, and a **face coloring** of a planar graph assigns a color to each face or region so that no two faces that share a boundary have the same color.

Vertex coloring is the starting point of the subject, and other coloring problems can be transformed into a vertex version. For example, an edge coloring of a graph is just a vertex coloring of its line graph, and a face coloring of a planar graph is just a vertex coloring of its planar dual. However, non-vertex coloring problems are often stated and studied *as is*. That is partly for perspective, and partly because some problems are best studied in non-vertex form, as for instance is edge coloring.

The convention of using colors originates from coloring the countries of a map, where each face is literally colored. This was generalized to coloring the faces of a graph embedded in



the plane. By planar duality it became coloring the vertices, and in this form it generalizes to all graphs. In mathematical and computer representations it is typical to use the first few positive or nonnegative integers as the "colors". In general one can use any finite set as the "color set". The nature of the coloring problem depends on the number of colors but not on what they are.

Graph coloring enjoys many practical applications as well as theoretical challenges. Beside the classical types of problems, different limitations can also be set on the graph, or on the way a color is assigned, or even on the color itself. It has even reached popularity with the general public in the form of the popular number puzzle Sudoku. Graph coloring is still a very active field of research.

Note: Many terms used in this article are defined in the glossary of graph theory.

History

The first results about graph coloring deal almost exclusively with planar graphs in the form of the coloring of *maps*. While trying to color a map of the counties of England, Francis Guthrie postulated the four color conjecture, noting that four colors were sufficient to color the map so that no regions sharing a common border received the same color. Guthrie's brother passed on the question to his mathematics teacher Augustus de Morgan at University College, who mentioned it in a letter to William Hamilton in 1852. Arthur Cayley raised the problem at a meeting of the London Mathematical Society in 1879. The same year, Alfred Kempe published a paper that claimed to establish the result, and for a decade the four color problem was considered solved. For his accomplishment Kempe was elected a Fellow of the Royal Society and later President of the London Mathematical Society.^[1]

In 1890, Heawood pointed out that Kempe's argument was wrong. However, in that paper he proved the five color theorem, saying that every planar map can be colored with no more than *five* colors, using ideas of Kempe. In the following century, a vast amount of work and theories were developed to reduce the number of colors to four, until the four color theorem was finally proved in 1976 by Kenneth Appel and Wolfgang Haken. Perhaps surprisingly, the proof went back to the ideas of Heawood and Kempe and largely disregarded the intervening developments.^[2] The proof of the four color theorem is also noteworthy for being the first major computer-aided proof.

In 1912, George David Birkhoff introduced the chromatic polynomial to study the coloring problems, which was generalised to the Tutte polynomial by Tutte, important structures in algebraic graph theory. Kempe had already drawn attention to the general, non-planar case in 1879^[3], and many results on generalisations of planar graph coloring to surfaces of higher order followed in the early 20th century.

In 1960, Claude Berge formulated another conjecture about graph coloring, the *strong perfect graph conjecture*, originally motivated by an information-theoretic concept called the zero-error capacity of a graph introduced by Shannon. The conjecture remained unresolved for 40 years, until it was established as the celebrated strong perfect graph theorem in 2002 by Chudnovsky, Robertson, Seymour, Thomas 2002.

Graph coloring has been studied as an algorithmic problem since the early 1970s. The chromatic number problem is one of Karp's 21 NP-complete problems from 1972, around the time of various exponential-time algorithms based on backtracking and Zykov's deletion-contraction recurrence from 1949^[4]. One of the major applications of graph

coloring, register allocation in compilers was introduced in 1981.

Definition and terminology

Vertex coloring

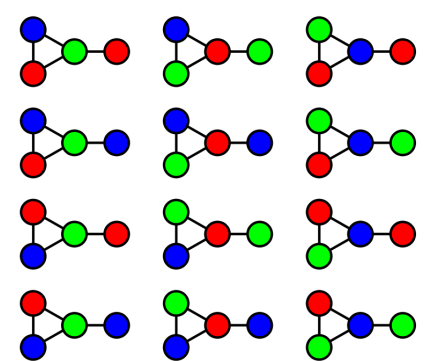
When used without any qualification, a **coloring** of a graph is almost always a *proper vertex coloring*, namely a labelling of the graph’s vertices with colors such that no two vertices sharing the same edge (graph theory) have the same color. Since a vertex with a loop (graph theory) could never be properly colored, it is understood that graphs in this context are loopless.

The terminology of using *colors* for vertex labels goes back to map coloring. Labels like *red* and *blue* are only used when the number of colors is small, and normally it is understood that the labels are drawn from the integers $\{1, 2, 3, \dots\}$.

A coloring using at most k colors is called a (proper) **k -coloring**. The smallest number of colors needed to color a graph G is called its **chromatic number**, $\chi(G)$. A graph that can be assigned a (proper) k -coloring is **k -colorable**, and it is **k -chromatic** if its chromatic number is exactly k . A subset of vertices assigned to the same color is called a *color class*, every such class forms an independent set. Thus, a k -coloring is the same as a partition of the vertex set into k independent sets, and the terms *k -partite* and *k -colorable* have the same meaning.

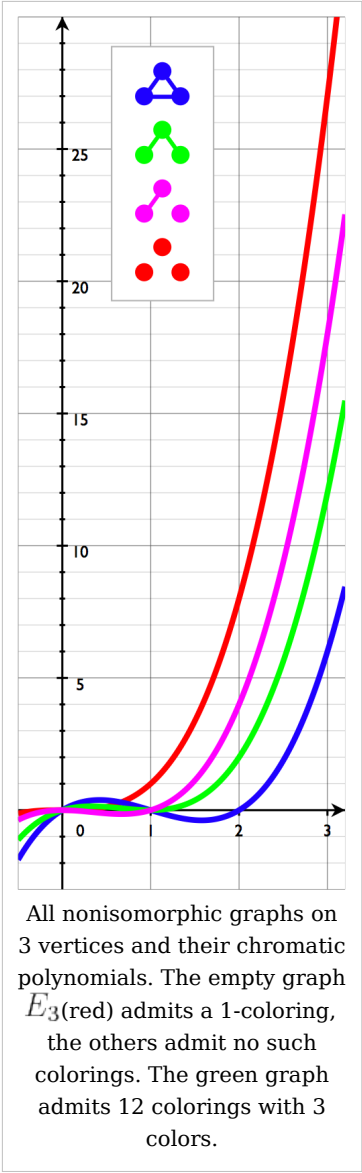
Chromatic polynomial

The **chromatic polynomial** counts the number of ways a graph can be colored using no more than a given number of colors. For example, using three colors, the graph in the image to the right can be colored in 12 ways. With only two colors, it cannot be colored at all. With four colors, it can be colored in $24 + 4 \times 12 = 72$ ways: using all four colors, there are $4! = 24$ valid colorings (*every* assignment of four colors to *any* 4-vertex graph is a proper coloring); and for every choice of three of the four colors, there are 12 valid 3-colorings. So, for the graph in the example, a table of the number of valid colorings would start like this:



This graph can be 3-colored in 12 different ways.

Available colors	1	2	3	4	...
Number of colorings	0	0	12	72	...



The chromatic polynomial includes at least as much information about the colorability of G as does the chromatic number. Indeed, χ is the smallest positive integer that is not a root of the chromatic polynomial,

$$\chi(G) = \min\{k: P(G, k) > 0\}$$

Chromatic polynomials for certain graphs

Triangle K_3	$t(t-1)(t-2)$
Complete graph K_n	$t(t-1)(t-2)\cdots(t-(n-1))$
Tree with n vertices	$t(t-1)^{n-1}$
Cycle C_n	$(t-1)^n + (-1)^n(t-1)$
Petersen graph	$t(t-1)(t-2)(t^7-12t^6+67t^5-230t^4+529t^3-814t^2+775t-352)$

Edge coloring

An **edge coloring** of a graph, is a proper coloring of the *edges*, meaning an assignment of colors to edges so that no vertex is incident to two edges of the same color. An edge coloring with k colors is called a k -edge-coloring and is equivalent to the problem of partitioning the edge set into k matchings. The smallest number of colors needed for an edge coloring of a graph G is the **chromatic index**, or **edge chromatic number**, $\chi'(G)$. A **Tait coloring** is a 3-edge coloring of a cubic graph. The four color theorem is equivalent to the assertion that every planar cubic bridgeless graph admits a Tait coloring.

Properties

Bounds on the chromatic number

Assigning distinct colors to distinct vertices always yields a proper coloring, so

$$1 \leq \chi(G) \leq n.$$

The only graphs that can be 1-colored are edgeless graphs, and the complete graph K_n of n vertices requires $\chi(K_n) = n$ colors. In an optimal coloring there must be at least one of the graph's m edges between every pair of color classes, so

$$\chi(G)(\chi(G) - 1) \leq 2m.$$

If G contains a clique of size k , then at least k colors are needed to color that clique; in other words, the chromatic number is at least the clique number:

$$\chi(G) \geq \omega(G).$$

For interval graphs this bound is tight.

The 2-colorable graphs are exactly the bipartite graphs, including trees and forests. By the four color theorem, every planar graph can be 4-colored.

A greedy coloring shows that every graph can be colored with one more color than the maximum vertex degree,

$$\chi(G) \leq \Delta(G) + 1.$$

Complete graphs have $\chi(G) = n$ and $\Delta(G) = n - 1$, and odd cycles have $\chi(G) = 3$ and $\Delta(G) = 2$, so for these graphs this bound is best possible. In all other cases, the bound can be slightly improved; Brooks' theorem^[5] states that

Brooks' theorem: $\chi(G) \leq \Delta(G)$ for a connected, simple graph G , unless G is a complete graph or an odd cycle

Graphs with high chromatic number

Graphs with large cliques have high chromatic number, but the opposite is not true. The Grötzsch graph is an example of a 4-chromatic graph without a triangle, and the example can be generalised to the Mycielskians.

Mycielski's Theorem: There exist triangle-free graphs with arbitrarily high chromatic number.

From Brooks's theorem, graphs with high chromatic number must have high maximum degree. Another local property that leads to high chromatic number is the presence of a large clique. But colorability is not an entirely local phenomenon: A graph with high girth looks locally like a tree, because all cycles are long, but its chromatic number need not be 2:

Theorem (Erdős): There exist graphs of arbitrarily high girth and chromatic number.

Bounds on the chromatic index

An edge coloring of G is a vertex coloring of its line graph $L(G)$, and vice versa. Thus,

$$\chi(G) = \chi'(L(G))$$

There is a strong relationship between edge colorability and the graph's maximum degree $\Delta(G)$. Since all edges incident to the same vertex need their own color, we have

$$\chi'(G) \geq \Delta(G)$$

Moreover,

König's theorem: $\chi'(G) = \Delta(G)$ if G is bipartite.

In general, the relationship is even stronger than what Brooks's theorem gives for vertex coloring:

Vizing's Theorem: A graph of maximal degree Δ has edge-chromatic number Δ or $\Delta + 1$.

Other properties

For planar graphs, vertex colorings are essentially dual to nowhere-zero flows.

About infinite graphs, much less is known. The following is one of the few results about infinite graph coloring:

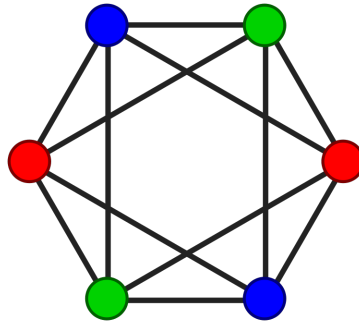
If all finite subgraphs of an infinite graph G are k -colorable, then so is G . (de Bruijn and Erdős 1951)

Open problems

The chromatic number of the plane, where two points are adjacent if they have unit distance, is unknown, although it is one of 4, 5, 6, or 7. Other open problems concerning the chromatic number of graphs include the Hadwiger conjecture stating that every graph with chromatic number k has a complete graph on k vertices as a minor, and the Erdős-Faber-Lovász conjecture bounding the chromatic number of unions of complete graphs that have at exactly one vertex in common to each pair.

When Birkhoff and Lewis introduced the chromatic polynomial in their attack on the four-color theorem, they conjectured that for planar graphs G , the polynomial $P(G, t)$ has no zeros in the region $[4, \infty)$. Although it is known that such a chromatic polynomial has no zeros in the region $[5, \infty)$ and that $P(G, 4) \neq 0$, their conjecture is still unresolved. It also remains an unsolved problem to characterize graphs which have the same chromatic polynomial and to determine which polynomials are chromatic.

Algorithms



Decision	
Name	Graph coloring, vertex coloring, k -coloring
Input	Graph G with n vertices. Integer k
Output	Does G admit a proper vertex coloring with k colors?
Running time	$O(2^n n)$
Complexity	NP-complete
Reduction from	3-Satisfiability
Garey-Johnson	GT4
Optimisation	
Name	Chromatic number
Input	Graph G with n vertices.
Output	$\chi(G)$
Complexity	NP-hard
Approximability	$O(n(\log n)^{-3}(\log \log n)^2)$
Inapproximability	$n^{1-\epsilon}$ unless P=NP
Counting problem	
Name	Chromatic polynomial
Input	Graph G with n vertices. Integer k
Output	The number $P(G, k)$ of proper k -colorings of G
Running time	$O(2^n n)$
Complexity	#P-complete
Approximability	FPRAS for restricted cases
Inapproximability	No PTAS unless P=NP

Efficient algorithms

Determining if a graph can be colored with 2 colors is equivalent to determining whether or not the graph is bipartite, and thus polynomial time-computable. More generally, the chromatic number and a corresponding coloring of perfect graphs can be computed in polynomial time using semidefinite programming. Closed formulas for chromatic polynomial are known for many classes of graphs, such as forest, chordal graphs, cycles, wheels, and ladders, so these can be evaluated in polynomial time.

Brute-force search

Brute-force search for a k -coloring considers every of the $\binom{n}{k}$ assignments of colors to vertices and checks for each if it is legal. To compute the chromatic number and the chromatic polynomial, this procedure is used for every k , for a total running time of $O((n+1)!)$, impractical for all but the smallest input graphs.

Contraction

The contraction G/uv of graph G is the graph obtained by identifying the vertices u and v , removing any edges between them, and replacing them with a single vertex w where any edges that were incident on u or v are redirected to w . This operation plays a major role in the analysis of graph coloring.

The chromatic number satisfies the recurrence relation:

$$\chi(G) = \min\{\chi(G + uv), \chi(G/uv)\}$$

due to Zykov,^[6] where u and v are nonadjacent vertices, $G + uv$ is the graph with the edge uv added. Several algorithms are based on evaluating this recurrence, the resulting computation tree is sometimes called a Zykov tree. The running time is based on the heuristic for choosing the vertices u and v .

The chromatic polynomial satisfies following recurrence relation

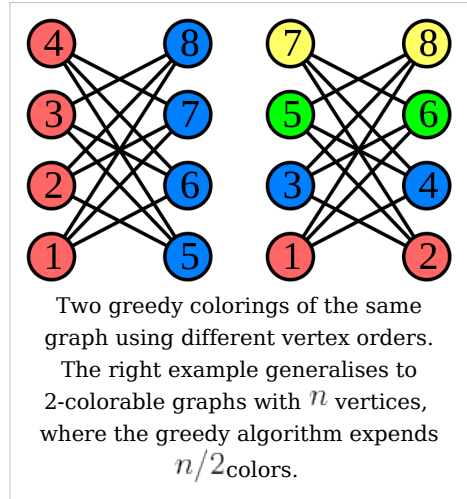
$$P(G - uv, k) = P(G/uv, k) + P(G, k)$$

where u and v are adjacent vertices and $G - uv$ is the graph with the edge uv removed. $P(G - uv, k)$ represents the number of possible proper colorings of the graph, when the vertices may have same or different colors. The number of proper colorings therefore come from the sum of two graphs. If the vertices u and v have different colors, then we can as well consider a graph, where u and v are adjacent. If u and v have the same colors, we may as well consider a graph, where u and v are contracted. Tutte's curiosity about which other graph properties satisfied this recurrence led him to discover a bivariate generalization of the chromatic polynomial, the Tutte polynomial.

The expressions give rise to a recursive procedure, called the *deletion-contraction algorithm*, which forms the basis of many algorithms for graph coloring. The running time satisfies the same recurrence relation as the Fibonacci numbers, so in the worst case, the algorithm runs in time within a polynomial factor of $((1 + \sqrt{5})/2)^{n+m} = O(1.6180)^{n+m}$ [7]. The analysis can be improved to within a polynomial factor of the number $t(G)$ of spanning trees of the input graph [8]. In practice, branch and bound strategies and graph isomorphism rejection are employed to avoid some recursive calls, the running time depends on the heuristic used to pick the vertex pair.

Greedy coloring

The greedy algorithm considers the vertices in a specific order $v_1, \dots, v_n$ and assigns to v_i the smallest available color not used by v_i 's neighbours among $v_1, \dots, v_{i-1}$, adding a fresh color if needed. The quality of the resulting coloring depends on the chosen ordering. There exists an ordering that leads to a greedy coloring with the optimal number of $\chi(G)$ colors. On the other hand, greedy colorings can be arbitrarily bad; for example, the crown graph on n vertices can be 2-colored, but has an ordering that leads to a greedy coloring with $n/2$ colors.



If the vertices are ordered according to their degrees, the resulting greedy coloring uses at most $\max_i \min\{d(x_i) + 1, i\}$ colors, at most one more than the graph's maximum degree. This heuristic is sometimes called the Welsh-Powell algorithm^[9]. Another heuristic establishes the ordering dynamically while the algorithm proceeds, choosing next the vertex adjacent to the largest number of different colors^[10]. Many other graph coloring heuristics are similarly based on greedy coloring for a specific static or dynamic strategy of ordering the vertices, these algorithms are sometimes called **sequential coloring** algorithms.

Computational complexity

Graph coloring is computationally hard. It is NP-complete to decide if a given graph admits a k -coloring for given k except for the cases $k = 1$ and $k = 2$. Especially, it is NP-hard to compute the chromatic number. Graph coloring remains NP-complete even on planar graphs of degree at most 4^[11].

The best known approximation algorithm computes a coloring of size at most within a factor $O(n(\log n)^{-3}(\log \log n)^2)$ of the chromatic number.^[12] For all $\epsilon > 0$ it is NP-hard to approximate the chromatic number within $n^{1-\epsilon}$.^[13]

It is NP-hard to color a 3-colorable graph with 4 colors^[14] and a k -colorable graph with $k^{(\log k)/25}$ colors for sufficiently large constant k .^[15]

Computing the coefficients of the chromatic polynomial is #P-hard. In fact, even computing the value of $\chi(G, k)$ is #P-hard at any rational point k except for $k = 1$ and $k = 2$.^[16] There is no FPRAS for evaluating the chromatic polynomial at any rational point $k \geq 1.5$ except for $k = 2$ unless NP=RP^[17].

For edge coloring, the proof of Vizing's result gives an algorithm that uses at most $\Delta + 1$ colors. However, deciding between the two candidate values for the edge chromatic number is NP-complete.^[18] In terms of approximation algorithms, Vizing's algorithm shows that the edge chromatic number can be approximated within $4/3$, and the hardness result shows that no $(4/3 - \epsilon)$ -algorithm exists for any $\epsilon > 0$ unless P=NP. These are among the oldest results in the literature of approximation algorithms, even though neither paper makes explicit use of that notion^[19].

Parallel and distributed algorithms

In the field of distributed algorithms, graph coloring is closely related to the problem of *symmetry breaking*. In a symmetric graph, a deterministic distributed algorithm cannot find a proper vertex coloring. Some auxiliary information is needed in order to break symmetry.

A standard assumption is that initially each node has a *unique identifier*, for example, from the set $\{1, 2, \dots, n\}$ where n is the number of nodes in the graph. Put otherwise, we assume that we are given an n -coloring. The challenge is to *reduce* the number of colors from n to, e.g., $\Delta + 1$.

A straightforward distributed version of the greedy algorithm for $(\Delta + 1)$ -coloring requires $\Omega(n)$ communication rounds in the worst case – information may need to be propagated from one side of the network to another side. However, much faster algorithms exist, at least if the maximum degree Δ is small.

The simplest interesting case is an n -cycle. Richard Cole and Uzi Vishkin^[20] show that there is a distributed algorithm that reduces the number of colors from n to $O(\log n)$ in one synchronous communication step. By iterating the same procedure, it is possible to obtain a 3-coloring of an n -cycle in $O(\log^* n)$ communication steps (assuming that we have unique node identifiers).

The function $\log^*$, iterated logarithm, is an extremely slowly growing function, "almost constant". Hence the result by Cole and Vishkin raised the question of whether there is a *constant-time* distributed algorithm for 3-coloring an n -cycle. Linial^[21] showed that this is not possible: any deterministic distributed algorithm requires $\Omega(\log^* n)$ communication steps to reduce an n -coloring to a 3-coloring in an n -cycle.

The technique by Cole and Vishkin can be applied in arbitrary bounded-degree graphs as well; the running time is $\text{poly}(\Delta) + O(\log^* n)$.^[22]

Applications

Scheduling

Vertex coloring models to a number of scheduling problems.^[23] In the cleanest form, a given set of jobs need to be assigned to time slots, each job requires one such slot. Jobs can be scheduled in any order, but pairs of jobs may be in *conflict* in the sense that they may not be assigned to the same time slot, for example because they both rely on a shared resource. The corresponding graph contains a vertex for every job and an edge for every conflicting pair of jobs. The chromatic number of the graph is exactly the minimum *makespan*, the optimal time to finish all jobs without conflicts.

Details of the scheduling problem define the structure of the graph. For example, when assigning aircrafts to flights, the resulting conflict graph is an interval graph, so the coloring problem can be solved efficiently. In bandwidth allocation to radio stations, the resulting conflict graph is a unit disk graph, so the coloring problem is 3-approximable.

Register allocation

A compiler is a computer program that translates one computer language into another. To improve the execution time of the resulting code, one of the techniques of compiler optimization is register allocation, where the most frequently used values of the compiled program are kept in the fast processor registers. Ideally, values are assigned to registers so that they can all reside in the registers when they are used.

The textbook approach to this problem is to model it as a graph coloring problem.^[24] The compiler constructs an *interference graph*, where vertices are symbolic registers and an edge connects two nodes if they are needed at the same time. If the graph can be colored with k colors then the variables can be stored in k registers.

Other applications

The problem of coloring a graph has found a number of applications, including pattern matching.

The recreational puzzle Sudoku can be seen as completing a 9-coloring on given specific graph with 81 vertices.

Other colorings

Ramsey theory

An important class of *improper* coloring problems is studied in Ramsey theory, where the graph's edges are assigned to colors, and there is no restriction on the colors of incident edges. A simple example is the friendship theorem says that in any coloring of the edges of K_6 the complete graph of six vertices there will be a monochromatic triangle; often illustrated by saying that any group of six people either has three mutual strangers or three mutual acquaintances. Ramsey theory is concerned with generalisations of this idea to seek regularity amid disorder, finding general conditions for the existence of monochromatic subgraphs with given structure.

Other colorings

List coloring

Each vertex chooses from a list of colors

List edge-coloring

Each edge chooses from a list of colors

Total coloring

Vertices and edges are colored

Harmonious coloring

Every pair of colors appears on at most one edge

Complete coloring

Every pair of colors appears on at least one edge

Exact coloring

Every pair of colors appears on exactly one edge

Acyclic coloring

Every 2-chromatic subgraph is acyclic

Strong coloring

Every color appears in every partition of equal size exactly once

Strong edge coloring

Edges are colored such that each color class induces a matching (equivalent to coloring the square of the line graph)

Equitable coloring

The sizes of color classes differ by at most one

Sum-coloring

The criterion of minimalization is the sum of colors

T-coloring

Distance between two colors of adjacent vertices must not belong to fixed set T

Rank coloring

If two vertices have the same color i , then every path between them contain a vertex with color greater than i

Interval edge-coloring

A color of edges meeting in a common vertex must be contiguous

Circular coloring

Motivated by task systems in which production proceeds in a cyclic way

Path coloring

Models a routing problem in graphs

Fractional coloring

Vertices may have multiple colors, and on each edge the sum of the color parts of each vertex is not greater than one

Oriented coloring

Takes into account orientation of edges of the graph

Cocoloring

An improper vertex coloring where every color class induces an independent set or a clique

Subcoloring

An improper vertex coloring where every color class induces a union of cliques

Weak coloring

An improper vertex coloring where every non-isolated node has at least one neighbor with a different color

Coloring can also be considered for signed graphs and gain graphs.

See also

- Uniquely colorable graph
- Critical graph
- Graph homomorphism
- Mathematics of Sudoku

Notes

- [1] M. Kubale, *History of graph coloring*, in Kubale (2004)
- [2] van Lint, Wilson & 2001 (Chap. 33)
- [3] Jensen, Toft & 1995 (p. 2)
- [4] Zykov (1949)
- [5] Brooks (1941)
- [6] Zykov (1949)
- [7] Wilf (1986)
- [8] Sekine, Imai & Tani (1995)
- [9] Welsh & Powell (1967)
- [10] Brèlaz (1979)
- [11] Garey, Johnson & Stockmeyer (1974)
- [12] Halldórsson (1993)
- [13] Zuckerman (2007)
- [14] Guruswami & Khanna (2000)
- [15] Khot (2001)
- [16] Jaeger, Vertigan & Welsh (1990)
- [17] Goldberg & Jerrum (2008)
- [18] Holyer (1981)
- [19] Crescenzi & Kann (1998)
- [20] Cole & Vishkin (1986), see also Cormen, Leiserson & Rivest (1990, Section 30.5)
- [21] Linial (1992)
- [22] Goldberg, Plotkin & Shannon (1988)
- [23] Marx (2004)
- [24] Chaitin (1982)

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External links

- *Graph Coloring Page* (<http://www.cs.ualberta.ca/~joe/Coloring/index.html>) by Joseph Culberson (graph coloring programs)
 - (<http://vispo.com/software/CoLoRaTiOn/>) by Jim Andrews and Mike Fellows is a graph coloring puzzle
 - GF-Graph Coloring Program (<http://gf8.piranhode.de>)
 - Links to Graph Coloring source codes (http://www.adaptivebox.net/research/bookmark/gcpcodes_link.html)
 - Code for efficiently computing Tutte, Chromatic and Flow Polynomials by Gary Haggard, David J. Pearce and Gordon Royle: (<http://www.mcs.vuw.ac.nz/~djp/tutte/>)
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Graph theory

In mathematics and computer science, **graph theory** is the study of *graphs*: mathematical structures used to model pairwise relations between objects from a certain collection. A "graph" in this context refers to a collection of vertices or 'nodes' and a collection of *edges* that connect pairs of vertices. A graph may be *undirected*, meaning that there is no distinction between the two vertices associated with each edge, or its edges may be *directed* from one vertex to another; see graph (mathematics) for more detailed definitions

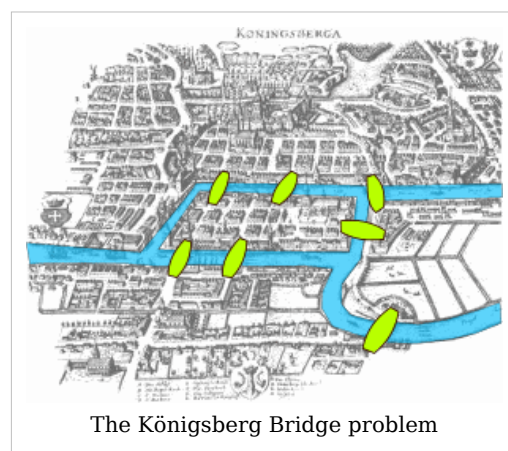
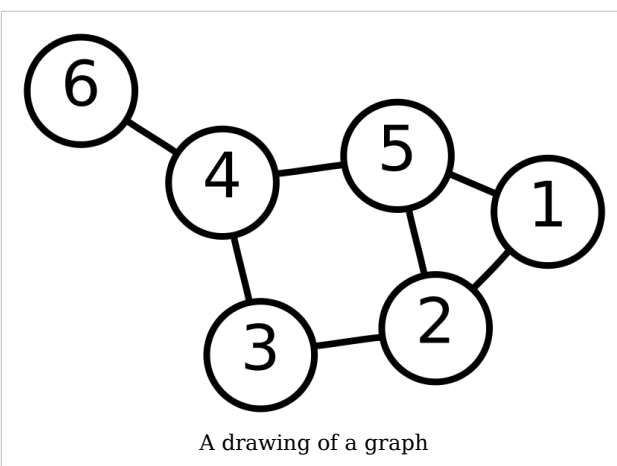
and for other variations in the types of graphs that are commonly considered. The graphs studied in graph theory should not be confused with "graphs of functions" and other kinds of graphs.

Please refer to Glossary of graph theory for some basic definitions in graph theory.

History

The paper written by Leonhard Euler on the *Seven Bridges of Königsberg* and published in 1736 is regarded as the first paper in the history of graph theory.^[1] This paper, as well as the one written by Vandermonde on the *knight problem*, carried on with the *analysis situs* initiated by Leibniz. Euler's formula relating the number of edges, vertices, and faces of a convex polyhedron was studied and generalized by Cauchy^[2] and L'Huillier,^[3] and is at the origin of topology.

More than one century after Euler's paper on the bridges of Königsberg and while Listing introduced topology, Cayley was led by the study of particular analytical forms arising from differential calculus to study a particular class of graphs, the *trees*. This study had many implications in theoretical chemistry. The involved techniques mainly concerned the enumeration of graphs having particular properties. Enumerative graph theory then rose from the results of Cayley and the fundamental results published by Pólya between 1935 and 1937 and the generalization of these by De Bruijn in 1959. Cayley linked his results on trees with the contemporary studies of chemical composition.^[4] The fusion of the ideas coming from mathematics with those coming from chemistry is at the origin of a part of the standard terminology of graph theory. In particular, the term "graph" was introduced by Sylvester in a paper published in 1878 in *Nature*.^[5]



One of the most famous and productive problems of graph theory is the four color problem: "Is it true that any map drawn in the plane may have its regions colored with four colors, in such a way that any two regions having a common border have different colors?" This problem was first posed by Francis Guthrie in 1852 and its first written record is in a letter of De Morgan addressed to Hamilton the same year. Many incorrect proofs have been proposed, including those by Cayley, Kempe, and others. The study and the generalization of this problem by Tait, Heawood, Ramsey and Hadwiger led to the study of the colorings of the graphs embedded on surfaces with arbitrary genus. Tait's reformulation generated a new class of problems, the *factorization problems*, particularly studied by Petersen and Kőnig. The works of Ramsey on colorations and more specially the results obtained by Turán in 1941 was at the origin of another branch of graph theory, *extremal graph theory*.

The four color problem remained unsolved for more than a century. A proof produced in 1976 by Kenneth Appel and Wolfgang Haken,^{[6] [7]} which involved checking the properties of 1,936 configurations by computer, was not fully accepted at the time due to its complexity. A simpler proof considering only 633 configurations was given twenty years later by Robertson, Seymour, Sanders and Thomas.^[8]

The autonomous development of topology from 1860 and 1930 fertilized graph theory back through the works of Jordan, Kuratowski and Whitney. Another important factor of common development of graph theory and topology came from the use of the techniques of modern algebra. The first example of such a use comes from the work of the physicist Gustav Kirchhoff, who published in 1845 his Kirchhoff's circuit laws for calculating the voltage and current in electric circuits.

The introduction of probabilistic methods in graph theory, especially in the study of Erdős and Rényi of the asymptotic probability of graph connectivity, gave rise to yet another branch, known as *random graph theory*, which has been a fruitful source of graph-theoretic results.

Drawing graphs

Graphs are represented graphically by drawing a dot for every vertex, and drawing an arc between two vertices if they are connected by an edge. If the graph is directed, the direction is indicated by drawing an arrow.

A graph drawing should not be confused with the graph itself (the abstract, non-graphical structure) as there are several ways to structure the graph drawing. All that matters is which vertices are connected to which others by how many edges and not the exact layout. In practice it is often difficult to decide if two drawings represent the same graph. Depending on the problem domain some layouts may be better suited and easier to understand than others.

Graph-theoretic data structures

There are different ways to store graphs in a computer system. The data structure used depends on both the graph structure and the algorithm used for manipulating the graph. Theoretically one can distinguish between list and matrix structures but in concrete applications the best structure is often a combination of both. List structures are often preferred for sparse graphs as they have smaller memory requirements. Matrix structures on the other hand provide faster access for some applications but can consume huge

amounts of memory .

List structures

Incidence list

The edges are represented by an array containing pairs (ordered if directed) of vertices (that the edge connects) and possibly weight and other data. Vertices connected by an edge are said to be *adjacent*.

Adjacency list

Much like the incidence list, each vertex has a list of which vertices it is adjacent to. This causes redundancy in an undirected graph: for example, if vertices A and B are adjacent, A's adjacency list contains B, while B's list contains A. Adjacency queries are faster, at the cost of extra storage space.

Matrix structures

Incidence matrix

The graph is represented by a matrix of size $|V|$ (number of vertices) by $|E|$ (number of edges) where the entry [vertex, edge] contains the edge's endpoint data (simplest case: 1 - connected, 0 - not connected).

Adjacency matrix

This is the n by n matrix A , where n is the number of vertices in the graph. If there is an edge from some vertex x to some vertex y , then the element $a_{x,y}$ is 1 (or in general the number of xy edges), otherwise it is 0. In computing, this matrix makes it easy to find subgraphs, and to reverse a directed graph.

Laplacian matrix or Kirchhoff matrix or Admittance matrix

This is defined as $D - A$, where D is the diagonal degree matrix. It explicitly contains both adjacency information and degree information.

Distance matrix

A symmetric n by n matrix D whose element $d_{x,y}$ is the length of a shortest path between x and y ; if there is no such path $d_{x,y} = \text{infinity}$. It can be derived from powers of A : $d_{x,y} = \min\{n \mid A^n[x, y] \neq 0\}$.

Problems in graph theory

Enumeration

There is a large literature on graphical enumeration: the problem of counting graphs meeting specified conditions. Some of this work is found in Harary and Palmer (1973).

Subgraphs, induced subgraphs, and minors

A common problem, called the subgraph isomorphism problem, is finding a fixed graph as a subgraph in a given graph. One reason to be interested in such a question is that many graph properties are *hereditary* for subgraphs, which means that a graph has the property if and only if all subgraphs have it too. Unfortunately, finding maximal subgraphs of a certain kind is often an NP-complete problem.

- Finding the largest complete graph is called the clique problem (NP-complete).

A similar problem is finding induced subgraphs in a given graph. Again, some important graph properties are hereditary with respect to induced subgraphs, which means that a graph has a property if and only if all induced subgraphs also have it. Finding maximal induced subgraphs of a certain kind is also often NP-complete. For example,

- Finding the largest edgeless induced subgraph, or independent set, called the independent set problem (NP-complete).

Still another such problem, the *minor containment problem*, is to find a fixed graph as a minor of a given graph. A minor or **subcontraction** of a graph is any graph obtained by taking a subgraph and contracting some (or no) edges. Many graph properties are hereditary for minors, which means that a graph has a property if and only if all minors have it too. A famous example:

- A graph is planar if it contains as a minor neither the complete bipartite graph $K_{3,3}$ (See the Three-cottage problem) nor the complete graph K_5 .

Another class of problems has to do with the extent to which various species and generalizations of graphs are determined by their *point-deleted subgraphs*, for example:

- The reconstruction conjecture

Graph coloring

Many problems have to do with various ways of coloring graphs, for example:

- The four-color theorem
- The strong perfect graph theorem
- The Erdős-Faber-Lovász conjecture (unsolved)
- The total coloring conjecture (unsolved)
- The list coloring conjecture (unsolved)
- The Hadwiger conjecture (graph theory) (unsolved)

Route problems

- Hamiltonian path and cycle problems
 - Minimum spanning tree
 - Route inspection problem (also called the "Chinese Postman Problem")
 - Seven Bridges of Königsberg
 - Shortest path problem
 - Steiner tree
 - Three-cottage problem
 - Traveling salesman problem (NP-complete)
-

Network flow

There are numerous problems arising especially from applications that have to do with various notions of flows in networks, for example:

- Max flow min cut theorem

Visibility graph problems

- Museum guard problem

Covering problems

Covering problems are specific instances of subgraph-finding problems, and they tend to be closely related to the clique problem or the independent set problem.

- Set cover problem
- Vertex cover problem

Applications

Applications of graph theory are primarily, but not exclusively, concerned with labeled graphs and various specializations of these.

Structures that can be represented as graphs are ubiquitous, and many problems of practical interest can be represented by graphs. The link structure of a website could be represented by a directed graph: the vertices are the web pages available at the website and a directed edge from page *A* to page *B* exists if and only if *A* contains a link to *B*. A similar approach can be taken to problems in travel, biology, computer chip design, and many other fields. The development of algorithms to handle graphs is therefore of major interest in computer science. There, the transformation of graphs is often formalized and represented by graph rewrite systems. They are either directly used or properties of the rewrite systems (e.g. confluence) are studied.

A graph structure can be extended by assigning a weight to each edge of the graph. Graphs with weights, or weighted graphs, are used to represent structures in which pairwise connections have some numerical values. For example if a graph represents a road network, the weights could represent the length of each road. A digraph with weighted edges in the context of graph theory is called a network.

Networks have many uses in the practical side of graph theory, network analysis (for example, to model and analyze traffic networks). Within network analysis, the definition of the term "network" varies, and may often refer to a simple graph.

Many applications of graph theory exist in the form of network analysis. These split broadly into three categories. Firstly, analysis to determine structural properties of a network, such as the distribution of vertex degrees and the diameter of the graph. A vast number of graph measures exist, and the production of useful ones for various domains remains an active area of research. Secondly, analysis to find a measurable quantity within the network, for example, for a transportation network, the level of vehicular flow within any portion of it. Thirdly, analysis of dynamical properties of networks.

Graph theory is also used to study molecules in chemistry and physics. In condensed matter physics, the three dimensional structure of complicated simulated atomic structures can be studied quantitatively by gathering statistics on graph-theoretic properties related to the

topology of the atoms. For example, Franzblau's shortest-path (SP) rings. In chemistry a graph makes a natural model for a molecule, where vertices represent atoms and edges bonds. This approach is especially used in computer processing of molecular structures, ranging from chemical editors to database searching.

Graph theory is also widely used in sociology as a way, for example, to measure actors' prestige or to explore diffusion mechanisms, notably through the use of social network analysis software.

See also

- Gallery of named graphs
- Glossary of graph theory
- List of graph theory topics
- Publications in graph theory

Related topics

- Graph property
- Algebraic graph theory
- Conceptual graph
- Data structure
- Disjoint-set data structure
- Entitative graph
- Existential graph
- Graph data structure
- Graph algebras
- Graph coloring
- Graph drawing
- Graph equation
- Graph rewriting
- Logical graph
- Loop
- Null graph
- Quantum graph
- Spectral graph theory
- Strongly regular graphs
- Tree data structure

Algorithms

- Bellman-Ford algorithm
 - Dijkstra's algorithm
 - Ford-Fulkerson algorithm
 - Kruskal's algorithm
 - Nearest neighbour algorithm
 - Prim's algorithm
 - Depth-first search
 - Breadth-first search
-

Subareas

- Algebraic graph theory
- Geometric graph theory
- Extremal graph theory
- Probabilistic graph theory
- Topological graph theory

Related areas of mathematics

- Combinatorics
- Group theory
- Knot theory
- Ramsey theory

Generalizations

- Hypergraph
- Abstract simplicial complex

Prominent graph theorists

- Berge, Claude
 - Bollobás, Béla
 - Chung, Fan
 - Dirac, Gabriel Andrew
 - Erdős, Paul
 - Euler, Leonhard
 - Faudree, Ralph
 - Graham, Ronald
 - Harary, Frank
 - Heawood, Percy John
 - Kőnig, Dénes
 - Lovász, László
 - Nešetřil, Jaroslav
 - Rényi, Alfréd
 - Ringel, Gerhard
 - Robertson, Neil
 - Seymour, Paul
 - Szemerédi, Endre
 - Thomassen, Carsten
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 - Tutte, W. T.
 - Tyshkevich, Regina
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External links

Online textbooks

- Graph Theory with Applications (<http://www.ecp6.jussieu.fr/pageperso/bondy/books/gtwa/gtwa.html>) (1976) by Bondy and Murty
- Encyclopaedia Britannica, Graph Theory (<http://www.britannica.com/eb/article-9037754/graph-theory>)
- Phase Transitions in Combinatorial Optimization Problems, Section 3: Introduction to Graphs (<http://arxiv.org/pdf/cond-mat/0602129>) (2006) by Hartmann and Weigt
- An Introduction to Graph Algorithms (<http://www.cs.auckland.ac.nz/~ute/220ft/graphalg/graphalg.html>) 1999 by Waltraut Ute Lorch based on Dr Michael Dinneen's lecture notes
- Digraphs: Theory Algorithms and Applications (<http://www.cs.rhul.ac.uk/books/dbook/>) 2007 by Jorgen Bang-Jensen and Gregory Gutin
- Graph Theory, by Reinhard Diestel (<http://www.math.uni-hamburg.de/home/diestel/books/graph.theory/GraphTheoryIII.pdf>)

Other resources

- More people and publications at: *Graph Theory Resources* (<http://www.math.gatech.edu/~sanders/graphtheory/>)
- Graph theory tutorial (<http://www.utm.edu/departments/math/graph/>)
- Image gallery: graphs (<http://web.archive.org/web/20060206155001/http://www.nd.edu/~networks/gallery.htm>)
- GraphViz open source software to produce graph images from a description of the graph (<http://graphexploration.cond.org/>)
- GUESS Graph Exploration System(Open Source GPL)
- JGraphT (<http://www.jgrapht.org>) an open source Java graph theory library
- Annas (<http://annas.googlecode.com>) an Open source Java graph and algorithm library
- Boost Graph Library (BGL) (<http://www.boost.org/libs/graph/doc/index.html>) an open source C++ graph theory library
- QuickGraph (<http://quickgraph.codeplex.com/>) an open source C# graph theory library based on the design of the BGL (<http://www.boost.org/libs/graph/doc/index.html>)
- Ruby Graph Library (RGL) (<http://rubyforge.org/projects/rgl/>) an open source Ruby graph theory library based on the design of the BGL (<http://www.boost.org/libs/graph/doc/index.html>)
- LEMON (<http://lemon.cs.elte.hu>) another open source C++ graph theory library
- NetworkX (<http://networkx.lanl.gov/>) an open source Python graph theory library
- Eric W. Weisstein, *Graph Theory* (<http://mathworld.wolfram.com/GraphTheory.html>) at MathWorld.
- (<http://orgnet.com/SocialLifeOfRouters.pdf>) Graph theory applied to computer/social networks
- Concise, annotated list of graph theory resources for researchers (<http://www.babelgraph.org/links.html>)
- GTAD (Graph Toolkit for Algorithms and Drawings) (<http://gtad.sourceforge.net/>) C++ graph library
- GraphsJ (http://www.or.deis.unibo.it/staff_pages/martello/GraphsJ/GraphsJ.html) An open source and extendable didactic Java software for running graph algorithms
- igraph (<http://igraph.sourceforge.net/>) an open source social-network/graph-theory library written in C with bindings to Python, R, and Ruby.

Multicategory

In mathematics (especially category theory), a **multicategory** is a generalization of the concept of category that allows morphisms of multiple arity. If morphisms in a category are viewed as analogous to functions, then morphisms in a multicategory are analogous to functions of several variables.

Definition

A multicategory consists of

- a collection (often a proper class) of *objects*;
- for every finite sequence $(X_1, X_2, \dots, X_n)$ of objects (for $n := 0, 1, 2, \dots$) and object Y , a set of *morphisms* from $X_1, X_2, \dots$, and X_n to Y ; and
- for every object X , a special identity morphism (with $n := 1$) from X to X .

Additionally, there are composition operations: Given a sequence of sequences $(X_{1,1}, X_{1,2}, \dots, X_{1,n_1}; X_{2,1}, X_{2,2}, \dots, X_{2,n_2}; \dots; X_{m,1}, X_{m,2}, \dots, X_{m,n_m})$ of objects, a sequence $(Y_1, Y_2, \dots, Y_m)$ of objects, and an object Z : if

- f_1 is a morphism from $X_{1,1}, X_{1,2}, \dots$, and X_{1,n_1} to Y_1 ;
- f_2 is a morphism from $X_{2,1}, X_{2,2}, \dots$, and X_{2,n_2} to Y_2 ;
- ...;
- f_m is a morphism from $X_{m,1}, X_{m,2}, \dots$, and X_{m,n_m} to Y_m ; and
- g is a morphism from $Y_1, Y_2, \dots$, and Y_m to Z :

then there is a composite morphism $g(f_1, f_2, \dots, f_m)$ from $X_{1,1}, X_{1,2}, \dots, X_{1,n_1}, X_{2,1}, X_{2,2}, \dots, X_{2,n_2}, \dots, X_{m,1}, X_{m,2}, \dots$, and X_{m,n_m} to Z . **This must satisfy certain axioms:**

- If m is 1, Z is Y , and g is the identity morphism for Y , then $g(f)$ must equal f ;
- if n_1 is 1, n_2 is 1, ..., n_m is 1, X_1 is Y_1 , X_2 is Y_2 , ..., X_m is Y_m , f_1 is the identity morphism for Y_1 , f_2 is the identity morphism for Y_2 , ..., and f_m is the identity morphism for Y_m , then $g(f_1, f_2, \dots, f_m)$ must equal g ; and
- an associativity condition (involving a further level of composition) that takes a long time to write down.

Examples

There is a multicategory whose objects are (small) sets, where a morphism from the sets $X_1, X_2, \dots$, and X_n to the set Y is an n -ary function, that is a function from the Cartesian product $X_1 \times X_2 \times \dots \times X_n$ to Y .

There is a multicategory whose objects are vector spaces (over the rational numbers, say), where a morphism from the vector spaces $X_1, X_2, \dots$, and X_n to the vector space Y is a multilinear operator, that is a linear transformation from the tensor product $X_1 \otimes X_2 \otimes \dots \otimes X_n$ to Y .

More generally, given any monoidal category $\mathbf{C}$, there is a multicategory whose objects are objects of $\mathbf{C}$, where a morphism from the $\mathbf{C}$ -objects $X_1, X_2, \dots$, and X_n to the $\mathbf{C}$ -object Y is a $\mathbf{C}$ -morphism from the monoidal product of $X_1, X_2, \dots$, and X_n to Y .

An operad is a multicategory with one unique object; except in degenerate cases, such a multicategory does not come from a monoidal category. (The term "operad" is often reserved for *symmetric* multicategories; terminology varies. [1])

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Metatheory

A **metatheory** or **meta-theory** is a theory whose subject matter is some other theory. In other words it is a theory about a theory. Statements made in the metatheory about the theory are called metatheorems.

According to the systemic TOGA meta-theory ^[1], a meta-theory may refer to the specific point of view on a theory and to its subjective meta-properties, but not to its application domain. In the above sense, a theory **T** of the domain **D** is a meta-theory if **D** is a theory or a set of theories. A general theory is not a meta-theory because its domain **D** are not theories.

The following is an example of a meta-theoretical statement:^[2]

“ Any physical theory is always provisional, in the sense that it is only a hypothesis; you can never prove it. No matter how many times the results of experiments agree with some theory, you can never be sure that the next time the result will not contradict the theory. On the other hand, you can disprove a theory by finding even a single observation that disagrees with the predictions of the theory. ”

Meta-theory belongs to the philosophical specialty of epistemology and metamathematics, as well as being an object of concern to the area in which the individual theory is conceived. An emerging domain of meta-theories is systemics.

Taxonomy

Examining groups of related theories, a first finding may be to identify classes of theories, thus specifying a taxonomy of theories. A proof engendered by a metatheory is called a *metatheorem*.

History

The concept burst upon the scene of twentieth-century philosophy as a result of the work of the German mathematician David Hilbert, who in 1905 published a proposal for proof of the consistency of mathematics, creating the field of metamathematics. His hopes for the success of this proof were dashed by the work of Kurt Gödel who in 1931 proved this to be unattainable by his incompleteness theorems. Nevertheless, his program of unsolved mathematical problems, out of which grew this metamathematical proposal, continued to influence the direction of mathematics for the rest of the twentieth century.

The study of metatheory became widespread during the rest of that century by its application in other fields, notably scientific linguistics and its concept of metalanguage.

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- [2] Stephen Hawking in *A Brief History of Time*

See also

- meta-
- meta-knowledge
- Metalogic
- Metamathematics

External links

- Meta-theoretical Issues (2003), Lyle Flint (<http://www.bsu.edu/classes/flint/comm360/metatheo.html>)

Metalogic

Metalogic is the study of the metatheory of logic. While *logic* is the study of the manner in which logical systems can be used to decide the correctness of arguments, metalogic studies the properties of the logical systems themselves.^[1] According to Geoffrey Hunter, while logic concerns itself with the "truths of logic," metalogic concerns itself with the theory of "sentences used to express truths of logic."^[2]

The basic objects of study in metalogic are formal languages, formal systems, and their interpretations. The study of interpretation of formal systems is the branch of mathematical logic known as model theory, while the study of deductive apparatus is the branch known as proof theory.

History

Metalogical questions have been asked since the time of Aristotle. However, it was only with the rise of formal languages in the late 19th and early 20th century that investigations into the foundations of logic began to flourish. In 1904, David Hilbert observed that in investigating the foundations of mathematics that logical notions are presupposed, and therefore a simultaneous account of metalogical and metamathematical principles was required. Today, metalogic and metamathematics are largely synonymous with each other, and both have been substantially subsumed by mathematical logic in academia.

Important distinctions in metalogic

Metalanguage-Object language

In metalogic, formal languages are sometimes called *object languages*. The language used to make statements about an object language is called a *metalanguage*. This distinction is a key difference between logic and metalogic. While logic deals with *proofs in a formal system*, expressed in some formal language, metalogic deals with *proofs about a formal system* which are expressed in a metalanguage about some object language.

Syntax-semantics

In metalogic, 'syntax' has to do with formal languages or formal systems without regard to any interpretation of them, whereas, 'semantics' has to do with interpretations of formal languages. The term 'syntactic' has a slightly wider scope than 'proof-theoretic', since it may be applied to properties of formal languages without any deductive systems, as well as to formal systems. 'Semantic' is synonymous with 'model-theoretic'.

Use-mention

In metalogic, the words 'use' and 'mention', in both their noun and verb forms, take on a technical sense in order to identify an important distinction.^[2] The *use-mention distinction* (sometimes referred to as the *words-as-words distinction*) is the distinction between *using* a word (or phrase) and *mentioning* it. Usually it is indicated that an expression is being mentioned rather than used by enclosing it in quotation marks, printing it in italics, or setting the expression by itself on a line. The enclosing in quotes of an expression gives us the name of an expression, for example:

'Metalogic' is the name of this article.

This article is about metalogic.

Type-token

The *type-token distinction* is a distinction in metalogic, that separates an abstract concept from the objects which are particular instances of the concept. For example, the particular bicycle in your garage is a token of the type of thing known as "The bicycle." Whereas, the bicycle in your garage is in a particular place at a particular time, that is not true of "the bicycle" as used in the sentence: "*The bicycle* has become more popular recently." This distinction is used to clarify the meaning of symbols of formal languages.

Overview

Formal language

A *formal language* is an organized set of symbols the essential feature of which is that it can be precisely defined in terms of just the shapes and locations of those symbols. Such a language can be defined, then, without any reference to any meanings of any of its expressions; it can exist before any interpretation is assigned to it -- that is, before it has any meaning. First order logic is expressed in some formal language. A formal grammar determines which symbols and sets of symbols are formulas in a formal language.

A formal language can be defined formally as a set A of strings (finite sequences) on a fixed alphabet α . Some authors, including Carnap, define the language as the ordered pair $\langle \alpha, A \rangle$.^[3] Carnap also requires that each element of α must occur in at least one string in A .

Formal grammar

A *formal grammar* (also called *formation rules*) is a precise description of a the well-formed formulas of a formal language. It is synonymous with the set of strings over the alphabet of the formal language which constitute well formed formulas. However, it does not describe their semantics (i.e. what they mean).

Formal systems

A *formal system* (also called a *logical calculus*, or a *logical system*) consists of a formal language together with a deductive apparatus (also called a *deductive system*). The deductive apparatus may consist of a set of transformation rules (also called *inference rules*) or a set of axioms, or have both. A formal system is used to derive one expression from one or more other expressions.

A *formal system* can be formally defined as an ordered triple $\langle \alpha, \mathcal{I}, \mathcal{D} \rangle$, where $\mathcal{D}$ is the relation of direct derivability. This relation is understood in a comprehensive sense such that the primitive sentences of the formal system are taken as directly derivable from the empty set of sentences. Direct derivability is a relation between a sentence and a finite, possibly empty set of sentences. Axioms are laid down in such a way that every first place member of $\mathcal{D}$ is a member of $\mathcal{I}$ and every second place member is a finite subset of $\mathcal{I}$.

It is also possible to define a *formal system* using only the relation $\mathcal{D}$. In this way we can omit $\mathcal{I}$, and α in the definitions of *interpreted formal language*, and *interpreted formal system*. However, this method can be more difficult to understand and work with.^[3]

Formal proofs

A *formal proof* is a sequences of well-formed formulas of a formal language, the last one of which is a theorem of a formal system. The theorem is a syntactic consequence of all the well formed formulae preceding it in the proof. For a well formed formula to qualify as part of a proof, it must be the result of applying a rule of the deductive apparatus of some formal system to the previous well formed formulae in the proof sequence.

Interpretations

An *interpretation* of a formal system is the assignment of meanings, to the symbols, and truth-values to the sentences of the formal system. The study of interpretations is called Formal semantics. *Giving an interpretation* is synonymous with *constructing a model*.

Results in metalogic

Results in metalogic consist of such things as formal proofs demonstrating the consistency, completeness, and decidability of particular formal systems.

Major results in metalogic include:

- Proof of the uncountability of the set of all subsets of the set of natural numbers (Cantor's theorem 1891)
- Löwenheim-Skolem theorem (Leopold Löwenheim 1915 and Thoralf Skolem 1919)
- Proof of the consistency of truth-functional propositional logic (Emil Post 1920)
- Proof of the semantic completeness of truth-functional propositional logic (Paul Bernays 1918)^[4], (Emil Post 1920)^[2]

- Proof of the syntactic completeness of truth-functional propositional logic (Emil Post 1920)^[2]
- Proof of the decidability of truth-functional propositional logic (Emil Post 1920)^[2]
- Proof of the consistency of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the semantic completeness of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the decidability of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the semantic completeness of first order predicate logic (Gödel's completeness theorem 1930)
- Proof of the consistency of first order predicate logic (David Hilbert and Wilhelm Ackermann 1928)
- Proof of the semantic completeness of first order predicate logic (Kurt Gödel 1930)
- Proof of the undecidability of first order predicate logic (Alonzo Church 1936)
- Gödel's first incompleteness theorem 1931
- Gödel's second incompleteness theorem 1931

See also

- Metamathematics
- Formal semantics

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 - [2] Hunter, Geoffrey, *Metalogic: An Introduction to the Metatheory of Standard First-Order Logic*, University of California Pres, 1971
 - [3] Rudolf Carnap (1958) *Introduction to Symbolic Logic and its Applications*, p. 102.
 - [4] Hao Wang, *Reflections on Kurt Gödel*
-

Metamathematics

Metamathematics is the study of mathematics itself using mathematical methods. This study produces metatheories, which are mathematical theories about other mathematical theories. Metamathematical metatheorems about mathematics itself were originally differentiated from ordinary mathematical theorems in the 19th century, to focus on what was then called the foundational crisis of mathematics. Richard's paradox (Richard 1905) concerning certain 'definitions' of real numbers in the English language is an example of the sort of contradictions which can easily occur if one fails to distinguish between mathematics and metamathematics.

The term "metamathematics" is sometimes used as a synonym for certain elementary parts of formal logic, including propositional logic and predicate logic.

History

Metamathematics was intimately connected to mathematical logic, so that the early histories of the two fields, during the late 19th and early 20th centuries, largely overlap. More recently, mathematical logic has often included the study of new pure mathematics, such as set theory, recursion theory, and pure model theory, which is not directly related to metamathematics.

Serious metamathematical reflection began with the work of Gottlob Frege, especially his *Begriffsschrift*.

David Hilbert was the first to invoke the term "metamathematics" with regularity (see Hilbert's program). In his hands, it meant something akin to contemporary proof theory, in which finitary methods are used to study various axiomatized mathematical theorems.

Other prominent figures in the field include Bertrand Russell, Thoralf Skolem, Emil Post, Alonzo Church, Stephen Kleene, Willard Quine, Paul Benacerraf, Hilary Putnam, Gregory Chaitin, and most important, Alfred Tarski and Kurt Gödel. In particular, Gödel's proof that, given any finite number of axioms for Peano arithmetic, there will be true statements about that arithmetic that cannot be proved from those axioms, a result known as the incompleteness theorem, is arguably the greatest achievement of metamathematics and the philosophy of mathematics to date.

Milestones

- Principia Mathematica (Whitehead and Russell 1925)
 - Gödel's completeness theorem, 1930
 - Gödel's incompleteness theorem, 1931
 - Tarski's definition of model-theoretic satisfaction, now called the T-schema
 - The proof of the impossibility of the Entscheidungsproblem, obtained independently in 1936–1937 by Church and Turing.
-

See also

- meta-
- model theory
- proof theory

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Categories

Category theories

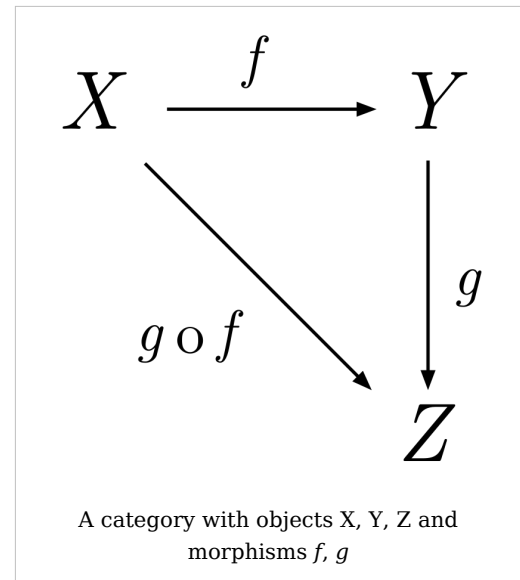
Category theory

In mathematics, **category theory** deals in an abstract way with mathematical structures and relationships between them: it abstracts from *sets* and *functions* to *objects* linked in diagrams by *morphisms* or *arrows*.

One of the simplest examples of a category (which is a very important concept in topology) is that of groupoid, defined as a category whose arrows or morphisms are all invertible. Categories now appear in most branches of mathematics and also in some areas of theoretical computer science where they correspond to types and mathematical physics where they can be used to describe vector spaces. Category theory provides both with a unifying notion and terminology. Categories were first

introduced by Samuel Eilenberg and Saunders Mac Lane in 1942–45, in connection with algebraic topology.

Category theory has several faces known not just to specialists, but to other mathematicians. A term dating from the 1940s, "general abstract nonsense", refers to its high level of abstraction, compared to more classical branches of mathematics. Homological algebra is category theory in its aspect of organising and suggesting manipulations in abstract algebra. Diagram chasing is a visual method of arguing with abstract "arrows" joined in diagrams. Note that arrows between categories are called functors, subject to specific defining commutativity conditions; moreover, categorical diagrams and sequences can be defined as functors (viz. Mitchell, 1965). An arrow between two functors is a natural transformation when it is subject to certain naturality or commutativity conditions. Both functors and natural transformations are key concepts in category theory, or the "real engines" of category theory. To paraphrase a famous sentence of the mathematicians who founded category theory: 'Categories were introduced to define functors, and functors were introduced to define natural transformations'. Topos theory is a form of abstract sheaf theory, with geometric origins, and leads to ideas such as pointless topology. A topos can also be considered as a specific type of category with two additional topos axioms.



Background

The study of categories is an attempt to *axiomatically* capture what is commonly found in various classes of related *mathematical structures* by relating them to the *structure-preserving functions* between them. A systematic study of category theory then allows us to prove general results about any of these types of mathematical structures from the axioms of a category.

Consider the following example. The class **Grp** of groups consists of all objects having a "group structure". More precisely, **Grp** consists of all sets G endowed with a binary operation satisfying a certain set of axioms. One can proceed to prove theorems about groups by making logical deductions from the set of axioms. For example, it is immediately proved from the axioms that the identity element of a group is unique.

Instead of focusing merely on the individual objects (e.g., groups) possessing a given structure, category theory emphasizes the morphisms – the structure-preserving mappings – *between* these objects; it turns out that by studying these morphisms, we are able to learn more about the structure of the objects. In the case of groups, the morphisms are the group homomorphisms. A group homomorphism between two groups "preserves the group structure" in a precise sense – it is a "process" taking one group to another, in a way that carries along information about the structure of the first group into the second group. The study of group homomorphisms then provides a tool for studying general properties of groups and consequences of the group axioms.

A similar type of investigation occurs in many mathematical theories, such as the study of continuous maps (morphisms) between topological spaces in topology (the associated category is called **Top**), and the study of smooth functions (morphisms) in manifold theory.

If one axiomatizes relations instead of functions, one obtains the theory of allegories.

Functors

Abstracting again, a category is *itself* a type of mathematical structure, so we can look for "processes" which preserve this structure in some sense; such a process is called a functor. A functor associates to every object of one category an object of another category, and to every morphism in the first category a morphism in the second.

In fact, what we have done is define a category *of categories and functors* – the objects are categories, and the morphisms (between categories) are functors.

By studying categories and functors, we are not just studying a class of mathematical structures and the morphisms between them; we are studying the *relationships between various classes of mathematical structures*. This is a fundamental idea, which first surfaced in algebraic topology. Difficult *topological* questions can be translated into *algebraic* questions which are often easier to solve. Basic constructions, such as the fundamental group or fundamental groupoid ^[1] of a topological space, can be expressed as fundamental functors ^[1] to the category of groupoids in this way, and the concept is pervasive in algebra and its applications.

Natural transformation

Abstracting yet again, constructions are often "naturally related" – a vague notion, at first sight. This leads to the clarifying concept of natural transformation, a way to "map" one functor to another. Many important constructions in mathematics can be studied in this context. "Naturality" is a principle, like general covariance in physics, that cuts deeper than is initially apparent.

Historical notes

In 1942–45, Samuel Eilenberg and Saunders Mac Lane were the first to introduce categories, functors, and natural transformations as part of their work in topology, especially algebraic topology. Their work was an important part of the transition from intuitive and geometric homology to axiomatic homology theory. Eilenberg and Mac Lane later wrote that their goal was to understand natural transformations; in order to do that, functors had to be defined, which required categories.

Stanislaw Ulam, and some writing on his behalf, have claimed that related ideas were current in the late 1930s in Poland. Eilenberg was Polish, and studied mathematics in Poland in the 1930s. Category theory is also, in some sense, a continuation of the work of Emmy Noether (one of Mac Lane's teachers) in formalizing abstract processes; Noether realized that in order to understand a type of mathematical structure, one needs to understand the processes preserving that structure. In order to achieve this understanding, Eilenberg and Mac Lane proposed an axiomatic formalization of the relation between structures and the processes preserving them.

The subsequent development of category theory was powered first by the computational needs of homological algebra, and later by the axiomatic needs of algebraic geometry, the field most resistant to being grounded in either axiomatic set theory or the Russell-Whitehead view of united foundations. General category theory, an extension of universal algebra having many new features allowing for semantic flexibility and higher-order logic, came later; it is now applied throughout mathematics.

Certain categories called *topoi* (singular *topos*) can even serve as an alternative to axiomatic set theory as a foundation of mathematics. These foundational applications of category theory have been worked out in fair detail as a basis for, and justification of, constructive mathematics. More recent efforts to introduce undergraduates to categories as a foundation for mathematics include Lawvere and Rosebrugh (2003) and Lawvere and Schanuel (1997).

Categorical logic is now a well-defined field based on type theory for intuitionistic logics, with applications in functional programming and domain theory, where a cartesian closed category is taken as a non-syntactic description of a lambda calculus. At the very least, category theoretic language clarifies what exactly these related areas have in common (in some abstract sense).

Categories, objects and morphisms

A category C consists of the following three mathematical entities:

- A class $\text{ob}(C)$, whose elements are called *objects*;
- A class $\text{hom}(C)$, whose elements are called morphisms or maps or *arrows*. Each morphism f has a unique *source object* a and *target object* b . We write $f: a \rightarrow b$, and we say " f is a morphism from a to b ". We write $\text{hom}(a, b)$ (or $\text{Hom}(a, b)$, or $\text{hom}_C(a, b)$, or $\text{Mor}(a, b)$, or $C(a, b)$) to denote the *hom-class* of all morphisms from a to b .
- A binary operation $\circ$, called *composition of morphisms*, such that for any three objects a, b , and c , we have $\text{hom}(a, b) \times \text{hom}(b, c) \rightarrow \text{hom}(a, c)$. The composition of $f: a \rightarrow b$ and $g: b \rightarrow c$ is written as $g \circ f$ or gf (some authors write fg), governed by two axioms:
 - Associativity: If $f: a \rightarrow b$, $g: b \rightarrow c$ and $h: c \rightarrow d$ then $h \circ (g \circ f) = (h \circ g) \circ f$, and
 - Identity: For every object x , there exists a morphism $1_x: x \rightarrow x$ called the *identity morphism* for x , such that for every morphism $f: a \rightarrow b$, we have $1_b \circ f = f = f \circ 1_a$.

From these axioms, it can be proved that there is exactly one identity morphism for every object. Some authors deviate from the definition just given by identifying each object with its identity morphism.

Relations among morphisms (such as $fg = h$) are often depicted using commutative diagrams, with "points" (corners) representing objects and "arrows" representing morphisms.

Properties of morphisms

Some morphisms have important properties. A morphism $f: a \rightarrow b$ is:

- a monomorphism (or *monic*) if $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$ for all morphisms $g_1, g_2: x \rightarrow a$.
- an epimorphism (or *epic*) if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for all morphisms $g_1, g_2: b \rightarrow x$.
- an isomorphism if there exists a morphism $g: b \rightarrow a$ with $f \circ g = 1_b$ and $g \circ f = 1_a$.^[2]
- an endomorphism if $a = b$. $\text{end}(a)$ denotes the class of endomorphisms of a .
- an automorphism if f is both an endomorphism and an isomorphism. $\text{aut}(a)$ denotes the class of automorphisms of a .

Functors

Functors are structure-preserving maps between categories. They can be thought of as morphisms in the category of all (small) categories.

A (**covariant**) functor F from a category C to a category D , written $F: C \rightarrow D$, consists of:

- for each object x in C , an object $F(x)$ in D ; and
- for each morphism $f: x \rightarrow y$ in C , a morphism $F(f): F(x) \rightarrow F(y)$,

such that the following two properties hold:

- For every object x in C , $F(1_x) = 1_{F(x)}$;
- For all morphisms $f: x \rightarrow y$ and $g: y \rightarrow z$, $F(g \circ f) = F(g) \circ F(f)$.

A **contravariant** functor $F: C \rightarrow D$, is like a covariant functor, except that it "turns morphisms around" ("reverses all the arrows"). More specifically, every morphism $f: x \rightarrow y$ in C must be assigned to a morphism $F(f): F(y) \rightarrow F(x)$ in D . In other words, a contravariant functor is a covariant functor from the opposite category C^{op} to D .

Natural transformations and isomorphisms

A *natural transformation* is a relation between two functors. Functors often describe "natural constructions" and natural transformations then describe "natural homomorphisms" between two such constructions. Sometimes two quite different constructions yield "the same" result; this is expressed by a natural isomorphism between the two functors.

If F and G are (covariant) functors between the categories C and D , then a natural transformation from F to G associates to every object x in C a morphism $\eta_x : F(x) \rightarrow G(x)$ in D such that for every morphism $f : x \rightarrow y$ in C , we have $\eta_y \circ F(f) = G(f) \circ \eta_x$; this means that the following diagram is commutative:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \eta_X \downarrow & & \downarrow \eta_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

The two functors F and G are called *naturally isomorphic* if there exists a natural transformation from F to G such that η_x is an isomorphism for every object x in C .

Universal constructions, limits, and colimits

Using the language of category theory, many areas of mathematical study can be cast into appropriate categories, such as the categories of all sets, groups, topologies, and so on. These categories surely have some objects that are "special" in a certain way, such as the empty set or the product of two topologies, yet in the definition of a category, objects are considered to be atomic, i.e., we *do not know* whether an object A is a set, a topology, or any other abstract concept – hence, the challenge is to define special objects without referring to the internal structure of those objects. But how can we define the empty set without referring to elements, or the product topology without referring to open sets?

The solution is to characterize these objects in terms of their relations to other objects, as given by the morphisms of the respective categories. Thus, the task is to find *universal properties* that uniquely determine the objects of interest. Indeed, it turns out that numerous important constructions can be described in a purely categorical way. The central concept which is needed for this purpose is called categorical *limit*, and can be dualized to yield the notion of a *colimit*.

Equivalent categories

It is a natural question to ask: under which conditions can two categories be considered to be "essentially the same", in the sense that theorems about one category can readily be transformed into theorems about the other category? The major tool one employs to describe such a situation is called *equivalence of categories*, which is given by appropriate functors between two categories. Categorical equivalence has found numerous applications

in mathematics.

Further concepts and results

The definitions of categories and functors provide only the very basics of categorical algebra; additional important topics are listed below. Although there are strong interrelations between all of these topics, the given order can be considered as a guideline for further reading.

- The functor category D^C has as objects the functors from C to D and as morphisms the natural transformations of such functors. The Yoneda lemma is one of the most famous basic results of category theory; it describes representable functors in functor categories.
- Duality: Every statement, theorem, or definition in category theory has a *dual* which is essentially obtained by "reversing all the arrows". If one statement is true in a category C then its dual will be true in the dual category C^{op} . This duality, which is transparent at the level of category theory, is often obscured in applications and can lead to surprising relationships.
- Adjoint functors: A functor can be left (or right) adjoint to another functor that maps in the opposite direction. Such a pair of adjoint functors typically arises from a construction defined by a universal property; this can be seen as a more abstract and powerful view on universal properties.

Higher-dimensional categories

Many of the above concepts, especially equivalence of categories, adjoint functor pairs, and functor categories, can be situated into the context of *higher-dimensional categories*. Briefly, if we consider a morphism between two objects as a "process taking us from one object to another", then higher-dimensional categories allow us to profitably generalize this by considering "higher-dimensional processes".

For example, a (strict) 2-category is a category together with "morphisms between morphisms", i.e., processes which allow us to transform one morphism into another. We can then "compose" these "bimorphisms" both horizontally and vertically, and we require a 2-dimensional "exchange law" to hold, relating the two composition laws. In this context, the standard example is **Cat**, the 2-category of all (small) categories, and in this example, bimorphisms of morphisms are simply natural transformations of morphisms in the usual sense. Another basic example is to consider a 2-category with a single object; these are essentially monoidal categories. Bicategories are a weaker notion of 2-dimensional categories in which the composition of morphisms is not strictly associative, but only associative "up to" an isomorphism.

This process can be extended for all natural numbers n , and these are called n -categories. There is even a notion of ω -category corresponding to the ordinal number ω .

Higher-dimensional categories are part of the broader mathematical field of higher-dimensional algebra, a concept introduced by Ronald Brown. For a conversational introduction to these ideas, see John Baez, 'A Tale of n -categories' (1996).^[3]

See also

- List of category theory topics
- Important publications in category theory
- Glossary of category theory
- Domain theory
- Enriched category theory
- Higher category theory
- Timeline of category theory and related mathematics
- Higher-dimensional algebra

Notes

- [1] <http://planetphysics.org/encyclopedia/FundamentalGroupoidFunctor.html>
- [2] Note that a morphism that is both epic and monic is not necessarily an isomorphism! For example, in the category consisting of two objects A and B , the identity morphisms, and a single morphism f from A to B , f is both epic and monic but is not an isomorphism.
- [3] <http://math.ucr.edu/home/baez/week73.html>

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External links

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- J. Adamek, H. Herrlich, G. Stecker, Abstract and Concrete Categories-The Joy of Cats (<http://katmat.math.uni-bremen.de/acc/acc.pdf>)
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- Baez, John, 1996, "The Tale of n -categories. (<http://math.ucr.edu/home/baez/week73.html>)" An informal introduction to higher order categories.
- The catsters (<http://www.youtube.com/user/TheCatsters>)" a Youtube channel about category theory.
- Category Theory (<http://planetmath.org/?op=getobj&from=objects&id=5622>) on PlanetMath
- Categories, Logic and the Foundations of Physics (<http://categorieslogicphysics.wikidot.com/>), Webpage dedicated to the use of Categories and Logic in the Foundations of Physics.

- Interactive Web page (<http://www.j-paine.org/cgi-bin/webcats/webcats.php>) which generates examples of categorical constructions in the category of finite sets. Written by Jocelyn Paine (<http://www.j-paine.org/>)

Algebraic category

In mathematics, specifically universal algebra, a **variety of algebras** is the class of all algebraic structures of a given signature satisfying a given set of identities. Equivalently, a variety is a class of algebraic structures of the same signature which is closed under the taking of homomorphic images, subalgebras and (direct) products. In the context of category theory, a variety of algebras is usually called a **finitary algebraic category**.

A **covariety** is the class of all coalgebraic structures of a given signature.

A variety of algebras should not be confused with an algebraic variety. Intuitively, a variety of algebras is an equationally defined **collection of algebras**, while an algebraic variety is an equationally defined **collection of elements from a single algebra**. The two are named alike by analogy, but they are formally quite distinct and their theories have little in common.

Birkhoff's theorem

Garrett Birkhoff proved equivalent the two definitions of variety given above, a result of fundamental importance to universal algebra and known as **Birkhoff's theorem** or as the **HSP theorem**. **H**, **S**, and **P** stand, respectively, for the closure operations of homomorphism, subalgebra, and product.

An equational class for some signature Σ is the collection of all models, in the sense of model theory, that satisfy some set E of *equations*, asserting equality between terms. A model *satisfies* these equations if they are true in the model for any valuation of the variables. The equations in E are then said to be identities of the model. Examples of such identities are the commutative law, characterizing commutative algebras, and the absorption law, characterizing lattices.

It is simple to see that the class of algebras satisfying some set of equations will be closed under the HSP operations. Proving the converse —classes of algebras closed under the HSP operations must be equational— is much harder.

Examples

The class of all semigroups forms a variety of algebras of signature (2). A sufficient defining equation is the associative law:

$$x(yz) = (xy)z.$$

It satisfies the HSP closure requirement, since any homomorphic image, any subset closed under multiplication and any direct product of semigroups is also a semigroup.

The class of groups forms a class of algebras of signature (2,1,0), the three operations being respectively *multiplication*, *inversion* and *identity*. Any subset of a group closed under multiplication, under inversion and under identity (ie. containing the identity) forms a subgroup. Likewise, the collection of groups is closed under homomorphic image and under direct product. Applying Birkhoff's theorem, this is sufficient to tell us that the groups form

a variety, and so it should be defined by a collection of identities. In fact, the familiar axioms of associativity, inverse and identity form one suitable set of identities:

$$x(yz) = (xy)z$$

$$1x = x1 = x$$

$$xx^{-1} = x^{-1}x = 1.$$

A **subvariety** is a subclass of a variety, closed under the operations H, S, P. Notice that although every group is a semigroup, the class of groups does **not** form a subvariety of the variety of semigroups. This is because not every sub**semigroup** of a group is a group.

The class of abelian groups, considered again with signature (2,1,0), also has the HSP closure properties. It forms a subvariety of the variety of groups, and can be defined equationally by the three group axioms above together with the commutativity law:

$$xy = yx.$$

Variety of finite algebras

Since varieties are closed under arbitrary cartesian products, all non-trivial varieties contain infinite algebras. It follows that the theory of varieties is of limited use in the study of finite algebras, where one must often apply techniques particular to the finite case. With this in mind, attempts have been made to develop a finitary analogue of the theory of varieties.

A **variety of finite algebras**, sometimes called a **pseudovariety**, is usually defined to be a class of finite algebras of a given signature, closed under the taking of homomorphic images, subalgebras and finitary direct products. There is no general finitary counterpart to Birkhoff's theorem, but in many cases the introduction of a more complex notion of equations allows similar results to be derived.

Pseudovarieties are of particular importance in the study of finite semigroups and hence in formal language theory. Eilenberg's theorem, often referred to as the *variety theorem* describes a natural correspondence between varieties of regular languages and pseudovarieties of finite semigroups.

Category theory

If A is a finitary algebraic category, then the forgetful functor

$$U : A \rightarrow \mathbf{Set}$$

is monadic. Even more, it is *strictly monadic*, in that the comparison functor

$$K : A \rightarrow \mathbf{Set}^T$$

is an isomorphism (and not just an equivalence).^[1] Here, $\mathbf{Set}^T$ is the Eilenberg-Moore category on $\mathbf{Set}$. In general, one says a category is an **algebraic category** if it is monadic over $\mathbf{Set}$. This is a more general notion than "finitary algebraic category" (the notion of "variety" used in universal algebra) because it admits such categories as **CABA** (complete atomic Boolean algebras) and **CSLat** (complete semilattices) whose signatures include infinitary operations. In those two cases the signature is large, meaning that it forms not a set but a proper class, because its operations are of unbounded arity. The algebraic category of sigma algebras also has infinitary operations, but their arity is countable whence its signature is small (forms a set).

See also

- Quasivariety

Notes

[1] Saunders Mac Lane, *Categories for the Working Mathematician*, Springer. (See p. 152)

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Categorical logic

Categorical logic is a branch of category theory within mathematics, adjacent to mathematical logic but in fact more notable for its connections to theoretical computer science. In broad terms, categorical logic represents both syntax and semantics by a category, and an interpretation by a functor. The categorical framework provides a rich conceptual background for logical and type-theoretic constructions. The subject has been recognisable in these terms since around 1970.

Categorical logic originated with Bill Lawvere's *Functorial Semantics of Algebraic Theories* (1963), and *Elementary Theory of the Category of Sets* (1964). Lawvere recognised the Grothendieck topos, introduced in algebraic topology as a generalised space, as a generalisation of the category of sets (*Quantifiers and Sheaves* (1970)). With Myles Tierney, Lawvere then developed the notion of elementary topos, thus establishing the fruitful field of topos theory, which provides a unified categorical treatment of the syntax and semantics of higher-order predicate logic. The resulting logic is formally intuitionistic. Andre Joyal is credited, in the term Kripke-Joyal semantics, with the observation that the sheaf models for predicate logic, provided by topos theory, generalise Kripke semantics. Joyal and others applied these models to study higher-order concepts such as the real numbers in the intuitionistic setting.

An analogous development was the link between the typed lambda calculus and cartesian-closed categories (Lawvere, Lambek, Scott), which provided a setting for the development of domain theory. Less expressive theories, from the mathematical logic viewpoint, have their own category theory counterparts. For example the concept of an algebraic theory leads to Gabriel-Ulmer duality. The view of categories as a generalisation unifying syntax and semantics has been productive in the study of logics and type theories for applications in computer science.

The founders of elementary topos theory were Lawvere and Tierney. Lawvere's writings, sometimes couched in a philosophical jargon, isolated some of the basic concepts as adjoint functors (which he explained as 'objective' in a Hegelian sense, not without some

justification). A subobject classifier is a strong property to ask of a category, since with cartesian closure and finite limits it gives a topos (axiom bashing shows how strong the assumption is). Lawvere's further work in the 1960s gave a theory of attributes, which in a sense is a subobject theory more in sympathy with type theory. Major influences subsequently have been Martin-Löf type theory from the direction of logic, type polymorphism and the calculus of constructions from functional programming, linear logic from proof theory, game semantics and the projected synthetic domain theory. The abstract categorical idea of fibration has been much applied.

To go back historically, the major irony here is that in large-scale terms, intuitionistic logic had reappeared in mathematics, in a central place in the Bourbaki-Grothendieck program, a generation after the messy Hilbert-Brouwer controversy had ended, with Hilbert the apparent winner. Bourbaki, or more accurately Jean Dieudonné, having laid claim to the legacy of Hilbert and the Göttingen school including Emmy Noether, had revived intuitionistic logic's credibility (although Dieudonné himself found Intuitionistic Logic ludicrous), as the logic of an arbitrary topos, where classical logic was that of 'the' topos of sets. This was one consequence, certainly unanticipated, of Grothendieck's relative point of view; and not lost on Pierre Cartier, one of the broadest of the core group of French mathematicians around Bourbaki and IHES. Cartier was to give a Séminaire Bourbaki exposition of intuitionistic logic.

In an even broader perspective, one might take category theory to be to the mathematics of the second half of the twentieth century, what measure theory was to the first half. It was Kolmogorov who applied measure theory to probability theory, the first convincing (if not the only) axiomatic approach. Kolmogorov was also a pioneer writer in the early 1920s on the formulation of intuitionistic logic, in a style entirely supported by the later categorical logic approach (again, one of the formulations, not the only one; the realizability concept of Stephen Kleene is also a serious contender here). Another route to categorical logic would therefore have been through Kolmogorov, and this is one way to explain the protean Curry-Howard isomorphism.

External links

- Categorical Logic ^[1] lecture notes by Steve Awodey ^[2]

See also

- Background and genesis of topos theory

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[1] <http://www.andrew.cmu.edu/user/awodey/catlog/>

[2] <http://www.andrew.cmu.edu/user/awodey/>

Category (mathematics)

In mathematics, a **category** is a fundamental and abstract way to describe mathematical entities and their relationships. A category is composed of a collection of abstract "objects" of any kind, linked together by a collection of abstract "arrows" of any kind that have a few basic properties (the ability to compose the arrows associatively and the existence of an identity arrow for each object). Two categories are the same if they have the same collection of objects, the same collection of arrows, and the same associative method of composing any two arrows. Two categories may also be considered "equivalent" for purposes of category theory, even if they are not precisely the same. Many well-known categories are conventionally identified by a short capitalized word or abbreviation in bold or italics such as **Set** (category of sets) or **Ring** (category of rings).

The notion of a *category* is the central idea within a branch of mathematics called *category theory*, which seeks to generalize all of mathematics in terms of such abstract objects and arrows, independent of the particular details of what the objects and arrows represent. Virtually every branch of modern mathematics can be described in terms of categories, and doing so often reveals deep insights and similarities between seemingly-different areas of mathematics. For more extensive motivational background and historical notes, see category theory and the list of category theory topics.

Definition

A **category** C consists of

- a class $\text{ob}(C)$ of **objects**:
- a class $\text{hom}(C)$ of **morphisms**, or **arrows**, or **maps**, between the objects. Each morphism f has a unique *source object* a and *target object* b where a and b are in $\text{ob}(C)$. We write $f: a \rightarrow b$, and we say " f is a morphism from a to b ". We write $\text{hom}(a, b)$ (or $\text{hom}_C(a, b)$ when there may be confusion about which category $\text{hom}(a, b)$ refers to) to denote the **hom-class** of all morphisms from a to b . (Some authors write $\text{Mor}(a, b)$ or simply $C(a, b)$ instead.)
- for every three objects a, b and c , a binary operation $\text{hom}(a, b) \times \text{hom}(b, c) \rightarrow \text{hom}(a, c)$ called *composition of morphisms*; the composition of $f: a \rightarrow b$ and $g: b \rightarrow c$ is written as $g \circ f$ or gf . (Some authors write fg or $f;g$.)

such that the following axioms hold:

- (associativity) if $f : a \rightarrow b$, $g : b \rightarrow c$ and $h : c \rightarrow d$ then $h \circ (g \circ f) = (h \circ g) \circ f$, and
- (identity) for every object x , there exists a morphism $1_x : x \rightarrow x$ called the *identity morphism* for x , such that for every morphism $f : a \rightarrow b$, we have $1_b \circ f = f = f \circ 1_a$.

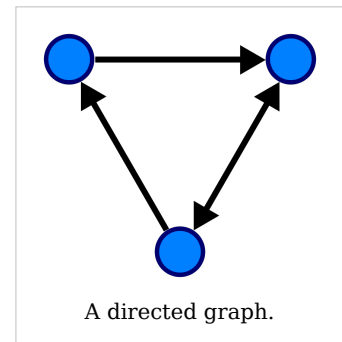
From these axioms, one can prove that there is exactly one identity morphism for every object. Some authors use a slight variation of the definition in which each object is identified with the corresponding identity morphism.

A **small category** is a category in which both $\text{ob}(C)$ and $\text{hom}(C)$ are actually sets and not proper classes. A category that is not small is said to be **large**. A **locally small category** is a category such that for all objects a and b , the hom-class $\text{hom}(a, b)$ is a set, called a **homset**. Many important categories in mathematics (such as the category of sets), although not small, are at least locally small.

Examples

The category **Set** consists of all sets together with all functions between sets, where composition is the usual function composition. **Set** is a large category.

Any directed graph generates a small category: the objects are the vertices of the graph, and the morphisms are the paths in the graph where composition of morphisms is concatenation of paths. Such a category is called the *free category* generated by the graph. This example demonstrates that morphisms need not be functions.



A *discrete category* is a category whose only morphisms are the identity morphisms. If I is a set, the *discrete category on I* is the small category that has the elements of I as objects and only the identity morphisms as morphisms. The composition law is forced, because there is at most one morphism from any object to another. Discrete categories are the simplest kind of category.

Any monoid forms a small category with a single object x . (Here, x is any fixed set.) The morphisms from x to x are precisely the elements of the monoid, and the categorical composition of morphisms is given by the monoid operation. One may view categories as generalizations of monoids; several definitions and theorems about monoids may be generalized for categories.

Any group can be seen as a category with a single object in which every morphism is invertible (for every morphism f there is a morphism g that is both left and right inverse to f under composition) by viewing the group as acting on itself by left multiplication.

Any preordered set $(P, \leq)$ forms a small category, where the objects are the members of P , the morphisms are arrows pointing from x to y when $x \leq y$. Between any two objects there can be at most one morphism. The existence of identity morphisms and the composability of the morphisms are guaranteed by the reflexivity and the transitivity of the preorder. By the same argument, any partially ordered set and any equivalence relation can be seen as a small category. Any ordinal number can be seen as a category when viewed as a ordered set.

The category **Rel** consists of all sets, with binary relations as morphisms. Abstracting from relations instead of functions yields allegories instead of categories.

The category **Cat** consists of all small categories with functors.

Concrete categories

The following are examples of concrete categories, obtained by adding some type of structure onto **Set**, and requiring that morphisms are functions that respect this added structure; the morphism composition is ordinary function composition.

Category	Objects	Morphisms
Ord	preordered sets	monotonic functions
Mag	magmas	magma homomorphisms
Grp	groups	group homomorphisms
Ab	abelian groups	group homomorphisms
Ring	rings	ring homomorphisms
R-Mod	R-Modules, where R is a Ring	module homomorphisms
Vect_K	vector spaces over the field K	K-linear maps
Top	topological spaces	continuous functions
Met	metric spaces	short maps
Uni	uniform spaces	uniformly continuous functions
Man^p	smooth manifolds	p-times continuously differentiable maps

Fiber bundles with bundle maps between them form a concrete category.

Construction of new categories

Dual category

Any category C can itself be considered as a new category in a different way: the objects are the same as those in the original category but the arrows are those of the original category reversed. This is called the *dual* or *opposite category* and is denoted C^{op} .

Product categories

If C and D are categories, one can form the *product category* $C \times D$: the objects are pairs consisting of one object from C and one from D , and the morphisms are also pairs, consisting of one morphism in C and one in D . Such pairs can be composed componentwise.

Types of morphisms

A morphism $f : a \rightarrow b$ is called

- a *monomorphism* (or *monic*) if $fg_1 = fg_2$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : x \rightarrow a$.
- an *epimorphism* (or *epic*) if $g_1f = g_2f$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : b \rightarrow x$.
- a **bimorphism** if it is both a monomorphism and an epimorphism.
- a *retraction* if it has a right inverse, i.e. if there exists a morphism $g : b \rightarrow a$ with $fg = 1_b$.
- a *section* if it has a left inverse, i.e. if there exists a morphism $g : b \rightarrow a$ with $gf = 1_a$.
- an *isomorphism* if it has an inverse, i.e. if there exists a morphism $g : b \rightarrow a$ with $fg = 1_b$ and $gf = 1_a$.
- an *endomorphism* if $a = b$. The class of endomorphisms of a is denoted $\text{end}(a)$.
- an *automorphism* if f is both an endomorphism and an isomorphism. The class of automorphisms of a is denoted $\text{aut}(a)$.

Every retraction is an epimorphism. Every section is a monomorphism. The following three statements are equivalent:

- f is a monomorphism and a retraction;
- f is an epimorphism and a section;
- f is an isomorphism.

Relations among morphisms (such as $fg = h$) can most conveniently be represented with commutative diagrams, where the objects are represented as points and the morphisms as arrows.

Types of categories

- In many categories, e.g. **Ab** or **Vect**_K, the hom-sets $\text{hom}(a, b)$ are not just sets but actually abelian groups, and the composition of morphisms is compatible with these group structures; i.e. is bilinear. Such a category is called preadditive. If, furthermore, the category has all finite products and coproducts, it is called an additive category. If all morphisms have a kernel and a cokernel, and all epimorphisms are cokernels and all monomorphisms are kernels, then we speak of an abelian category. A typical example of an abelian category is the category of abelian groups.
- A category is called complete if all limits exist in it. The categories of sets, abelian groups and topological spaces are complete.
- A category is called cartesian closed if it has finite direct products and a morphism defined on a finite product can always be represented by a morphism defined on just one of the factors. Examples include **Set** and **CPO**, the category of complete partial orders with Scott-continuous functions.
- A topos is a certain type of cartesian closed category in which all of mathematics can be formulated (just like classically all of mathematics is formulated in the category of sets). A topos can also be used to represent a logical theory.
- A groupoid is a category in which every morphism is an isomorphism. Groupoids are generalizations of groups, group actions and equivalence relations.

See also

- Enriched category
- Higher category theory
- Table of mathematical symbols

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External links

- Chris Hillman, Categorical primer ^[1], formal introduction to Category Theory.
- Homepage of the Categories mailing list ^[2], with extensive list of resources
- *Category Theory* section of Alexandre Stefanov's list of free online mathematics resources ^[3]

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Abelian category

In mathematics, an **abelian category** is a category in which morphisms and objects can be added and in which kernels and cokernels exist and have desirable properties. The motivating prototype example of an abelian category is the category of abelian groups, **Ab**. The theory originated in a tentative attempt to unify several cohomology theories by Alexander Grothendieck. Abelian categories are very *stable* categories, for example they are regular and they satisfy the snake lemma. The class of Abelian categories is closed under several categorical constructions, for example, the category of chain complexes of an Abelian category, or the category of functors from a small category to an Abelian category are Abelian as well. These stability properties make them inevitable in homological algebra and beyond; the theory has major applications in algebraic geometry, cohomology and pure category theory.

Definitions

A category is **abelian** if

- it has a zero object,
- it has all *pullbacks* and *pushouts*, and
- all monomorphisms and epimorphisms are normal.

By a theorem of Peter Freyd, this definition is equivalent to the following "piecemeal" definition:

- A category is *preadditive* if it is enriched over the monoidal category **Ab** of abelian groups. This means that all hom-sets are abelian groups and the composition of morphisms is bilinear.
- A preadditive category is *additive* if every finite set of objects has a biproduct. This means that we can form finite direct sums and direct products.
- An additive category is *preabelian* if every morphism has both a kernel and a cokernel.
- Finally, a preabelian category is **abelian** if every monomorphism and every epimorphism is normal. This means that every monomorphism is a kernel of some morphism, and every epimorphism is a cokernel of some morphism.

Note that the enriched structure on hom-sets is a *consequence* of the three axioms of the first definition. This highlights the foundational relevance of the category of Abelian groups in the theory and its canonical nature.

The concept of exact sequence arises naturally in this setting, and it turns out that exact functors, i.e. the functors preserving exact sequences in various senses, are the relevant functors between Abelian categories. This *exactness* concept has been axiomatized in the theory of exact categories, forming a very special case of regular categories.

Examples

- As mentioned above, the category of all abelian groups is an abelian category. The category of all finitely generated abelian groups is also an abelian category, as is the category of all finite abelian groups.
- If R is a ring, then the category of all left (or right) modules over R is an abelian category. In fact, it can be shown that any small abelian category is equivalent to a full subcategory of such a category of modules (*Mitchell's embedding theorem*).
- If R is a left-noetherian ring, then the category of finitely generated left modules over R is abelian. In particular, the category of finitely generated modules over a noetherian commutative ring is abelian; in this way, abelian categories show up in commutative algebra.
- As special cases of the two previous examples: the category of vector spaces over a fixed field k is abelian, as is the category of finite-dimensional vector spaces over k .
- If X is a topological space, then the category of all (real or complex) vector bundles on X is not usually an abelian category, as there can be monomorphisms that are not kernels.
- If X is a topological space, then the category of all sheaves of abelian groups on X is an abelian category. More generally, the category of sheaves of abelian groups on a Grothendieck site is an abelian category. In this way, abelian categories show up in algebraic topology and algebraic geometry.
- If **C** is a small category and **A** is an abelian category, then the category of all functors from **C** to **A** forms an abelian category (the morphisms of this category are the natural

transformations between functors). If $\mathbf{C}$ is small and preadditive, then the category of all additive functors from $\mathbf{C}$ to $\mathbf{A}$ also forms an abelian category. The latter is a generalization of the R -module example, since a ring can be understood as a preadditive category with a single object.

Grothendieck's axioms

In his Tôhoku article, Grothendieck listed four additional axioms (and their duals) that an abelian category $\mathbf{A}$ might satisfy. These axioms are still in common use to this day. They are the following:

- AB3) For every set $\{A_i\}$ of objects of $\mathbf{A}$, the coproduct $\coprod A_i$ exists in $\mathbf{A}$ (i.e. $\mathbf{A}$ is cocomplete).
- AB4) $\mathbf{A}$ satisfies AB3), and the coproduct of a family of monomorphisms is a monomorphism.
- AB5) $\mathbf{A}$ satisfies AB3), and filtered colimits of exact sequences are exact.

and their duals

- AB3*) For every set $\{A_i\}$ of objects of $\mathbf{A}$, the product $\prod A_i$ exists in $\mathbf{A}$ (i.e. $\mathbf{A}$ is complete).
- AB4*) $\mathbf{A}$ satisfies AB3*), and the product of a family of epimorphisms is an epimorphism.
- AB5*) $\mathbf{A}$ satisfies AB3*), and filtered limits of exact sequences are exact.

Axioms AB1) and AB2) were also given. They are what make an additive category abelian. Specifically:

- AB1) Every morphism has a kernel and a cokernel.
- AB2) For every morphism f , the canonical morphism from $\text{coim } f$ to $\text{im } f$ is an isomorphism.

Grothendieck also gave axioms AB6) and AB6*).

Elementary properties

Given any pair A, B of objects in an abelian category, there is a special zero morphism from A to B . This can be defined as the zero element of the hom-set $\text{Hom}(A, B)$, since this is an abelian group. Alternatively, it can be defined as the unique composition $A \rightarrow 0 \rightarrow B$, where 0 is the zero object of the abelian category.

In an abelian category, every morphism f can be written as the composition of an epimorphism followed by a monomorphism. This epimorphism is called the *coimage* of f , while the monomorphism is called the *image* of f .

Subobjects and quotient objects are well-behaved in abelian categories. For example, the poset of subobjects of any given object A is a bounded lattice.

Every abelian category $\mathbf{A}$ is a module over the monoidal category of finitely generated abelian groups; that is, we can form a tensor product of a finitely generated abelian group G and any object A of $\mathbf{A}$. The abelian category is also a comodule; $\text{Hom}(G, A)$ can be interpreted as an object of $\mathbf{A}$. If $\mathbf{A}$ is complete, then we can remove the requirement that G be finitely generated; most generally, we can form finitary enriched limits in $\mathbf{A}$.

Related concepts

Abelian categories are the most general setting for homological algebra. All of the constructions used in that field are relevant, such as exact sequences, and especially short exact sequences, and derived functors. Important theorems that apply in all abelian categories include the five lemma (and the short five lemma as a special case), as well as the snake lemma (and the nine lemma as a special case).

History

Abelian categories were introduced by Alexander Grothendieck in his famous Tôhoku paper in the middle of the 1950s in order to unify various cohomology theories. At the time, there was a cohomology theory for sheaves, and a cohomology theory for groups. The two were defined completely differently, but they had formally almost identical properties. In fact, much of category theory was developed as a language to study these similarities. Grothendieck managed to unify the two theories: they both arise as derived functors on abelian categories; on the one hand the abelian category of sheaves of abelian groups on a topological space, on the other hand the abelian category of G -modules for a given group G .

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Topos

In mathematics, a **topos** (plural "topoi" or "toposes") is a type of category that behaves like the category of sheaves of sets on a topological space. For a discussion of the history of topos theory, see the article Background and genesis of topos theory.

Grothendieck topoi (topoi in geometry)

Since the introduction of sheaves into mathematics in the 1940s a major theme has been to study a space by studying sheaves on that space. This idea was expounded by Alexander Grothendieck by introducing the notion of a **topos**. The main utility of this notion is in the abundance of situations in mathematics where topological intuition is very effective but an honest topological space is lacking; it is sometimes possible to find a topos formalizing the intuition. The greatest single success of this programmatic idea to date has been the introduction of the étale topos of a scheme.

Equivalent formulations

Let C be a category. A theorem of Giraud states that the following are equivalent:

- There is a small category D and an inclusion $C \hookrightarrow \text{Presh}(D)$ that admits a finite-limit-preserving left adjoint.
- C is the category of sheaves on a Grothendieck site.
- C satisfies Giraud's axioms, below.

A category with these properties is called a "(Grothendieck) topos". Here $\text{Presh}(D)$ denotes the category of contravariant functors from D to the category of sets; such a contravariant functor is frequently called a presheaf.

Giraud's axioms

Giraud's axioms for a category C are:

- C has a small set of generators, and admits all small colimits. Furthermore, colimits commute with fiber products.
- Sums in C are disjoint. In other words, the fiber product of X and Y over their sum is the initial object in C .
- All equivalence relations in C are effective.

The last axiom needs the most explanation. If X is an object of C , an equivalence relation R on X is a map $R \rightarrow X \times X$ in C such that all the maps $\text{Hom}(Y, R) \rightarrow \text{Hom}(Y, X) \times \text{Hom}(Y, X)$ are equivalence relations of sets. Since C has colimits we may form the coequalizer of the two maps $R \rightarrow X$; call this X/R . The equivalence relation is effective if the canonical map

$$R \rightarrow X \times_{X/R} X$$

is an isomorphism.

Examples

Giraud's theorem already gives "sheaves on sites" as a complete list of examples. Note, however, that nonequivalent sites often give rise to equivalent topoi. As indicated in the introduction, sheaves on ordinary topological spaces motivate many of the basic definitions and results of topos theory.

The category of sets is an important special case: it plays the role of a point in topos theory. Indeed, a set may be thought of as a sheaf on a point.

More exotic examples, and the *raison d'être* of topos theory, come from algebraic geometry. To a scheme and even a stack one may associate an étale topos, an fppf topos, a Nisnevich topos...

Counterexamples

Topos theory is, in some sense, a generalization of classical point-set topology. One should therefore expect to see old and new instances of pathological behavior. For instance, there is an example due to Pierre Deligne of a nontrivial topos that has no points (see below).

Geometric morphisms

If X and Y are topoi, a *geometric morphism* $u: X \rightarrow Y$ is a pair of adjoint functors (u^*, u_*) such that u^* preserves finite limits. Note that u^* automatically preserves colimits by virtue of having a right adjoint.

By Freyd's adjoint functor theorem, to give a geometric morphism $X \rightarrow Y$ is to give a functor $u^*: Y \rightarrow X$ that preserves finite limits and all small colimits. Thus geometric morphisms between topoi may be seen as analogues of maps of locales.

If X and Y are topological spaces and u is a continuous map between them, then the pullback and pushforward operations on sheaves yield a geometric morphism between the associated topoi.

Points of topoi

A point of a topos X is a geometric morphism from the topos of sets to X .

If X is an ordinary space and x is a point of X , then the functor that takes a sheaf F to its stalk F_x has a right adjoint (the "skyscraper sheaf" functor), so an ordinary point of X also determines a topos-theoretic point. These may be constructed as the pullback-pushforward along the continuous map $x: 1 \rightarrow X$.

Essential geometric morphisms

A geometric morphism (u^*, u_*) is *essential* if u^* has a further left adjoint $u_!$, or equivalently (by the adjoint functor theorem) if u^* preserves not only finite but all small limits.

Ringed topoi

A **ringed topos** is a pair (X, R) , where X is a topos and R is a commutative ring object in X . Most of the constructions of ringed spaces go through for ringed topoi. The category of R -module objects in X is an abelian category with enough injectives. A more useful abelian category is the subcategory of quasi-coherent R -modules: these are R -modules that admit a presentation.

Another important class of ringed topoi, besides ringed spaces, are the étale topoi of Deligne-Mumford stacks.

Homotopy theory of topoi

Michael Artin and Barry Mazur associated to any topos a pro-simplicial set. Using this inverse system of simplicial sets one may *sometimes* associate to a homotopy invariant in classical topology an inverse system of invariants in topos theory.

The pro-simplicial set associated to the étale topos of a scheme is a pro-finite simplicial set. Its study is called étale homotopy theory.

Elementary toposes (toposes in logic)

Introduction

A traditional axiomatic foundation of mathematics is set theory, in which all mathematical objects are ultimately represented by sets (even functions which map between sets). More recent work in category theory allows this foundation to be generalized using toposes; each topos completely defines its own mathematical framework. The category of sets forms a familiar topos, and working within this topos is equivalent to using traditional set theoretic mathematics. But one could instead choose to work with many alternative toposes. A standard formulation of the axiom of choice makes sense in any topos, and there are toposes in which it is invalid. Constructivists will be interested to work in a topos without the law of excluded middle. If symmetry under a particular group G is of importance, one can use the topos consisting of all G -sets.

It is also possible to encode an algebraic theory, such as the theory of groups, as a topos. The individual models of the theory, i.e. the groups in our example, then correspond to functors from the encoding topos to the category of sets that respect the topos structure.

Formal definition

When used for foundational work a topos will be defined axiomatically; set theory is then treated as a special case of topos theory. Building from category theory, there are multiple equivalent definitions of a topos. The following has the virtue of being concise, if not illuminating:

A topos is a category which has the following two properties:

- All limits taken over finite index categories exist.
- Every object has a power object.

From this one can derive that

- All colimits taken over finite index categories exist.
- The category has a subobject classifier.
- Any two objects have an exponential object.
- The category is cartesian closed.

In many applications, the role of the subobject classifier is pivotal, whereas power objects are not. Thus some definitions reverse the roles of what is defined and what is derived.

Explanation

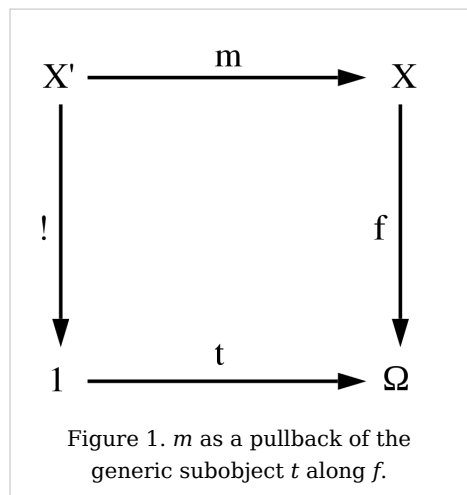
A topos as defined above can be understood as a cartesian closed category for which the notion of subobject of an object has an elementary or first-order definition. This notion, as a natural categorical abstraction of the notions of subset of a set, subgroup of a group, and more generally subalgebra of any algebraic structure, predates the notion of topos. It is definable in any category, not just toposes, in second-order language, i.e. in terms of classes of morphisms instead of individual morphisms, as follows. Given two monics m, n from respectively Y and Z to X , we say that $m \leq n$ when there exists a morphism $p: Y \rightarrow Z$ for which $np = m$, inducing a preorder on monics to X . When $m \leq n$ and $n \leq m$ we say that m and n are equivalent. The subobjects of X are the resulting equivalence classes of the monics to it.

In a topos "subobject" becomes, at least implicitly, a first-order notion, as follows.

As noted above, a topos is a category C having all finite limits and hence in particular the empty limit or final object 1 . It is then natural to treat morphisms of the form $x: 1 \rightarrow X$ as *elements* $x \in X$. Morphisms $f: X \rightarrow Y$ thus correspond to functions mapping each element $x \in X$ to the element $fx \in Y$, with application realized by composition.

One might then think to define a subobject of X as an equivalence class of monics $m: X' \rightarrow X$ having the same image or range $\{mx \mid x \in X'\}$. The catch is that two or more morphisms may correspond to the same function, that is, we cannot assume that C is concrete in the sense that the functor $C(1, -): C \rightarrow \mathbf{Set}$ is faithful. For example the category **Grph** of graphs and their associated homomorphisms is a topos whose final object 1 is the graph with one vertex and one edge (a self-loop), but is not concrete because the elements $1 \rightarrow G$ of a graph G correspond only to the self-loops and not the other edges, nor the vertices without self-loops. Whereas the second-order definition makes G and its set of self-loops (with their vertices) distinct subobjects of G (unless every edge is, and every vertex has, a self-loop), this image-based one does not. This can be addressed for the graph example and related examples via the Yoneda Lemma as described in the Examples section below, but this then ceases to be first-order. Toposes provide a more abstract, general, and first-order solution.

As noted above a topos C has a subobject classifier Ω , namely an object of C with an element $t \in \Omega$, the *generic subobject* of C , having the property that every monic $m: X' \rightarrow X$ arises as a pullback of the generic subobject along a unique morphism $f: X \rightarrow \Omega$, as per Figure 1. Now the pullback of a monic is a monic, and all elements including t are monics since there is only one morphism to 1 from any given object, whence the pullback of t along $f: X \rightarrow \Omega$ is a monic. The monics to X are therefore in bijection with the pullbacks of t along morphisms from X to Ω . The latter morphisms partition the monics into equivalence classes each determined by a morphism $f: X \rightarrow \Omega$, the characteristic morphism of that class, which we take to be the subobject of X characterized or named by f .



All this applies to any topos, whether or not concrete. In the concrete case, namely $C(1, -)$ faithful, for example the category of sets, the situation reduces to the familiar behavior of functions. Here the monics $m: X' \rightarrow X$ are exactly the injections (one-one functions) from X' to X , and those with a given image $\{mx \mid x \in X'\}$ constitute the subobject of X

corresponding to the morphism $f: X \rightarrow \Omega$ for which $f^{-1}(t)$ is that image. The monics of a subobject will in general have many domains, all of which however will be in bijection with each other.

To summarize, this first-order notion of subobject classifier implicitly defines for a topos the same equivalence relation on monics to X as had previously been defined explicitly by the second-order notion of subobject for any category. The notion of equivalence relation on a class of morphisms is itself intrinsically second-order, which the definition of topos neatly sidesteps by explicitly defining only the notion of subobject *classifier* Ω , leaving the notion of subobject of X as an implicit consequence characterized (and hence namable) by its associated morphism $f: X \rightarrow \Omega$.

Further examples

If C is a small category, then the functor category $\mathbf{Set}^C$ (consisting of all covariant functors from C to sets, with natural transformations as morphisms) is a topos. For instance, the category **Grph** of graphs of the kind permitting multiple directed edges between two vertices is a topos. A graph consists of two sets, an edge set and a vertex set, and two functions s, t between those sets, assigning to every edge e its source $s(e)$ and target $t(e)$. **Grph** is thus equivalent to the functor category $\mathbf{Set}^C$, where C is the category with two objects E and V and two morphisms $s, t: E \rightarrow V$ giving respectively the source and target of each edge.

The categories of finite sets, of finite G -sets (actions of a group G on a finite set), and of finite graphs are also toposes.

The Yoneda Lemma asserts that C^{op} embeds in $\mathbf{Set}^C$ as a full subcategory. In the graph example the embedding represents C^{op} as the subcategory of $\mathbf{Set}^C$ whose two objects are V' as the one-vertex no-edge graph and E' as the two-vertex one-edge graph (both as functors), and whose two nonidentity morphisms are the two graph homomorphisms from V' to E' (both as natural transformations). The natural transformations from V' to an arbitrary graph (functor) G constitute the vertices of G while those from E' to G constitute its edges. Although $\mathbf{Set}^C$, which we can identify with **Grph**, is not made concrete by either V' or E' alone, the functor $U: \mathbf{Grph} \rightarrow \mathbf{Set}^2$ sending object G to the pair of sets $(\mathbf{Grph}(V', G), \mathbf{Grph}(E', G))$ and morphism $h: G \rightarrow H$ to the pair of functions $(\mathbf{Grph}(V', h), \mathbf{Grph}(E', h))$ is faithful. That is, a morphism of graphs can be understood as a *pair* of functions, one mapping the vertices and the other the edges, with application still realized as composition but now with multiple sorts of *generalized* elements. This shows that the traditional concept of a concrete category as one whose objects have an underlying set can be generalized to cater for a wider range of toposes by allowing an object to have multiple underlying sets, that is, to be multisorted.

See also

- Background and genesis of topos theory
- Category theory
- Intuitionistic type theory

References

Some gentle papers

- John Baez: "Topos theory in a nutshell." ^[1] A gentle introduction.
- Stephen Vickers: "Toposes pour les nuls" ^[2] and "Toposes pour les vraiment nuls." ^[3] Elementary and even more elementary introductions to toposes as generalized spaces.
- Illusie, Luc, "<http://www.ams.org/notices/200409/what-is-illusie.pdf> What is a ... topos?", *Notices of the AMS*, <http://www.ams.org/notices/200409/what-is-illusie.pdf>

The following texts are easy-paced introductions to toposes and the basics of category theory. They should be suitable for those knowing little mathematical logic and set theory, even non-mathematicians.

- F. William Lawvere and Stephen H. Schanuel (1997) *Conceptual Mathematics: A First Introduction to Categories*. Cambridge University Press. An "introduction to categories for computer scientists, logicians, physicists, linguists, etc." (cited from cover text).
- F. William Lawvere and Robert Rosebrugh (2003) *Sets for Mathematics*. Cambridge University Press. Introduces the foundations of mathematics from a categorical perspective.

Grothendieck foundational work on toposes:

- Grothendieck and Verdier: *Théorie des topos et cohomologie étale des schémas* (known as SGA4)". New York/Berlin: Springer, ?? (Lecture notes in mathematics, 269–270)

The following monographs include an introduction to some or all of topos theory, but do not cater primarily to beginning students. Listed in (perceived) order of increasing difficulty.

- Colin McLarty (1992) *Elementary Categories, Elementary Toposes*. Oxford Univ. Press. A nice introduction to the basics of category theory, topos theory, and topos logic. Assumes very few prerequisites.
- Robert Goldblatt (1984) *Topoi, the Categorical Analysis of Logic* (Studies in logic and the foundations of mathematics, 98). North-Holland. A good start. Reprinted 2006 by Dover Publications, and available online ^[4] at Robert Goldblatt's homepage. ^[5]
- John Lane Bell (2005) *The Development of Categorical Logic*. Handbook of Philosophical Logic, Volume 12. Springer. Version available online ^[6] at John Bell's homepage. ^[7]
- Saunders Mac Lane and Ieke Moerdijk (1992) *Sheaves in Geometry and Logic: a First Introduction to Topos Theory*. Springer Verlag. More complete, and more difficult to read.
- Michael Barr and Charles Wells (1985) *Toposes, Triples and Theories*. Springer Verlag. Corrected online version at <http://www.cwru.edu/artsci/math/wells/pub/ttt.html> ^[8]. More concise than *Sheaves in Geometry and Logic*, but hard on beginners.

Reference works for experts, less suitable for first introduction

- Francis Borceux (1994) *Handbook of Categorical Algebra 3: Categories of Sheaves*, Volume 52 of the *Encyclopedia of Mathematics and its Applications*. Cambridge University Press. The third part of "Borceux' remarkable magnum opus", as Johnstone

has labelled it. Still suitable as an introduction, though beginners may find it hard to recognize the most relevant results among the huge amount of material given.

- Peter T. Johnstone (1977) *Topos Theory*, L. M. S. Monographs no. 10. Academic Press. For a long time the standard compendium on topos theory. However, even Johnstone describes this work as "far too hard to read, and not for the faint-hearted."
- Peter T. Johnstone (2002) *Sketches of an Elephant: A Topos Theory Compendium*. Oxford Science Publications. As of early 2006, two of the scheduled three volumes of this overwhelming compendium were available.

Books that target special applications of topos theory

- Maria Cristina Pedicchio and Walter Tholen, eds. (2004) *Categorical Foundations: Special Topics in Order, Topology, Algebra, and Sheaf Theory*. Volume 97 of the *Encyclopedia of Mathematics and its Applications*. Cambridge University Press. Includes many interesting special applications.

Online encyclopedias:

- Topos^[9] on PlanetMath

References

- [1] <http://math.ucr.edu/home/baez/topos.html>
- [2] <http://www.cs.bham.ac.uk/~sjv/#papers>
- [3] <http://www.cs.bham.ac.uk/~sjv/TopPLVN.pdf>
- [4] <http://historical.library.cornell.edu/cgi-bin/cul.math/docviewer?did=Gold010&id=3>
- [5] <http://www.mcs.vuw.ac.nz/~rob/>
- [6] <http://publish.uwo.ca/~jbell/catlogprime.pdf>
- [7] <http://publish.uwo.ca/~jbell/>
- [8] <http://www.cwru.edu/artsci/math/wells/pub/ttt.html>
- [9] <http://planetmath.org/?op=getobj&from=objects&id=8796>

Heyting algebra

In mathematics, **Heyting algebras** are special partially ordered sets that constitute a generalization of Boolean algebras, named after Arend Heyting. Heyting algebras arise as models of intuitionistic logic, a logic in which the law of excluded middle does not in general hold. Complete Heyting algebras are a central object of study in pointless topology.

Formal definition

A Heyting algebra H is a bounded lattice such that for all a and b in H there is a greatest element x of H such that

$$a \wedge x \leq b.$$

This element is the **relative pseudo-complement** of a with respect to b , and is denoted $a \rightarrow b$. We write 1 and 0 for the largest and the smallest element of H , respectively.

In any Heyting algebra, one defines the **pseudo-complement** $\neg x$ of any element x by setting $\neg x = (x \rightarrow 0)$. By definition, $a \wedge \neg a = 0$, and $\neg a$ is the largest element having this property. However, it is not in general true that $a \vee \neg a = 1$, thus $\neg$ is only a pseudo-complement, not a true complement, as would be the case in a Boolean algebra.

A **complete Heyting algebra** is a Heyting algebra that is a complete lattice.

A **subalgebra** of a Heyting algebra H is a subset H_1 of H containing 0 and 1 and closed under the operations $\wedge$, $\vee$ and $\rightarrow$. It follows that it is also closed under $\neg$. A subalgebra is made into a Heyting algebra by the induced operations.

Alternative definitions

Lattice-theoretic definitions

An equivalent definition of Heyting algebras can be given by considering the mappings

$$f_a : H \rightarrow H \text{ defined by } f_a(x) = a \wedge x,$$

for some fixed a in H . A bounded lattice H is a Heyting algebra if and only if all mappings f_a are the lower adjoint of a monotone Galois connection. In this case the respective upper adjoints g_a are given by $g_a(x) = a \rightarrow x$, where $\rightarrow$ is defined as above.

Yet another definition is as a residuated lattice whose monoid operation is $\wedge$. The monoid unit must then be the top element 1. Commutativity of this monoid implies that the two residuals coincide as $a \rightarrow b$.

Bounded lattice with an implication operation

Given a bounded lattice A with largest and smallest elements 1 and 0, and a binary operation $\rightarrow$, these together form a Heyting algebra if and only if the following hold:

1. $a \rightarrow a = 1$
2. $a \wedge (a \rightarrow b) = a \wedge b$
3. $b \wedge (a \rightarrow b) = b$
4. $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$ (distributive law for $\rightarrow$)

Characterization using the axioms of intuitionist logic

This characterization of Heyting algebras makes the proof of the basic facts concerning the relationship between intuitionist propositional calculus and Heyting algebras immediate. (For these facts, see the sections "Provable identities" and "Universal constructions.") One should think of the element 1 as meaning, intuitively, "provably true." Compare with the axioms at Intuitionistic logic#Axiomatization.

Given a set A with three binary operations $\rightarrow$, $\wedge$ and $\vee$, and two distinguished elements 0 and 1, then A is a Heyting algebra for these operations (and the relation $\leq$ defined by the condition that $a \leq b$ when $a \rightarrow b = 1$) if and only if the following conditions hold for any elements x , y and z of A :

EQUIV: if $x \rightarrow y = 1$ and $y \rightarrow x = 1$, then $x = y$ ("equivalent formulas should be identified");

MODUS-PONENS: if $1 \rightarrow y = 1$, then $y = 1$ ("provably true formulas are closed under modus ponens");

THEN-1: $x \rightarrow (y \rightarrow x) = 1$

THEN-2: $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$

AND-1: $x \wedge y \rightarrow x = 1$

AND-2: $x \wedge y \rightarrow y = 1$

AND-3: $x \rightarrow (y \rightarrow (x \wedge y)) = 1$

OR-1: $x \rightarrow x \vee y = 1$

OR-2: $y \rightarrow x \vee y = 1$

OR-3: $(x \rightarrow z) \rightarrow ((y \rightarrow z) \rightarrow (x \vee y \rightarrow z)) = 1$

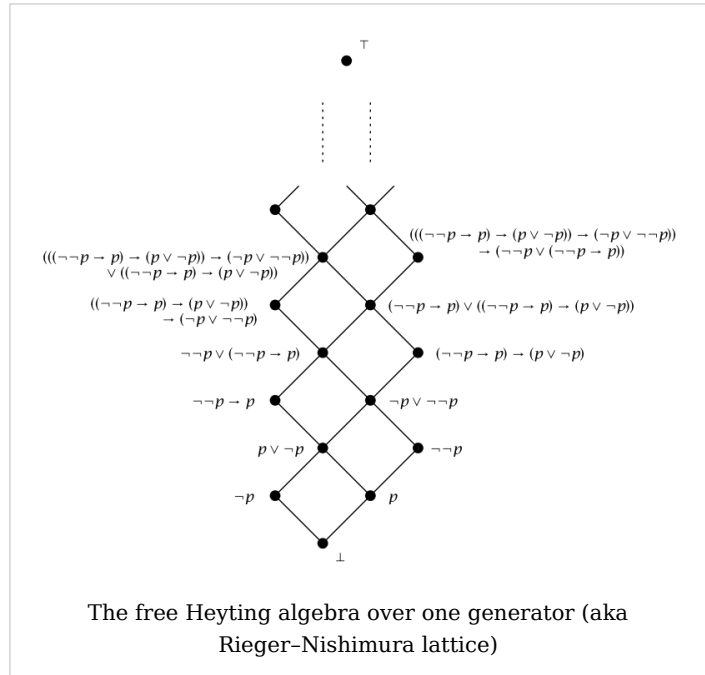
FALSE: $0 \rightarrow x = 1$

Finally, we define $\neg x$ to be $x \rightarrow 0$.

Of course, if a different set of axioms were chosen for logic, we could modify ours accordingly.

Examples

- Every Boolean algebra is a Heyting algebra, with $p \rightarrow q$ given by $\neg p \vee q$.
- Every totally ordered set that is a bounded lattice is also a Heyting algebra, where $p \rightarrow q$ is equal to q when $p > q$, and 1 otherwise.
- The simplest Heyting algebra that is not already a Boolean algebra is the totally ordered set $\{0, \frac{1}{2}, 1\}$ with $\rightarrow$ defined as above. Notice that $\frac{1}{2} \vee \neg \frac{1}{2} = \frac{1}{2} \vee (\frac{1}{2} \rightarrow 0) = \frac{1}{2} \vee 0 = \frac{1}{2}$ falsifies the law of excluded middle.



- Every topology provides a complete Heyting algebra in the form of its open set lattice. In this case, the element $A \rightarrow B$ is the interior of the union of A_c and B , where A_c denotes the complement of the open set A . Not all complete Heyting algebras are of this form. These issues are studied in pointless topology, where complete Heyting algebras are also called **frames** or **locales**.
- The Lindenbaum algebra of propositional intuitionistic logic is a Heyting algebra.
- The global elements of the subobject classifier Ω of an elementary topos form a Heyting algebra; it is the Heyting algebra of truth values of the intuitionistic higher-order logic induced by the topos.

Properties

General properties

The ordering $\leq$ on a Heyting algebra H can be recovered from the operation $\rightarrow$ as follows: for any elements a, b of H , $a \leq b$ if and only if $a \rightarrow b = 1$.

In contrast to some many-valued logics, Heyting algebras share the following property with Boolean algebras: if negation has a fixed point (i.e. $\neg a = a$ for some a), then the Heyting algebra is the trivial one-element Heyting algebra.

Provable identities

Given a formula $F(A_1, A_2, \dots, A_n)$ of propositional calculus (using, in addition to the variables, the connectives $\wedge, \vee, \neg, \rightarrow$, and the constants 0 and 1), it is a fact, proved early on in any study of Heyting algebras, that the following two conditions are equivalent:

1. the formula F is provably true in intuitionist propositional calculus;
2. the identity $F(a_1, a_2, \dots, a_n) = 1$ is true for any Heyting algebra H and any elements $a_1, a_2, \dots, a_n$ in H .

The implication $1 \rightarrow 2$ is extremely useful and is the principal practical method for proving identities in Heyting algebras. In practice, one frequently uses the deduction theorem in such proofs.

Since for any a and b in a Heyting algebra H we have $a \leq b$ if and only if $a \rightarrow b = 1$, it follows from $1 \rightarrow 2$ that whenever a formula $F \rightarrow G$ is provably true, we have $F(a_1, a_2, \dots, a_n) \leq G(a_1, a_2, \dots, a_n)$ for any Heyting algebra H , and any elements $a_1, a_2, \dots, a_n$ of H . (It follows from the deduction theorem that $F \rightarrow G$ is provable [from nothing] if and only if G is a provable consequence of F .) In particular, if F and G are provably equivalent, then $F(a_1, a_2, \dots, a_n) = G(a_1, a_2, \dots, a_n)$, since $\leq$ is an order relation.

$1 \rightarrow 2$ can be proved by examining the logical axioms of the system of proof and verifying that their value is 1 in any Heyting algebra, and then verifying that the application of the rules of inference to expressions with value 1 in a Heyting algebra results in expressions with value 1. For example, let us choose the system of proof having modus ponens as its sole rule of inference, and whose axioms are the Hilbert-style ones given at Intuitionistic logic#Axiomatization. Then the facts to be verified follow immediately from the axiom-like definition of Heyting algebras given above.

$1 \rightarrow 2$ also provides a method for proving that certain propositional formulas, though tautologies in classical logic, *cannot* be proved in intuitionist propositional logic. In order to prove that some formula $F(A_1, A_2, \dots, A_n)$ is not provable, it is enough to exhibit a Heyting algebra H and elements $a_1, a_2, \dots, a_n$ of H such that $F(a_1, a_2, \dots, a_n) \neq 1$.

If one wishes to avoid mention of logic, then in practice it becomes necessary to prove as a lemma a version of the deduction theorem valid for Heyting algebras: for any elements a, b and c of a Heyting algebra H , we have $(a \wedge b) \rightarrow c = a \rightarrow (b \rightarrow c)$.

For more on the implication $2 \rightarrow 1$, see the section "Universal constructions" below.

Distributivity

Heyting algebras are always distributive. Specifically, we always have the identities

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

The distributive law is sometimes stated as an axiom, but in fact it follows from the existence of relative pseudo-complements. The reason is that, being the lower adjoint of a Galois connection, $\wedge$ preserves all existing suprema. Distributivity in turn is just the preservation of binary suprema by $\wedge$.

By a similar argument, the following infinite distributive law holds in any complete Heyting algebra:

$$x \wedge \bigvee Y = \bigvee \{x \wedge y : y \in Y\}$$

for any element x in H and any subset Y of H . Conversely, any complete lattice satisfying the above infinite distributive law is a complete Heyting algebra, with

$$a \rightarrow b = \bigvee \{c \mid a \wedge c \leq b\}$$

being its relative pseudo-complement operation.

Regular and complemented elements

An element x of a Heyting algebra H is called **regular** if either of the following equivalent conditions hold:

1. $x = \neg\neg x$.
2. $x = \neg y$ for some y in H .

The equivalence of these conditions can be restated simply as the identity $\neg\neg\neg x = \neg x$, valid for all x in H .

Elements x and y of a Heyting algebra H are called **complements** to each other if $x \wedge y = 0$ and $x \vee y = 1$. If it exists, any such y is unique and must in fact be equal to $\neg x$.

We call an element x **complemented** if it admits a complement. It is true that if x is complemented, then so is $\neg x$, and then x and $\neg x$ are complements to each other. However, confusingly, even if x is not complemented, $\neg x$ may nonetheless have a complement (not equal to x). In any Heyting algebra, the elements 0 and 1 are complements to each other.

Any complemented element of a Heyting algebra is regular, though the converse is not true in general. In particular, 0 and 1 are always regular.

For any Heyting algebra H , the following conditions are equivalent:

1. H is a Boolean algebra;
2. every x in H is regular;^[1]
3. every x in H is complemented.^[2]

In this case, the element $a \rightarrow b$ is equal to $\neg a \vee b$.

The regular (resp. complemented) elements of any Heyting algebra H constitute a Boolean algebra H_{reg} (resp. H_{comp}), in which the operations $\wedge$, $\neg$ and $\rightarrow$, as well as the constants 0 and 1, coincide with those of H . In the case of H_{comp} , the operation $\vee$ is also the same, hence H_{comp} is a subalgebra of H . In general however, H_{reg} will not be a subalgebra of H , because its join operation $\vee_{\text{reg}}$ may differ from $\vee$. For $x, y \in H_{\text{reg}}$, we have $x \vee_{\text{reg}} y = \neg(\neg x \wedge \neg y)$. See below for necessary and sufficient conditions in order for $\vee_{\text{reg}}$ to coincide with $\vee$.

The De Morgan laws in a Heyting algebra

One of the two De Morgan laws is satisfied in every Heyting algebra, namely

$$\neg(x \vee y) = \neg x \wedge \neg y, \text{ for all } x, y \in H.$$

However, the other De Morgan law does not always hold. We have instead a weak de Morgan law:

$$\neg(x \wedge y) = \neg\neg(\neg x \vee \neg y), \text{ for all } x, y \in H.$$

The following statements are equivalent for all Heyting algebras H :

1. H satisfies both De Morgan laws;
2. $\neg(x \wedge y) = \neg x \vee \neg y$, for all $x, y \in H$;
3. $\neg(x \wedge y) = \neg x \vee \neg y$, for all regular $x, y \in H$;

4. $\neg\neg(x \vee y) = \neg\neg x \vee \neg\neg y$, for all x, y in H ;
5. $\neg\neg(x \vee y) = x \vee y$, for all regular x, y in H ;
6. $\neg(\neg x \wedge \neg y) = x \vee y$, for all regular x, y in H ;
7. $\neg x \vee \neg\neg x = 1$, for all $x \in H$.

Condition 2 is the other De Morgan law. Condition 6 says that the join operation $\vee_{\text{reg}}$ on the Boolean algebra H_{reg} of regular elements of H coincides with the operation $\vee$ of H . Condition 7 states that every regular element is complemented, i.e., $H_{\text{reg}} = H_{\text{comp}}$.

We prove the equivalence. $1 \rightarrow 2$, $2 \rightarrow 3$ and $4 \rightarrow 5$ are trivial. $3 \rightarrow 4$ and $5 \rightarrow 6$ result simply from the first De Morgan law and the definition of regular elements. 7 results from 6 taking $\neg x$ and $\neg\neg x$ for x and y and using the identity $a \wedge \neg a = 0$. $2 \rightarrow 1$ follows from the first De Morgan law, and $7 \rightarrow 6$ results from the fact that the join operation $\vee$ on the subalgebra H_{comp} is just the restriction of $\vee$ to H_{comp} , taking into account the characterizations we have given of conditions 6 and 7. The implication $5 \rightarrow 2$ is a trivial consequence of the weak De Morgan law, taking $\neg x$ and $\neg y$ for x and y in 5.

Heyting algebras satisfying the above properties are related to De Morgan logic in the same way Heyting algebras in general are related to intuitionist logic.

Heyting algebra morphisms

Definition

Given two Heyting algebras H_1 and H_2 and a mapping $f: H_1 \rightarrow H_2$, we say that f is a **morphism** of Heyting algebras if, for any elements x and y in H_1 , we have:

1. $f(x \wedge y) = f(x) \wedge f(y)$;
2. $f(x \vee y) = f(x) \vee f(y)$;
3. $f(1) = 1$;
4. $f(0) = 0$;
5. $f(x \rightarrow y) = f(x) \rightarrow f(y)$; and
6. $[f(\neg x) = \neg f(x)]$.

We put condition 6 in brackets because it follows from the others, as $\neg x$ is just $x \rightarrow 0$, and one may or may not wish to consider $\neg$ to be a basic operation.

It follows from conditions 3 and 5 (or 1 alone, or 2 alone) that f is an increasing function, that is, that $f(x) \leq f(y)$ whenever $x \leq y$.

Assume H_1 and H_2 are structures with operations $\rightarrow, \wedge, \vee$ (and possibly $\neg$) and constants 0 and 1, and f is a surjective mapping from H_1 to H_2 with properties 1 through 5 (or 1 through 6) above. Then if H_1 is a Heyting algebra, so too is H_2 . This follows from the characterization of Heyting algebras as bounded lattices (thought of as algebraic structures rather than partially ordered sets) with an operation $\rightarrow$ satisfying certain identities.

Properties

The identity map $f(x) = x$ from any Heyting algebra to itself is a morphism, and the composite $g \circ f$ of any two morphisms f and g is a morphism. Hence Heyting algebras form a category.

Examples

Given a Heyting algebra H and any subalgebra H_1 , the inclusion mapping $i:H_1 \rightarrow H$ is a morphism.

For any Heyting algebra H , the map $x \mapsto \neg\neg x$ defines a morphism from H onto the Boolean algebra of its regular elements H_{reg} . This is *not* in general a morphism from H to itself, since the join operation of H_{reg} may be different from that of H .

Quotients

Let H be a Heyting algebra, and let $F \subseteq H$. We call F a **filter** on H if it satisfies the following properties:

1. $1 \in F$;
2. if $x, y \in F$, then $x \wedge y \in F$;
3. if $x \in F$, $y \in H$, and $x \leq y$, then $y \in F$.

The intersection of any set of filters on H is again a filter. Therefore, given any subset S of H there is a smallest filter containing S . We call it the filter **generated** by S . If S is empty, $F = \{1\}$. Otherwise, F is equal to the set of x in H such that there exist $y_1, y_2, \dots, y_n$ in S with $y_1 \wedge y_2 \wedge \dots \wedge y_n \leq x$.

If H is a Heyting algebra and F is a filter on H , we define a relation $\sim$ on H as follows: we write $x \sim y$ whenever $x \rightarrow y$ and $y \rightarrow x$ both belong to F . Then $\sim$ is an equivalence relation; we write H/F for the quotient set. There is a unique Heyting algebra structure on H/F such that the canonical surjection $p_F:H \rightarrow H/F$ becomes a Heyting algebra morphism. We call the Heyting algebra H/F the **quotient** of H by F .

Let S be a subset of a Heyting algebra H and let F be the filter generated by S . Then H/F satisfies the following universal property:

Given any morphism of Heyting algebras $f:H \rightarrow H'$ satisfying $f(y) = 1$ for every y in S , f factors uniquely through the canonical surjection $p_F:H \rightarrow H/F$. That is, there is a unique morphism $f':H/F \rightarrow H'$ satisfying $f'p_F = f$. The morphism f' is said to be *induced* by f .

Let $f:H_1 \rightarrow H_2$ be a morphism of Heyting algebras. The **kernel** of f , written $\ker f$, is the set $f^{-1}[\{1\}]$. It is a filter on H_1 . (Care should be taken because this definition, if applied to a morphism of Boolean algebras, is dual to what would be called the kernel of the morphism viewed as a morphism of rings.) By the foregoing, f induces a morphism $f':H_1/(\ker f) \rightarrow H_2$. It is an isomorphism of $H_1/(\ker f)$ onto the subalgebra $f[H_1]$ of H_2 .

Universal constructions

Heyting algebra of propositional formulas in n variables up to intuitionist equivalence

The implication $2 \rightarrow 1$ in the section "Provable identities" is proved by showing that the result of the following construction is itself a Heyting algebra:

1. Consider the set L of propositional formulas in the variables $A_1, A_2, \dots, A_n$.
2. Endow L with a preorder $\sqsubseteq$ by defining $F \sqsubseteq G$ if G is an (intuitionist) logical consequence of F , that is, if G is provable from F . It is immediate that $\sqsubseteq$ is a preorder.
3. Consider the equivalence relation $F \equiv G$ induced by the preorder $F \sqsubseteq G$. (It is defined by $F \equiv G$ if and only if $F \sqsubseteq G$ and $G \sqsubseteq F$. In fact, $\equiv$ is the relation of (intuitionist) logical equivalence.)
4. Let H_0 be the quotient set $L/\equiv$. This will be the desired Heyting algebra.
5. We write $[F]$ for the equivalence class of a formula F . Operations $\rightarrow, \wedge, \vee$ and $\neg$ are defined in an obvious way on L . Verify that given formulas F and G , the equivalence classes $[F \rightarrow G], [F \wedge G], [F \vee G]$ and $[\neg F]$ depend only on $[F]$ and $[G]$. This defines operations $\rightarrow, \wedge, \vee$ and $\neg$ on the quotient set $H_0 = L/\equiv$. Further define 1 to be the class of provably true statements, and set $0 = [\perp]$.
6. Verify that H_0 , together with these operations, is a Heyting algebra. We do this using the axiom-like definition of Heyting algebras. H_0 satisfies conditions THEN-1 through FALSE because all formulas of the given forms are axioms of intuitionist logic. MODUS-PONENS follows from the fact that if a formula $\Box \rightarrow F$ is provably true, where $\Box$ is provably true, then F is provably true (by application of the rule of inference modus ponens). Finally, EQUIV results from the fact that if $F \rightarrow G$ and $G \rightarrow F$ are both provably true, then F and G are provable from each other (by application of the rule of inference modus ponens), hence $[F] = [G]$.

As always under the axiom-like definition of Heyting algebras, we define $\leq$ on H_0 by the condition that $x \leq y$ if and only if $x \rightarrow y = 1$. Since, by the deduction theorem, a formula $F \rightarrow G$ is provably true if and only if G is provable from F , it follows that $[F] \leq [G]$ if and only if $F \sqsubseteq G$. In other words, $\leq$ is the order relation on $L/\equiv$ induced by the preorder $\sqsubseteq$ on L .

Free Heyting algebra on an arbitrary set of generators

In fact, the preceding construction can be carried out for any set of variables $\{A_i: i \in I\}$ (possibly infinite). One obtains in this way the *free* Heyting algebra on the variables $\{A_i\}$, which we will again denote by H_0 . It is free in the sense that given any Heyting algebra H given together with a family of its elements $\{a_i: i \in I\}$, there is a unique morphism $f: H_0 \rightarrow H$ satisfying $f([A_i]) = a_i$. The uniqueness of f is not difficult to see, and its existence results essentially from the implication $1 \rightarrow 2$ of the section "Provable identities" above, in the form of its corollary that whenever F and G are provably equivalent formulas, $F(\Box a_i) = G(\Box a_i)$ for any family of elements $\{a_i\}$ in H .

Heyting algebra of formulas equivalent with respect to a theory T

Given a set of formulas T in the variables $\{A_i\}$, viewed as axioms, the same construction could have been carried out with respect to a relation $F \sqsubseteq G$ defined on L to mean that G is a provable consequence of F and the set of axioms T . Let us denote by H_T the Heyting algebra so obtained. Then H_T satisfies the same universal property as H_0 above, but with respect to Heyting algebras H and families of elements $\sqsubseteq a_i$ satisfying the property that $J(\sqsubseteq a_i) = 1$ for any axiom $J(\sqsubseteq A_i)$ in T . (Let us note that H_T , taken with the family of its elements $\sqsubseteq [A_i]$, itself satisfies this property.) The existence and uniqueness of the morphism is proved the same way as for H_0 , except that one must modify the implication 1 $\rightarrow$ 2 in "Provable identities" so that 1 reads "provably true from T ," and 2 reads "any elements $a_1, a_2, \dots, a_n$ in H satisfying the formulas of T ."

The Heyting algebra H_T that we have just defined can be viewed as a quotient of the free Heyting algebra H_0 on the same set of variables, by applying the universal property of H_0 with respect to H_T and the family of its elements $\sqsubseteq [A_i]$.

Every Heyting algebra is isomorphic to one of the form H_T . To see this, let H be any Heyting algebra, and let $\sqsubseteq a_i: i \in I$ be a family of elements generating H (for example, any surjective family). Now consider the set T of formulas $J(\sqsubseteq A_i)$ in the variables $\sqsubseteq A_i: i \in I$ such that $J(\sqsubseteq a_i) = 1$. Then we obtain a morphism $f: H_T \rightarrow H$ by the universal property of H_T which is clearly surjective. It is not difficult to show that f is injective.

Comparison to Lindenbaum algebras

The constructions we have just given play an entirely analogous role with respect to Heyting algebras to that of Lindenbaum algebras with respect to Boolean algebras. In fact, The Lindenbaum algebra B_T in the variables $\{A_i\}$ with respect to the axioms T is just our $H_{T \cup T_1}$, where T_1 is the set of all formulas of the form $\neg \neg F \rightarrow F$, since the additional axioms of T_1 are the only ones that need to be added in order to make all classical tautologies provable.

Heyting algebras as applied to intuitionistic logic

If one interprets the axioms of the intuitionistic propositional logic as terms of a Heyting algebra, then they will evaluate to the largest element, 1, in *any* Heyting algebra under any assignment of values to the formula's variables. For instance, $(P \wedge Q) \rightarrow P$ is, by definition of the pseudo-complement, the largest element x such that $P \wedge Q \wedge x \leq P$. This inequation is satisfied for any x , so the largest such x is 1.

Furthermore the rule of modus ponens allows us to derive the formula Q from the formulas P and $P \rightarrow Q$. But in any Heyting algebra, if P has the value 1, and $P \rightarrow Q$ has the value 1, then it means that $P \wedge 1 \leq Q$, and so $1 \wedge 1 \leq Q$; it can only be that Q has the value 1.

This means that if a formula is deducible from the laws of intuitionistic logic, being derived from its axioms by way of the rule of modus ponens, then it will always have the value 1 in all Heyting algebras under any assignment of values to the formula's variables. However one can construct a Heyting algebra in which the value of Peirce's law is not always 1. Consider the 3-element algebra $\{0, \frac{1}{2}, 1\}$ as given above. If we assign $\frac{1}{2}$ to P and 0 to Q , then the value of Peirce's law $((P \rightarrow Q) \rightarrow P) \rightarrow P$ is $\frac{1}{2}$. It follows that Peirce's law cannot be intuitionistically derived. See Curry-Howard isomorphism for the general context of what this implies in type theory.

The converse can be proven as well: if a formula always has the value 1, then it is deducible from the laws of intuitionistic logic, so the *intuitionistically valid* formulas are exactly those that always have a value of 1. This is similar to the notion that *classically valid* formulas are those formulas that have a value of 1 in the two-element Boolean algebra under any possible assignment of true and false to the formula's variables — that is, they are formulas which are tautologies in the usual truth-table sense. A Heyting algebra, from the logical standpoint, is then a generalization of the usual system of truth values, and its largest element 1 is analogous to 'true'. The usual two-valued logic system is a special case of a Heyting algebra, and the smallest non-trivial one, in which the only elements of the algebra are 1 (true) and 0 (false).

Word problem

The word problem on free Heyting algebras is difficult.^[3] The only known results are that the free Heyting algebra on one generator is infinite, and that the free complete Heyting algebra on one generator exists (and has one more element than the free Heyting algebra).

Notes

[1] Rutherford (1965), Th.26.2 p.78.

[2] Rutherford (1965), Th.26.1 p.78.

[3] Peter T. Johnstone, *Stone Spaces*, (1982) Cambridge University Press, Cambridge, ISBN 0-521-23893-5. (See paragraph 4.11)

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See also

- List of Boolean algebra topics

External links

- Heyting algebra (GFDLd)

Subobject classifier

In category theory, a **subobject classifier** is a special object Ω of a category; intuitively, the subobjects of an object X correspond to the morphisms from X to Ω . As the name suggests, what a **subobject classifier** does is to identify/classify subobjects of a given object according to which elements belong to the subobject in question. Because of this role, the subobject classifier is also referred to as the **truth value object**. In fact the way in which the **subobject classifier** classifies subobjects of a given object, is by assigning the values true to elements belonging to the subobject in question, and false to elements not belonging to the subobject. This is why the **subobject classifier** is widely used in the categorical description of logic.

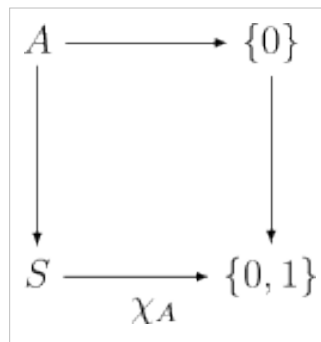
Introductory example

As an example, the set $\Omega = \{0,1\}$ is a subobject classifier in the category of sets and functions: to every subset $j:U \rightarrow X$ we can assign the function χ_j from X to Ω that maps precisely the elements of U to 1 (see characteristic function). Every function from X to Ω arises in this fashion from precisely one subset U .

To render this example more clear let us consider a subset A of S ($A \sqsubseteq S$), where S is a set. The notion of being a subset can be expressed mathematically using the so-called characteristic function: $\chi_A:S \rightarrow \{0,1\}$, which is defined as follows:

$$\chi_A(x) = \begin{cases} 0, & \text{if } x \notin A \\ 1, & \text{if } x \in A \end{cases}$$

(Here we interpret 1 as true and 0 as false.) The role of the characteristic function is to determine which elements belong or not to a certain subset. Since in any category subobjects are identified as monic arrows, we identify the value true with the arrow: **true**: $\{0\} \rightarrow \{0, 1\}$ which maps 0 to 1. Given this definition it can be easily seen that the subset A can be uniquely defined through the characteristic function $A = \chi_A^{-1}(1)$. Therefore the diagram



is a pullback.

The above example of subobject classifier in **Set** is very useful because it enables us to easily prove the following axiom:

Axiom: Given a category **C**, then there exists an isomorphism,

$$y: \text{Sub}_{\mathbf{C}}(X) \cong \text{Hom}_{\mathbf{C}}(X, \Omega) \cong X \cong \mathbf{C}$$

In **Set** this axiom can be restated as follows:

Axiom: The collection of all subsets of S denoted by $\mathcal{P}(S)$, and the collection of all maps from S to the set $\{0, 1\} = 2$ denoted by 2^S are isomorphic i.e. the function $y : \mathcal{P}(S) \rightarrow 2^S$, which in terms of single elements of $\mathcal{P}(S)$ is $A \mapsto \chi_A$, is a bijection.

The above axiom implies the alternative definition of a subobject classifier:

Definition: Ω is a **subobject classifier** iff there is a one to one correspondence between subobject of X and morphisms from X to Ω .

Definition

For the general definition, we start with a category $\mathbf{C}$ that has a terminal object, which we denote by 1 . The object Ω of $\mathbf{C}$ is a subobject classifier for $\mathbf{C}$ if there exists a morphism

$$1 \rightarrow \Omega$$

with the following property:

for each monomorphism $j: U \rightarrow X$ there is a unique morphism $\chi_j: X \rightarrow \Omega$ such that the following commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad} & 1 \\ j \downarrow & & \downarrow \\ X & \xrightarrow{\chi_j} & \Omega \end{array}$$

is a pullback diagram — that is, U is the limit of the diagram:

$$\begin{array}{ccc} & & 1 \\ & & \downarrow \\ X & \xrightarrow{\chi_j} & \Omega \end{array}$$

The morphism χ_j is then called the **classifying morphism** for the subobject represented by j .

Further examples

Every topos has a subobject classifier. For the topos of sheaves of sets on a topological space X , it can be described in these terms: For any open set U of X , $\Omega(U)$ is the set of all open subsets of U . Roughly speaking an assertion inside this topos is variably true or false, and its truth value from the viewpoint of an open subset U is the open subset of U where the assertion is true.

For a small category $\mathbf{C}$, the **subobject classifier** in the **topos of presheaves** $\mathcal{S}^{\mathbf{C}^{op}}$ is given as follows. For any $c \in \mathbf{C}$, $\Omega(c)$ is the set of sieves on c .

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- [1] <http://topos-physics.org/>

Generator

Generator may refer to:

- Electrical generator
- Engine-generator, an electrical generator, but with its own engine.
- Generator (mathematics), any of several closely related usages in mathematics.

Music

- "Generator" (song), a song by The Foo Fighters
- "Generator" (The Holloways song), a song by The Holloways
- *Generator* (Bad Religion album), an album by punk band Bad Religion, and its opening track
- *Generator* (Aborym album)
- The Generators, a punk rock band formed in Los Angeles in 1997 by members of the defunct Warner Bros. band Schlepprock
- Kix (band), an American glam metal band also known as the Generators

In **computing**:

- Generator (computer science), a specialized routine that acts like an iterator
- A program which produces a stream of data
- Pseudorandom number generator, producer of a sequence of random or nearly-random numbers
- Prime number generator, a producer of the ordered sequence of prime numbers
- A zero-generator, the pseudo-device `/dev/zero` outputs a never ending stream of zero-valued bytes
- Code generator, a program which creates source code as its output
- anything that creates source code automatically in generative programming
- Natural language generator, a program that produces human language from a machine representation
- Generator matrix, a matrix whose rows can generate all the elements of a linear code

In **popular culture**:

- A Generator in the anime *Generator Gawl* is a metal-organic hybrid organism with far greater powers than that of a human.

See also

- Generate
 - Ginerator
-

Exact sequence

In mathematics, especially in homological algebra and other applications of abelian category theory, as well as in differential geometry and group theory, an **exact sequence** is a (finite or infinite) sequence of objects and morphisms between them such that the image of one morphism equals the kernel of the next.

Definition

To be precise, fix an abelian category (such as the category of abelian groups or the category of vector spaces over a given field) or some other category with kernels and cokernels (such as the category of all groups). Choose an index set of consecutive integers. Then for each integer i in the index set, let A_i be an object in the category and let f_i be a morphism from A_i to A_{i+1} . This defines a sequence of objects and morphisms.

The sequence is *exact* at A_i if the image of f_{i-1} is equal to the kernel of f_i :

$$\text{im } f_{i-1} = \ker f_i.$$

The sequence itself is *exact* if it is exact at each object (except at the very first and the very last object, where exactness doesn't make sense).

Example

Consider the following sequence of abelian groups:

$$\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$$

Here the hook arrow $\hookrightarrow$ indicates that the map $2\cdot$ from $\mathbb{Z}$ to $\mathbb{Z}$ is a monomorphism, and the two-headed arrow $\twoheadrightarrow$ indicates an epimorphism (the map $\text{mod } 2$). This is an exact sequence because the image $2\mathbb{Z}$ of the monomorphism is the kernel of the epimorphism.

This sequence may also be written without using special symbols for monomorphism and epimorphism:

$$0 \rightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$

Here 0 denotes the trivial abelian group with a single element, the map from $\mathbb{Z}$ to $\mathbb{Z}$ is multiplication by 2, and the map from $\mathbb{Z}$ to the factor group $\mathbb{Z}/2\mathbb{Z}$ is given by reducing integers modulo 2. This is indeed an exact sequence:

- the image of the map $0 \rightarrow \mathbb{Z}$ is $\{0\}$, and the kernel of multiplication by 2 is also $\{0\}$, so the sequence is exact at the first $\mathbb{Z}$.
- the image of multiplication by 2 is $2\mathbb{Z}$, and the kernel of reducing modulo 2 is also $2\mathbb{Z}$, so the sequence is exact at the second $\mathbb{Z}$.
- the image of reducing modulo 2 is all of $\mathbb{Z}/2\mathbb{Z}$, and the kernel of the zero map is also all of $\mathbb{Z}/2\mathbb{Z}$, so the sequence is exact at the position $\mathbb{Z}/2\mathbb{Z}$.

Another example, from differential geometry, especially relevant for work on the Maxwell equations:

$$\mathbb{H}_1 \xrightarrow{\text{grad}} \mathbb{H}_{\text{curl}} \xrightarrow{\text{curl}} \mathbb{H}_{\text{div}} \xrightarrow{\text{div}} \mathbb{L}_2,$$

based on the fact that

$$\begin{aligned} \text{curl}(\text{grad } f) &= \nabla \times (\nabla f) = 0 \\ \text{div}(\text{curl } \vec{v}) &= \nabla \cdot \nabla \times \vec{v} = 0 \end{aligned}$$

on properly defined Hilbert spaces.

Special cases

To make sense of the definition, it is helpful to consider what it means in relatively simple cases where the sequence is finite and begins or ends with 0.

- The sequence $0 \rightarrow A \rightarrow B$ is exact at A if and only if the map from A to B has kernel $\{0\}$, i.e. if and only if that map is a monomorphism (one-to-one).
- Dually, the sequence $B \rightarrow C \rightarrow 0$ is exact at C if and only if the image of the map from B to C is all of C , i.e. if and only if that map is an epimorphism (onto).
- A consequence of these last two facts is that the sequence $0 \rightarrow X \rightarrow Y \rightarrow 0$ is exact if and only if the map from X to Y is an isomorphism.

When dealing with exact sequences of groups, it is common to write 1 instead of 0 for the trivial group with a single element.

Important are **short exact sequences**, which are exact sequences of the form

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

By the above, we know that for any such short exact sequence, f is a monomorphism and g is an epimorphism. Furthermore, the image of f is equal to the kernel of g . It is helpful to think of A as a subobject of B with f being the embedding of A into B , and of C as the corresponding factor object B/A , with the map g being the natural projection from B to B/A (whose kernel is exactly A).

Facts

The splitting lemma states that if the above short exact sequence admits a morphism $t: B \rightarrow A$ such that $t \circ f$ is the identity on A or a morphism $u: C \rightarrow B$ such that $g \circ u$ is the identity on C , then B is a twisted direct sum of A and C . (For groups, a twisted direct sum is a semidirect product; in an abelian category, every twisted direct sum is an ordinary direct sum.) In this case, we say that the short exact sequence *splits*.

The snake lemma shows how a commutative diagram with two exact rows gives rise to a longer exact sequence. The nine lemma is a special case.

The five lemma gives conditions under which the middle map in a commutative diagram with exact rows of length 5 is an isomorphism; the short five lemma is a special case thereof applying to short exact sequences.

The importance of short exact sequences is underlined by the fact that every exact sequence results from "weaving together" several overlapping short exact sequences. Consider for instance the exact sequence

$$A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_4 \rightarrow A_5 \rightarrow A_6,$$

which implies that there exist objects C_k in the category such that

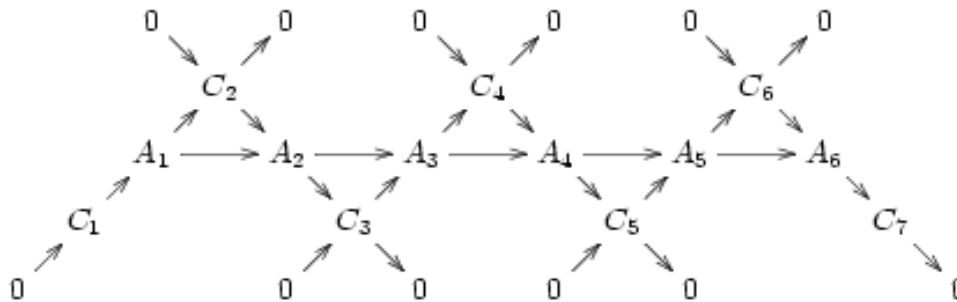
$$C_k \cong \ker(A_k \rightarrow A_{k+1}) \cong \operatorname{im}(A_{k-1} \rightarrow A_k).$$

Suppose in addition that the cokernel of each morphism exists, and is isomorphic to the image of the next morphism in the sequence:

$$C_k \cong \operatorname{coker}(A_{k-2} \rightarrow A_{k-1})$$

(This is true for a number of interesting categories, including any abelian category such as the abelian groups; but it is not true for all categories that allow exact sequences, and in

particular is not true for the category of groups, in which $\text{coker } f : G \rightarrow H$ is not $H/\text{im } f$ but $H/\langle \text{im } f \rangle^H$ (the conjugate closure of $\text{im } f$.) Then we obtain a commutative diagram in which all the diagonals are short exact sequences:



Note that the only portion of this diagram that depends on the cokernel condition is the object C_7 and the final pair of morphisms $A_6 \rightarrow C_7 \rightarrow 0$. If there exists any object A_{k+1} and morphism $A_k \rightarrow A_{k+1}$ such that $A_{k-1} \rightarrow A_k \rightarrow A_{k+1}$ is exact, then the exactness of $0 \rightarrow C_k \rightarrow A_k \rightarrow C_{k+1} \rightarrow 0$ is ensured. Again taking the example of the category of groups, the fact that $\text{im } f$ is the kernel of some homomorphism on H implies that it is a normal subgroup, which coincides with its conjugate closure; thus $\text{coker } f$ is isomorphic to the image $H/\text{im } f$ of the next morphism.

Conversely, given any list of overlapping short exact sequences, their middle terms form an exact sequence in the same manner.

Applications of exact sequences

In the theory of abelian categories, short exact sequences are often used as a convenient language to talk about sub- and factor objects.

The extension problem is essentially the question, given the end terms A and C of a short exact sequence, what possibilities exist for the middle term B ? In the category of groups, this is equivalent to the question, what groups B have A as a normal subgroup and C as the corresponding factor group? This problem is important in the classification of groups. See also Outer automorphism group.

Notice that in an exact sequence, the composition $f_{i+1} \circ f_i$ maps A_i to 0 in A_{i+2} , so every exact sequence is a chain complex. Furthermore, only f_i -images of elements of A_i are mapped to 0 by f_{i+1} , so the homology of this chain complex is trivial. More succinctly:

Exact sequences are precisely those chain complexes which are acyclic.

Given any chain complex, its homology can therefore be thought of as a measure of the degree to which it fails to be exact.

If we take a series of short exact sequences linked by chain complexes (that is, a short exact sequence of chain complexes, or from another point of view, a chain complex of short exact sequences), then we can derive from this a **long exact sequence** (i.e. an exact sequence indexed by the natural numbers) on homology by application of the zig-zag lemma. It comes up in algebraic topology in the study of relative homology; the Mayer-Vietoris sequence is another example. Long exact sequences induced by short exact sequences are also characteristic of derived functors.

Exact functors are functors that transform exact sequences into exact sequences.

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External links

- Exact sequence ^[1] on PlanetMath
- Eric W. Weisstein, *Exact Sequence* ^[2] at MathWorld.
- Eric W. Weisstein, *Short Exact Sequence* ^[3] at MathWorld.

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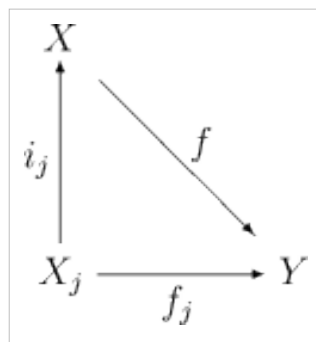
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 [2] <http://mathworld.wolfram.com/ExactSequence.html>
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Coproduct

In category theory, the **coproduct**, or **categorical sum**, is the category-theoretic construction which subsumes the disjoint union of sets and of topological spaces, the free product of groups, and the direct sum of modules and vector spaces. The coproduct of a family of objects is essentially the "least specific" object to which each object in the family admits a morphism. It is the category-theoretic dual notion to the categorical product, which means the definition is the same as the product but with all arrows reversed. Despite this innocuous-looking change in the name and notation, coproducts can be and typically are dramatically different from products.

Definition

The formal definition is as follows: Let C be a category and let $\{X_j : j \in J\}$ be an indexed family of objects in C . The coproduct of the set $\{X_j\}$ is an object X together with a collection of morphisms $i_j : X_j \rightarrow X$ (called *canonical injections* although they need not be injections or even monic) which satisfy a universal property: for any object Y and any collection of morphisms $f_j : X_j \rightarrow Y$, there exists a unique morphism f from X to Y such that $f_j = f \circ i_j$. That is, the following diagram commutes (for each j):



The coproduct of the family $\{X_j\}$ is often denoted

$$X = \coprod_{j \in J} X_j$$

or

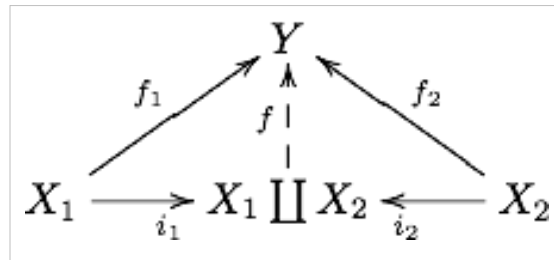
$$X = \bigoplus_{j \in J} X_j.$$

Sometimes the morphism f may be denoted

$$f = \coprod_{j \in J} f_j : \coprod_{j \in J} X_j \rightarrow Y$$

to indicate its dependence on the individual f_j .

If the family of objects consists of only two members the product is usually written $X_1 \amalg X_2$ or $X_1 \sqcup X_2$ or sometimes simply $X_1 + X_2$, and the diagram takes the form:



The unique arrow f making this diagram commute is then correspondingly denoted $f_1 \amalg f_2$ or $f_1 \sqcup f_2$ or $f_1 + f_2$ or $[f_1, f_2]$.

Examples

The coproduct in the category of sets is simply the **disjoint union** with the maps i_j being the inclusion maps. Unlike direct products, coproducts in other categories are not all obviously based on the notion for sets, because unions don't behave well with respect to preserving operations (e.g. the union of two groups need not be a group), and so coproducts in different categories can be dramatically different from each other. For example, the coproduct in the category of groups, called the **free product**, is quite complicated. On the other hand, in the category of abelian groups (and equally for vector spaces), the coproduct, called the **direct sum**, consists of the elements of the direct product which have only finitely many nonzero terms (this therefore coincides exactly with the direct product, in the case of finitely many factors). As a consequence, since most introductory linear algebra courses deal with only finite-dimensional vector spaces, nobody really hears much about direct sums until later on.

In the case of topological spaces coproducts are disjoint unions with their disjoint union topologies. That is it is a disjoint union of the underlying sets, and the open sets are sets *open in each of the spaces*, in a rather evident sense. In the category of pointed spaces, fundamental in homotopy theory, the coproduct is the wedge sum (which amounts to joining a collection of spaces with base points at a common base point).

Despite all this dissimilarity, there is still, at the heart of the whole thing, a disjoint union: the direct sum of abelian groups is the group generated by the "almost" disjoint union (disjoint union of all nonzero elements, together with a common zero), similarly for vector spaces: the space spanned by the "almost" disjoint union; the free product for groups is generated by the set of all letters from a similar "almost disjoint" union where no two elements from different sets are allowed to commute.

Discussion

The coproduct construction given above is actually a special case of a colimit in category theory. The coproduct in a category C can be defined as the colimit of any functor from a discrete category J into C . Not every family $\{X_j\}$ will have a coproduct in general, but if it does, then the coproduct is unique in a strong sense: if $i_j : X_j \rightarrow X$ and $k_j : X_j \rightarrow Y$ are two coproducts of the family $\{X_j\}$, then (by the definition of coproducts) there exists a unique isomorphism $f : X \rightarrow Y$ such that $i_j = k_j f$ for each j in J .

As with any universal property, the coproduct can be understood as a universal morphism. Let $\Delta : C \rightarrow C \times C$ be the diagonal functor which assigns to each object X the ordered pair (X, X) and to each morphism $f : X \rightarrow Y$ the pair (f, f) . Then the coproduct $X + Y$ in C is given by a universal morphism to the functor Δ from the object (X, Y) in $C \times C$.

The coproduct indexed by the empty set (that is, an *empty coproduct*) is the same as an initial object in C .

If J is a set such that all coproducts for families indexed with J exist, then it is possible to choose the products in a compatible fashion so that the coproduct turns into a functor $C^J \rightarrow C$. The coproduct of the family $\{X_j\}$ is then often denoted by $\coprod_j X_j$, and the maps i_j are known as the **natural injections**.

Letting $\text{Hom}_C(U, V)$ denote the set of all morphisms from U to V in C (that is, a hom-set in C), we have a natural isomorphism

$$\text{Hom}_C\left(\coprod_{j \in J} X_j, Y\right) \cong \prod_{j \in J} \text{Hom}_C(X_j, Y)$$

given by the bijection which maps every tuple of morphisms

$$(f_j)_{j \in J} \in \prod_{j \in J} \text{Hom}(X_j, Y)$$

(a product in **Set**, the category of sets, which is the Cartesian product, so it is a tuple of morphisms) to the morphism

$$\prod_{j \in J} f_j \in \text{Hom}\left(\coprod_{j \in J} X_j, Y\right).$$

That this map is a surjection follows from the commutativity of the diagram: any morphism f is the coproduct of the tuple

$$(f \circ i_j)_{j \in J}.$$

That it is an injection follows from the universal construction which stipulates the uniqueness of such maps. The naturality of the isomorphism is also a consequence of the diagram. Thus the contravariant hom-functor changes coproducts into products. Stated another way, the hom-functor, viewed as a functor from the opposite category C^{opp} to **Set** is continuous; it preserves limits (a coproduct in C is a product in C^{opp}).

If J is a finite set, say $J = \{1, \dots, n\}$, then the coproduct of objects $X_1, \dots, X_n$ is often denoted by $X_1 \sqcup \dots \sqcup X_n$. Suppose all finite coproducts exist in C , coproduct functors have been chosen as above, and 0 denotes the initial object of C corresponding to the empty coproduct. We then have natural isomorphisms

$$X \oplus (Y \oplus Z) \cong (X \oplus Y) \oplus Z \cong X \oplus Y \oplus Z$$

$$X \oplus 0 \cong 0 \oplus X \cong X$$

$$X \oplus Y \cong Y \oplus X.$$

These properties are formally similar to those of a commutative monoid; a category with finite coproducts is an example of a symmetric monoidal category.

If the category has a zero object Z , then we have unique morphism $X \rightarrow Z$ (since Z is terminal) and thus a morphism $X \sqcup Y \rightarrow Z \sqcup Y$. Since Z is also initial, we have a canonical isomorphism $Z \sqcup Y \cong Y$ as in the preceding paragraph. We thus have morphisms $X \sqcup Y \rightarrow X$ and $X \sqcup Y \rightarrow Y$, by which we infer a canonical morphism $X \sqcup Y \rightarrow X \times Y$. This may be extended by induction to a canonical morphism from any finite coproduct to the corresponding product. This morphism need not in general be an isomorphism; in **Grp** it is a proper epimorphism while in **Set**_{*} (the category of pointed sets) it is a proper monomorphism. In any preadditive category, this morphism is an isomorphism and the corresponding object is known as the biproduct. A category with all finite biproducts is known as an additive category.

If all families of objects indexed by J have coproducts in C , then the coproduct comprises a functor $C^J \rightarrow C$. Note that, like the product, this functor is *covariant*.

See also

- Product
- Limits and colimits
- Coequalizer
- Direct limit

External links

- Interactive Web page ^[1] which generates examples of coproducts in the category of finite sets. Written by Jocelyn Paine ^[2].

References

- [1] <http://www.j-paine.org/cgi-bin/webcats/webcats.php>
 [2] <http://www.j-paine.org/>

Product

Product may mean:

- Product (business), an item that ideally satisfies a market's want or need
 - Product (project management), a deliverable or set of deliverables that contribute to a business solution
- Product (biology), something manufactured by an organelle
- Product (chemistry), a substance found at the beginning of a chemical reaction
- Product (mathematics), the result of multiplying
- Product (category theory), a more abstract product
- Product (album), a musical recording by Norwegian group De Press
- .the .product, a notable "64K demo" by the demogroup Farbrausch

Pushout

A **pushout** is a student counseled or forced out of a school prior to graduation. Children are often pushed out of an educational institutions because their presence in the school creates difficulty in meeting some goal of the school. For example, in the case where funding for the school is dependent upon scholastic achievement of the students, if the school can get rid of low-performing students, average test scores on academic performance tests will go up, thus increasing funding.^[1] In Ontario, where the education system has zero tolerance towards violence, a student is pushed out province-wide. In some low-performing schools in Chicago combined dropout/pushout rates have exceeded 25% in one year.^[2]

Children are also pushed from schools because they present discipline problems or have become "too old." Even in cases where a child can legally remain in high school until they are 21, for example, they may be counseled out after they are over 18.

See also

- Dropout

References

[1] article, "To Cut Failure Rate, Schools Shed Students" (<http://www.nytimes.com/2003/07/31/nyregion/31PUSH.html>) by Tamar Lewin and Jennifer Medina, *New York Times*. July 31, 2003.

[2] "Enrollment drops point to pushout effect", *Catalyst, Voices of Chicago Educational Reform*. June, 1999 (<http://www.catalyst-chicago.org/06-99/069midyearchart.htm>)

Pullback

Pullback refers to two different, but related processes: precomposition and fiber-product.

Precomposition

Precomposition with a function probably provides the most elementary notion of pullback: in simple terms, a function f of a variable y , where y itself is a function of another variable x , may be written as a function of x . This is the pullback of f by the function $y(x)$.

$$f(y(x)) \equiv g(x)$$

It is such a fundamental process, that it is often passed over without mention, for instance in elementary calculus: this is sometimes called *omitting pullbacks*, and pervades areas as diverse as fluid mechanics and differential geometry.

However, it is not just functions that can be "pulled back" in this sense. Pullbacks can be applied to many other objects such as differential forms and their cohomology classes.

See:

- Pullback (differential geometry)
- Pullback (cohomology)

Fibre-product

The notion of pullback as a fibre-product ultimately leads to the very general idea of a categorical pullback, but it has important special cases: inverse image (and pullback) sheaves in algebraic geometry, and pullback bundles in algebraic topology and differential geometry.

See:

- Pullback (category theory)
- Inverse image sheaf
- Pullback bundle
- Fibred category

Relationship

The relation between the two notions of pullback can perhaps best be illustrated by sections of fibre bundles: if s is a section of a fibre bundle E over N , and f is a map from M to N , then the pullback (precomposition) $f^*s = s \circ f$ of s with f is a section of the pullback (fibre-product) bundle f^*E over M .

Limit

A **limit** can be:

- Limit (mathematics), including:
 - Limit of a function
 - Limit of a sequence
 - One-sided limit
 - Limit superior and limit inferior
 - Limit of a net
 - Limit point
 - Limit (category theory)
 - Direct limit
- A constraint (mathematical, physical, economical, legal, etc.) in the form of an inequality, such as:
 - Chandrasekhar limit
 - Greisen-Zatsepin-Kuzmin limit
 - Budget constraint
 - Speed limit
 - Age of consent
- An extreme value or boundary, such as:
 - High frequency limit
 - A limit order is a type of order to buy a security at no more (or sell at no less) than a specific price on an exchange.
- Other uses, such as:
 - Limit (music) in just intonation
 - In BDSM, limits are activities that a partner feels strongly about, and to which special attention must be paid.
 - `limits.h`, the header of a general purpose standard library of the C programming language
- The Limit, a 1980s band

See also

- Limited
-

Colimit

1. REDIRECT Limit (category theory)

Yoneda lemma

In mathematics, specifically in category theory, the **Yoneda lemma** is an abstract result on functors of the type *morphisms into a fixed object*. It is a vast generalisation of Cayley's theorem from group theory (viewing a group as a particular kind of category with just one object). It allows the embedding of any category into a category of functors defined on that category. It also clarifies how the embedded category, of representable functors and their natural transformations, relates to the other objects in the larger functor category. It is an important tool that underlies several modern developments in algebraic geometry and representation theory. It is named after Nobuo Yoneda.

Generalities

The Yoneda lemma suggests that instead of studying the (small) category C , one should study the category of all functors of C into **Set** (the category of sets with functions as morphisms). **Set** is a category we understand well, and a functor of C into **Set** can be seen as a "representation" of C in terms of known structures. The original category C is contained in this functor category, but new objects appear in the functor category which were absent and "hidden" in C . Treating these new objects just like the old ones often unifies and simplifies the theory.

This approach is akin to (and in fact generalizes) the common method of studying a ring by investigating the modules over that ring. The ring takes the place of the category C , and the category of modules over the ring is a category of functors defined on C .

Formal statement

General version

Yoneda's lemma concerns functors from a fixed category C to the category of sets, **Set**. If C is a locally small category (i.e. the hom-sets are actual sets and not proper classes), then each object A of C gives rise to a natural functor to **Set** called a hom-functor. This functor is denoted:

$$h_A = \text{Hom}(A, -),$$

The hom-functor h_A sends X to the set of morphisms $\text{Hom}(A, X)$.

Let F be an arbitrary functor from C to **Set**. Then Yoneda's lemma says that for each object A of C the natural transformations from h_A to F are in one-to-one correspondence with the elements of $F(A)$. That is,

$$\text{Nat}(h_A, F) \cong F(A).$$

Given a natural transformation Φ from h_A to F , the corresponding element of $F(A)$ is $u = \Phi_A(\text{id}_A)$.

There is a contravariant version of Yoneda's lemma which concerns contravariant functors from C to **Set**. This version involves the contravariant hom-functor

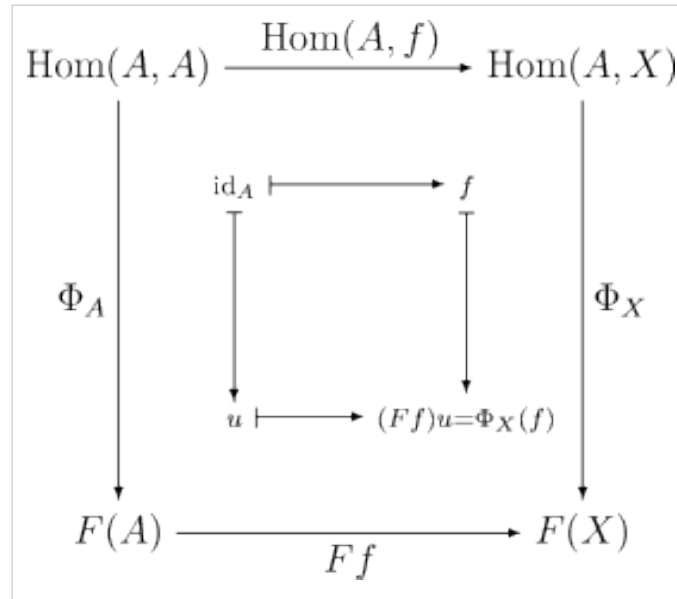
$$h'_A = \text{Hom}(-, A),$$

which sends X to the hom-set $\text{Hom}(X, A)$. Given an arbitrary contravariant functor G from C to **Set**, Yoneda's lemma asserts that

$$\text{Nat}(h'_A, G) \cong G(A).$$

Proof

The proof of Yoneda's lemma is indicated by the following commutative diagram:



This diagram shows that the natural transformation Φ is completely determined by $\Phi_A(\text{id}_A) = u$ since for each morphism $f: A \rightarrow X$ one has

$$\Phi_X(f) = (Ff)u.$$

Moreover, any element $u \in F(A)$ defines a natural transformation in this way. The proof in the contravariant case is completely analogous.

In this way, Yoneda's Lemma provides a complete classification of all natural transformations from the functor $\text{Hom}(A, -)$ to an arbitrary functor $F: C \rightarrow \text{Set}$.

The Yoneda embedding

An important special case of Yoneda's lemma is when the functor F from C to **Set** is another hom-functor h_B . In this case, the covariant version of Yoneda's lemma states that

$$\text{Nat}(h_A, h_B) \cong \text{Hom}(B, A).$$

That is, natural transformations between hom-functors are in one-to-one correspondence with morphisms (in the reverse direction) between the associated objects. Given a morphism $f: B \rightarrow A$ the associated natural transformation is denoted $\text{Hom}(f, -)$.

Mapping each object A in C to its associated hom-functor $h_A = \text{Hom}(A, -)$ and each morphism $f: B \rightarrow A$ to the corresponding natural transformation $\text{Hom}(f, -)$ determines a contravariant functor h from C to **Set** ^{C} , the functor category of all (covariant) functors from C to **Set**. One can interpret h as a covariant functor:

$$h: C^{\text{op}} \rightarrow \text{Set}^C.$$

The meaning of Yoneda's lemma in this setting is that the functor h is fully faithful, and therefore gives an embedding of C^{op} in the category of functors to **Set**. The collection of all functors $\{h_A, A \text{ in } C\}$ is a subcategory of **Set** ^{C} . Therefore, Yoneda embedding implies that the category C^{op} is isomorphic to the category $\{h_A, A \text{ in } C\}$.

The contravariant version of Yoneda's lemma states that

$$\text{Nat}(h'_A, h'_B) \cong \text{Hom}(A, B).$$

Therefore, h gives rise to a covariant functor from C to the category of contravariant functors to **Set**:

$$h': C \rightarrow \text{Set}^{C^{op}}.$$

Yoneda's lemma then states that any locally small category C can be embedded in the category of contravariant functors from C to **Set** via h . This is called the *Yoneda embedding*.

Preadditive categories, rings and modules

A *preadditive category* is a category where the morphism sets form abelian groups and the composition of morphisms is bilinear; examples are categories of abelian groups or modules. In a preadditive category, there is both a "multiplication" and an "addition" of morphisms, which is why preadditive categories are viewed as generalizations of rings. Rings are preadditive categories with one object.

The Yoneda lemma remains true for preadditive categories if we choose as our extension the category of *additive* contravariant functors from the original category into the category of abelian groups; these are functors which are compatible with the addition of morphisms and should be thought of as forming a *module category* over the original category. The Yoneda lemma then yields the natural procedure to enlarge a preadditive category so that the enlarged version remains preadditive — in fact, the enlarged version is an abelian category, a much more powerful condition. In the case of a ring R , the extended category is the category of all left modules over R , and the statement of the Yoneda lemma reduces to the well-known isomorphism

$$M \cong \text{Hom}_R(R, M) \quad \text{for all left modules } M \text{ over } R.$$

References

- Freyd, Peter (1964), *Abelian categories*, Harper and Row. Reprinted 2003 ^[1].
- Mac Lane, Saunders (1998), *Categories for the Working Mathematician*, Graduate Texts in Mathematics, **5**, Springer-Verlag, ISBN 0-387-98403-8.

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- [1] <http://www.tac.mta.ca/tac/reprints/articles/3/tr3abs.html>

Functor

In category theory, a branch of mathematics, a **functor** is a special type of mapping between categories. Functors can be thought of as morphisms in the category of small categories.

Functors were first considered in algebraic topology, where algebraic objects (like the fundamental group) are associated to topological spaces, and algebraic homomorphisms are associated to continuous maps. Nowadays, functors are used throughout modern mathematics to relate various categories. The word "functor" was borrowed by mathematicians from the philosopher Rudolf Carnap [Mac Lane, p. 30]. Carnap used the term "functor" to stand in relation to functions analogously as predicates stand in relation to properties. [See Carnap, *The Logical Syntax of Language*, p.13-14, 1937, Routledge & Kegan Paul.] For Carnap then, unlike modern category theory's use of the term, a functor is a linguistic item. For category theorists, a functor is a particular kind of function.

Definition

Let C and D be categories. A **functor** F from C to D is a mapping that

- associates to each object $X \in C$ an object $F(X) \in D$,
 - associates to each morphism $f : X \rightarrow Y \in C$ a morphism $F(f) : F(X) \rightarrow F(Y) \in D$
- such that the following two conditions hold:

- $F(id_X) = id_{F(X)}$ for every object $X \in C$
- $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$.

That is, functors must preserve identity morphisms and composition of morphisms.

A functor from a category to itself is called an **endofunctor**.

Covariance and contravariance

There are many constructions in mathematics which would be functors but for the fact that they "turn morphisms around" and "reverse composition". We then define a **contravariant functor** F from C to D as a mapping that

- associates to each object $X \in C$ an object $F(X) \in D$,
 - associates to each morphism $f : X \rightarrow Y \in C$ a morphism $F(f) : F(Y) \rightarrow F(X) \in D$
- such that
- $F(id_X) = id_{F(X)}$ for every object $X \in C$,
 - $F(g \circ f) = F(f) \circ F(g)$ for all morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$.

Note that contravariant functors reverse the direction of composition.

Ordinary functors are also called **covariant functors** in order to distinguish them from contravariant ones. Note that one can also define a contravariant functor as a *covariant* functor on the dual category C^{op} . Some authors prefer to write all expressions covariantly. That is, instead of saying $F : C \rightarrow D$ is a contravariant functor, they simply write $F : C^{op} \rightarrow D$ (or sometimes $F : C \rightarrow D^{op}$) and call it a functor.

Contravariant functors are also occasionally called *cofunctors*.

Examples

Constant functor: The functor $C \rightarrow D$ is one which maps every object of C to a fixed object X in D and every morphism in C to the identity morphism on X . Such a functor is called a *constant* or *selection* functor.

Diagonal functor: The diagonal functor is defined as the functor from D to the functor category D^C which sends each object in D to the constant functor at that object.

Limit functor: For a fixed index category J , if every functor $J \rightarrow C$ has a limit (for instance if C is complete), then the limit functor $C^J \rightarrow C$ assigns to each functor its limit. The existence of this functor can be proved by realizing that it is the right-adjoint to the diagonal functor and invoking the Freyd adjoint functor theorem. This requires a suitable version of the axiom of choice. Similar remarks apply to the colimit functor (which is covariant).

Power sets: The power set functor $P : \mathbf{Set} \rightarrow \mathbf{Set}$ maps each set to its power set and each function $f : X \rightarrow Y$ to the map which sends $U \subseteq X$ to its image $f(U) \subseteq Y$. One can also consider the contravariant power set functor which sends $f : X \rightarrow Y$ to the map which sends $V \subseteq Y$ to its inverse image $f^{-1}(V) \subseteq X$.

Dual vector space: The map which assigns to every vector space its dual space and to every linear map its dual or transpose is a contravariant functor from the category of all vector spaces over a fixed field to itself.

Fundamental group: Consider the category of pointed topological spaces, i.e. topological spaces with distinguished points. The objects are pairs (X, x_0) , where X is a topological space and x_0 is a point in X . A morphism from (X, x_0) to (Y, y_0) is given by a continuous map $f : X \rightarrow Y$ with $f(x_0) = y_0$.

To every topological space X with distinguished point x_0 , one can define the fundamental group based at x_0 , denoted $\pi_1(X, x_0)$. This is the group of homotopy classes of loops based at x_0 . If $f : X \rightarrow Y$ morphism of pointed spaces, then every loop in X with base point x_0 can be composed with f to yield a loop in Y with base point y_0 . This operation is compatible with the homotopy equivalence relation and the composition of loops, and we get a group homomorphism from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$. We thus obtain a functor from the category of pointed topological spaces to the category of groups.

In the category of topological spaces (without distinguished point), one considers homotopy classes of generic curves, but they cannot be composed unless they share an endpoint. Thus one has the **fundamental groupoid** instead of the fundamental group, and this construction is functorial.

Algebra of continuous functions: a contravariant functor from the category of topological spaces (with continuous maps as morphisms) to the category of real associative algebras is given by assigning to every topological space X the algebra $C(X)$ of all real-valued continuous functions on that space. Every continuous map $f : X \rightarrow Y$ induces an algebra homomorphism $C(f) : C(Y) \rightarrow C(X)$ by the rule $C(f)(\varphi) = \varphi \circ f$ for every φ in $C(Y)$.

Tangent and cotangent bundles: The map which sends every differentiable manifold to its tangent bundle and every smooth map to its derivative is a covariant functor from the category of differentiable manifolds to the category of vector bundles. Likewise, the map which sends every differentiable manifold to its cotangent bundle and every smooth map to its pullback is a contravariant functor.

Doing these constructions pointwise gives covariant and contravariant functors from the category of pointed differentiable manifolds to the category of real vector spaces.

Group actions/representations: Every group G can be considered as a category with a single object whose morphisms are the elements of G . A functor from G to **Set** is then nothing but a group action of G on a particular set, i.e. a G -set. Likewise, a functor from G to the category of vector spaces, $\mathbf{Vect}_K$, is a linear representation of G . In general, a functor $G \rightarrow C$ can be considered as an "action" of G on an object in the category C . If C is a group, then this action is a group homomorphism.

Lie algebras: Assigning to every real (complex) Lie group its real (complex) Lie algebra defines a functor.

Tensor products: If C denotes the category of vector spaces over a fixed field, with linear maps as morphisms, then the tensor product $V \otimes W$ defines a functor $C \times C \rightarrow C$ which is covariant in both arguments.

Forgetful functors: The functor $U : \mathbf{Grp} \rightarrow \mathbf{Set}$ which maps a group to its underlying set and a group homomorphism to its underlying function of sets is a functor. Functors like these, which "forget" some structure, are termed *forgetful functors*. Another example is the functor $\mathbf{Rng} \rightarrow \mathbf{Ab}$ which maps a ring to its underlying additive abelian group. Morphisms in **Rng** (ring homomorphisms) become morphisms in **Ab** (abelian group homomorphisms).

Free functors: Going in the opposite direction of forgetful functors are free functors. The free functor $F : \mathbf{Set} \rightarrow \mathbf{Grp}$ sends every set X to the free group generated by X . Functions get mapped to group homomorphisms between free groups. Free constructions exist for many categories based on structured sets. See free object.

Homomorphism groups: To every pair A, B of abelian groups one can assign the abelian group $\text{Hom}(A, B)$ consisting of all group homomorphisms from A to B . This is a functor which is contravariant in the first and covariant in the second argument, i.e. it is a functor $\mathbf{Ab}^{\text{op}} \times \mathbf{Ab} \rightarrow \mathbf{Ab}$ (where **Ab** denotes the category of abelian groups with group homomorphisms). If $f : A_1 \rightarrow A_2$ and $g : B_1 \rightarrow B_2$ are morphisms in **Ab**, then the group homomorphism $\text{Hom}(f, g) : \text{Hom}(A_2, B_1) \rightarrow \text{Hom}(A_1, B_2)$ is given by $\varphi \mapsto g \circ \varphi \circ f$. See Hom functor.

Representable functors: We can generalize the previous example to any category C . To every pair X, Y of objects in C one can assign the set $\text{Hom}(X, Y)$ of morphisms from X to Y . This defines a functor to **Set** which is contravariant in the first argument and covariant in the second, i.e. it is a functor $C^{\text{op}} \times C \rightarrow \mathbf{Set}$. If $f : X_1 \rightarrow X_2$ and $g : Y_1 \rightarrow Y_2$ are morphisms in C , then the group homomorphism $\text{Hom}(f, g) : \text{Hom}(X_2, Y_1) \rightarrow \text{Hom}(X_1, Y_2)$ is given by $\varphi \mapsto g \circ \varphi \circ f$.

Functors like these are called representable functors. An important goal in many settings is to determine whether a given functor is representable.

Presheaves: If X is a topological space, then the open sets in X form a partially ordered set $\text{Open}(X)$ under inclusion. Like every partially ordered set, $\text{Open}(X)$ forms a small category by adding a single arrow $U \rightarrow V$ if and only if $U \subseteq V$. Contravariant functors on $\text{Open}(X)$ are called *presheaves* on X . For instance, by assigning to every open set U the associative algebra of real-valued continuous functions on U , one obtains a presheaf of algebras on X .

Properties

Two important consequences of the functor axioms are:

- F transforms each commutative diagram in C into a commutative diagram in D ;
- if f is an isomorphism in C , then $F(f)$ is an isomorphism in D .

On any category C one can define the **identity functor** 1_C which maps each object and morphism to itself. One can also compose functors, i.e. if F is a functor from A to B and G is a functor from B to C then one can form the composite functor GF from A to C . Composition of functors is associative where defined. This shows that functors can be considered as morphisms in categories of categories, for example in the category of small categories.

A small category with a single object is the same thing as a monoid: the morphisms of a one-object category can be thought of as elements of the monoid, and composition in the category is thought of as the monoid operation. Functors between one-object categories correspond to monoid homomorphisms. So in a sense, functors between arbitrary categories are a kind of generalization of monoid homomorphisms to categories with more than one object.

Bifunctors and multifunctors

A **bifunctor** (also known as a **binary functor**) is a functor in *two* arguments. The Hom functor is a natural example; it is contravariant in one argument, covariant in the other.

Formally, a bifunctor is a functor whose domain is a product category. For example, the Hom functor is of the type $C^{\text{op}} \times C \rightarrow \mathbf{Set}$.

A **multifunctor** is a generalization of the functor concept to n variables. So, for example, a bifunctor is a multifunctor with $n=2$.

Relation to other categorical concepts

Let C and D be categories. The collection of all functors $C \rightarrow D$ form the objects of a category: the functor category. Morphisms in this category are natural transformations between functors.

Functors are often defined by universal properties; examples are the tensor product, the direct sum and direct product of groups or vector spaces, construction of free groups and modules, direct and inverse limits. The concepts of limit and colimit generalize several of the above.

Universal constructions often give rise to pairs of adjoint functors.

See also

Types of functors

- Additive functor: a functor between categories whose hom-sets are abelian groups is additive if it is a group homomorphism of the hom-sets
- Adjoint functors: functors F and G are adjoint if $\text{Hom}(FX, Y) \cong \text{Hom}(X, GY)$, where the isomorphism is natural in X and Y
- Derived functor: the image of a short exact sequence under a functor that is only half-exact can be extended to a long exact sequence, the objects of which are images of a derived functor
- Enriched functor
- Essentially surjective functor: a functor every object of whose codomain is isomorphic to the image of an object in the domain
- Exact functor: a functor that takes short exact sequences to short exact sequences
- Faithful functor: a functor that is injective on the set of morphisms with given domain and codomain
- Full functor: a functor that is surjective on the set of morphisms with given domain and codomain
- Smooth functor: a functor F from $K\text{-}\mathbf{Vect}$ to $K\text{-}\mathbf{Vect}$ such that $\text{Hom}(V, W) \rightarrow \text{Hom}(FV, FW)$ is smooth. Examples include V^* , $\Lambda^k V$, $\Sigma^k V$ and the like.

Other

- Diagram (category theory)
- Functor category
- Kan extension

References

- S. Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag: New York, 1971.

Adjoint functors

In mathematics, **adjoint functors** are pairs of functors which stand in a particular relationship with one another, called an **adjunction**. The relationship of adjunction is ubiquitous in mathematics, as it rigorously reflects the intuitive notions of optimization and efficiency. It is studied in generality by the branch of mathematics known as category theory, which helps to minimize the repetition of the same logical details separately in every subject.

In the most concise symmetric definition, an adjunction between categories C and D is a pair of functors,

$$F : C \leftarrow D \quad \text{and} \quad G : C \rightarrow D$$

and a family of bijections

$$\text{hom}_C(FY, X) \cong \text{hom}_D(Y, GX)$$

which is natural in the variables X and Y . The functor F is called a **left adjoint functor**, while G is called a **right adjoint functor**. The relationship “ F is left adjoint to G ” (or equivalently, “ G is right adjoint to F ”) is sometimes written

$$F \dashv G.$$

This definition and others are made precise below.

Introduction

“The slogan is ‘Adjoint functors arise everywhere’.” (Saunders Mac Lane, *Categories for the working mathematician*)

The long list of examples in this article is only a partial indication of how often an interesting mathematical construction is an adjoint functor. As a result, general theorems about left/right adjoint functors, such as the equivalence of their various definitions or the fact that they respectively preserve colimits/limits (which are also found in every area of math), can encode the details of many useful and otherwise non-trivial results.

Motivation

One good way to motivate adjoint functors is to vaguely explain what problem they solve, and how they solve it. (This motivation runs parallel to the definitions via universal morphisms below.)

Adjoint functors as formulaic solutions to optimization problems

It can be said an adjoint functor is a way of giving the *most efficient* solution to some problem via a method which is *formulaic*. For example, an elementary problem in ring theory is how to turn a rng (which is like a ring that might not have a multiplicative identity) into a ring. The *most efficient* way is to adjoin an element ‘1’ to the rng, adjoin no unnecessary extra elements (we will need to have $r+1$ for each r in the ring, clearly), and impose no relations in the newly formed ring that are not forced by axioms. Moreover, this construction is *formulaic* in the sense that it works in essentially the same way for any rng.

This is rather vague, though suggestive, and can be made precise in the language of category theory: a construction is *most efficient* if it satisfies a universal property, and is *formulaic* if it defines a functor. Universal properties come in two types: initial properties

and terminal properties. Since these are dual (opposite) notions, it is only necessary to discuss one of them.

The idea of using an initial property is to set up the problem in terms of some auxiliary category E , and then identify that what we want is to find an initial object of E . This has an advantage that the *optimization* — the sense that we are finding the *most efficient* solution — means something rigorous and is recognisable, rather like the attainment of a supremum. Picking the right category E is something of a knack: for example, take the given rng R , and make a category E whose *objects* are rng homomorphisms $R \rightarrow S$, with S a ring having a multiplicative identity. The *morphisms* in E are commutative triangles of the form $(R \rightarrow S_1, R \rightarrow S_2, S_1 \rightarrow S_2)$ where $S_1 \rightarrow S_2$ is a ring map (which preserves the identity). The assertion that an object $R \rightarrow R^*$ is initial in E means that the ring R^* is a *most efficient* solution to our problem.

The two facts that this method of turning rngs into rings is *most efficient* and *formulaic* can be expressed simultaneously by saying that it defines an *adjoint functor*.

The hidden symmetry of optimization problems

Continuing this discussion, suppose we *started* with the functor F , and posed the following (vague) question: is there a problem to which F is the most efficient solution?

The notion of that F is the *most efficient solution* to the problem posed by G is, in a certain rigorous sense, equivalent to the notion that G poses the *most difficult problem* which F solves.

This has the intuitive meaning that adjoint functors should occur in pairs, and in fact they do, but this is not trivial from the universal morphism definitions. The equivalent symmetric definitions involving adjunctions and the symmetric language of adjoint functors (we can say either F is left adjoint to G or G is right adjoint to F) have the advantage of making this fact explicit.

Formal definitions

There are various definitions for adjoint functors. Their equivalence is elementary but not at all trivial and in fact highly useful. This article provides several such definitions:

- The definitions via universal morphisms are easy to state, and require minimal verifications when constructing an adjoint functor or proving two functors are adjoint. They are also the most analogous to our intuition involving optimizations.
- The definition via counit-unit adjunction is convenient for proofs about functors which are known to be adjoint, because they provide formulas that can be directly manipulated.
- The definition via hom-sets makes symmetry the most apparent, and is the reason for using the word *adjoint*.

Adjoint functors arise everywhere, in all areas of mathematics. Their full usefulness lies in that the structure in any of these definitions gives rise to the structures in the others via a long but trivial series of deductions. Thus, switching between them makes implicit use of a great deal of tedious details that would otherwise have to be repeated separately in every subject area. For example, naturality and terminality of the counit can be used to prove that any right adjoint functor preserves limits.

A helpful writing convention

The theory of adjoints has the terms *left* and *right* at its foundation, and there are many components which live in one of two categories C and D which are under consideration. It can therefore be extremely helpful to choose letters in alphabetical order according to whether they live in the "lefthand" category C or the "righthand" category D , and also to write them down in this order whenever possible.

In this article for example, the letters X, F, f, ε will consistently denote things which live in the category C , the letters Y, G, g, η will consistently denote things which live in the category D , and whenever possible such things will be referred to in order from left to right (a functor $F:C \leftarrow D$ can be thought of as "living" where its outputs are, in C).

Definitions via universal morphisms

A functor $F : C \leftarrow D$ is a **left adjoint functor** if for each object X in C , there exists a terminal morphism from F to X . If $G : C \rightarrow D$ is another functor such that terminal morphisms to F can be taken of the form (GX, ε_X) for each X (so each ε_X is a morphism $F(GX) \rightarrow X$), then we say G is **right adjoint to F** .

A functor $G : C \rightarrow D$ is a **right adjoint functor** if for each object Y in D , there exists an initial morphism from Y to G . If $F : C \leftarrow D$ is another functor such that initial morphisms to G can be taken of the form (FY, η_Y) for each Y (so each η_Y is a morphism $Y \rightarrow G(FY)$), then we say F is **left adjoint to G** .

Remarks:

It is true, but not immediate from these definitions, that F is *left adjoint to G* if and only if G is *right adjoint to F* , and that every left adjoint functor is *left adjoint to* some right adjoint functor G which is essentially unique (and the dual statement). The equivalent symmetric definitions below make these facts less confusing to remember, and have more immediate implications because of the adjunction structures they invoke. On the other hand, these universal morphism definitions are useful for *establishing* adjoints, because they are technically minimalistic in their requirements, and intuitively meaningful in that finding universal morphisms is like optimization.

Definition via counit-unit adjunction

A **counit-unit adjunction** between two categories C and D consists of two functors $F : C \leftarrow D$ and $G : C \rightarrow D$ and two natural transformations

$$\varepsilon : FG \rightarrow 1_C$$

$$\eta : 1_D \rightarrow GF$$

respectively called the **counit** and the **unit** of the adjunction (terminology from universal algebra*), such that the compositions

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F$$

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$$

are the identity transformations 1_F and 1_G on F and G respectively.

In this situation we say that **F is left adjoint to G** and **G is right adjoint to F** , and may indicate this relationship by writing $(\varepsilon, \eta) : F \dashv G$, or simply $F \dashv G$.

In equation form, the above conditions on (ε, η) are the **counit-unit equations**

$$1_F = \varepsilon F \circ F\eta$$

$$1_G = G\varepsilon \circ \eta G$$

which mean that for each X in C and each Y in D ,

$$1_{FY} = \varepsilon_{FY} \circ F(\eta_Y)$$

$$1_{GX} = G(\varepsilon_X) \circ \eta_{GX}.$$

These equations are useful in reducing proofs about adjoint functors to algebraic manipulations. They are sometimes called the *zig-zag equations* because of the appearance of the corresponding string diagrams. A way to remember them is to first write down the nonsensical equation $1 = \varepsilon \circ \eta$ and then fill in either F or G in one of the two simple ways which make the compositions defined.

Note: The use of the prefix "co" in counit here is not consistent with the terminology of limits and colimits, because a colimit satisfies an *initial* property whereas the counit morphisms will satisfy *terminal* properties, and dually. The term *unit* here is borrowed from the theory of monads where it looks like the insertion of the identity 1 into a monoid.

Definition via hom-set adjunction

A **hom-set adjunction** between two categories C and D consists of two functors $F : C \leftarrow D$ and $G : C \rightarrow D$ and a natural isomorphism

$$\Phi : \text{hom}_C(F-, -) \rightarrow \text{hom}_D(-, G-).$$

This specifies a family of bijections

$$\Phi_{Y,X} : \text{hom}_C(FY, X) \rightarrow \text{hom}_D(Y, GX).$$

for all objects X in C and Y in D .

In this situation we say that **F is left adjoint to G** and **G is right adjoint to F** , and may indicate this relationship by writing $\Phi : F \dashv G$, or simply $F \dashv G$.

This definition is a logical compromise in that it is somewhat more difficult to satisfy than the universal morphism definitions, and has fewer immediate implications than the counit-unit definition. It is useful because of its obvious symmetry, and as a stepping-stone between the other definitions.

In order to interpret Φ as a *natural isomorphism*, one must recognize $\text{hom}_C(F-, -)$ and $\text{hom}_D(-, G-)$ as functors. In fact, they are both bifunctors from $D^{\text{op}} \times C$ to **Set** (the category of sets). For details, see the article on hom functors. Explicitly, the naturality of Φ means that for all morphisms $f : X \rightarrow X'$ in C and all morphisms $g : Y \rightarrow Y'$ in D the following diagram commutes:

$$\begin{array}{ccc}
 \text{Hom}_C(FY, X) & \xlongequal{\Phi_{Y,X}} & \text{Hom}_D(Y, GX) \\
 \downarrow \text{Hom}(Fg, f) & & \downarrow \text{Hom}(g, Gf) \\
 \text{Hom}_C(FY', X') & \xlongequal[\Phi_{Y',X'}]{} & \text{Hom}_D(Y', GX')
 \end{array}$$

The vertical arrows in this diagram are those induced by composition with f and g .

Adjunctions in full

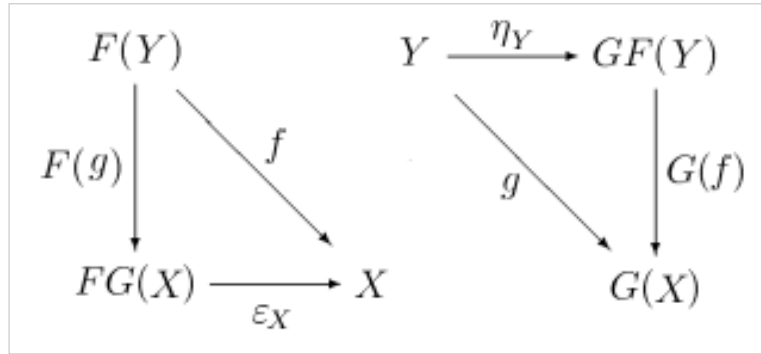
There are hence numerous functors and natural transformations associated with every adjunction, and only a small portion is sufficient to determine the rest.

An 'adjunction between categories C and D consists of

- A functor $F : C \leftarrow D$ called the **left adjoint**
- A functor $G : C \rightarrow D$ called the **right adjoint**
- A natural isomorphism $\Phi : \text{hom}_C(F-, -) \rightarrow \text{hom}_D(-, G-)$
- A natural transformation $\varepsilon : FG \rightarrow 1_C$ called the **counit**
- A natural transformation $\eta : 1_D \rightarrow GF$ called the **unit**

An equivalent formulation, where X denotes any object of C and Y denotes any object of D :

For every C -morphism $f : FY \rightarrow X$ there is a unique D -morphism $\Phi_{Y,X}(f) = g : Y \rightarrow GX$ such that the diagrams below commute, and for every D -morphism $g : Y \rightarrow GX$ there is a unique C -morphism $\Phi_{X,Y}^{-1}(g) = f : FY \rightarrow X$ in C such that the diagrams below commute:



From this assertion, one can recover that:

- The transformations ε , η , and Φ are related by the equations

$$f = \Phi_{Y,X}^{-1}(g) = \varepsilon_X \circ F(g) \in \text{hom}_C(F(Y), X)$$

$$g = \Phi_{Y,X}(f) = G(f) \circ \eta_Y \in \text{hom}_D(Y, G(X))$$

$$\Phi_{GX,X}^{-1}(1_{GX}) = \varepsilon_X \in \text{hom}_C(FG(X), X)$$

$$\Phi_{Y,FY}(1_{FY}) = \eta_Y \in \text{hom}_D(Y, GF(Y))$$

- The transformations ε , η satisfy the counit-unit equations

$$1_F = \varepsilon F \circ F \eta$$

$$1_G = G \varepsilon \circ \eta G$$

- Each pair (GX, ε_X) is a terminal morphism from F to X in C
- Each pair (FY, η_Y) is an initial morphism from Y to G in D

In particular, the equations above allow one to define (ε, η) in terms of Φ or vice versa. We will demonstrate the equivalence of these situations below.

Universal morphisms induce hom-set adjunction

Given a right adjoint functor $G : C \rightarrow D$ in the sense of initial morphisms, we can construct a functor $F : C \leftarrow D$ and hom-set adjunction

$$\Phi : \text{hom}_C(F-, -) \rightarrow \text{hom}_D(-, G-)$$

in the following steps:

- For each Y in D , choose an initial morphism (X_Y, η_Y) from Y to G , so we have $\eta_Y \rightarrow G(X_Y)$.
- Initiality of these morphisms allows us to construct a unique functor $F : C \leftarrow D$ such that $FY = X_Y$ and $\eta : 1_D \rightarrow GF$ is a natural transformation.
- For every morphism $g : Y \rightarrow GX$, initiality of (FY, η_Y) means we can let $\Psi_{Y,X}(g)$ be the unique morphism $f : FY \rightarrow X$ such that $G(f) \circ \eta_Y = g$.
- The map $\Psi_{Y,X} : \text{hom}_D(FY, X) \leftarrow \text{hom}_D(Y, GX)$ is injective by uniqueness and surjective because we can solve its defining equation for f . It is natural in X because η is natural, and natural in Y because G is a functor. Hence letting $\Phi_{Y,X} = \Psi_{Y,X}^{-1} : \text{hom}_C(FY, X) \rightarrow \text{hom}_D(Y, GX)$ gives a hom-set adjunction as required.

A similar argument allows one to construct a hom-set adjunction from the terminal morphisms to a left adjoint functor. (The construction that starts with a right adjoint is slightly more common, since the right adjoint in many adjoint pairs is a trivially defined inclusion or forgetful functor.)

Counit-unit adjunction induces hom-set adjunction

Given functors $F : C \leftarrow D$, $G : C \rightarrow D$, and a counit-unit adjunction $(\varepsilon, \eta) : F \dashv G$, we can construct a hom-set adjunction

$$\Phi : \text{hom}_D(F-, -) \rightarrow \text{hom}_C(-, G-)$$

in the following steps:

- For each $f : FY \rightarrow X$ and each $g : Y \rightarrow GX$, define

$$\Phi_{Y,X}(f) = G(f) \circ \eta_Y$$

$$\Psi_{Y,X}(g) = \varepsilon_X \circ F(g)$$
 The transformations Φ and Ψ are natural because η and ε are natural.
- Using, in order, that F is a functor, that ε is natural, and the counit-unit equation $1_{FY} = \varepsilon_{FY} \circ F(\eta_Y)$, we obtain

$$\begin{aligned} \Psi\Phi f &= \varepsilon_X \circ FG(f) \circ F(\eta_Y) \\ &= f \circ \varepsilon_{FY} \circ F(\eta_Y) \\ &= f \circ 1_{FY} = f \end{aligned}$$

hence $\Psi\Phi$ is the identity transformation.

- Dually, using that G is a functor, that η is natural, and the counit-unit equation $1_{GX} = G(\varepsilon_X) \circ \eta_{GX}$, we obtain

$$\begin{aligned} \Phi\Psi g &= G(\varepsilon_X) \circ GF(g) \circ \eta_Y \\ &= G(\varepsilon_X) \circ \eta_{GX} \circ g \\ &= 1_{GX} \circ g = g \end{aligned}$$

hence $\Phi\Psi$ is the identity transformation, so Φ is a natural isomorphism with inverse $\Phi^{-1} = \Psi$.

Hom-set adjunction induces all of the above

Given functors $F : C \leftarrow D$, $G : C \rightarrow D$, and a hom-set adjunction $\Phi : \text{hom}_C(F-, -) \rightarrow \text{hom}_D(-, G-)$, we can construct a counit-unit adjunction

$$(\varepsilon, \eta) : F \dashv G,$$

which defines families of initial and terminal morphisms, in the following steps:

- Let $\varepsilon_X = \Phi_{GX, X}^{-1}(1_{GX}) \in \text{hom}_D(FGX, X)$ for each X in C , where $1_{GX} \in \text{hom}_C(GX, GX)$ is the identity morphism.
- Let $\eta_Y = \Phi_{Y, FY}(1_{FY}) \in \text{hom}_C(Y, GFY)$ for each Y in D , where $1_{FY} \in \text{hom}_C(FY, FY)$ is the identity morphism.
- The bijectivity and naturality of Φ imply that each (GX, ε_X) is a terminal morphism from X to F in C , and each (FY, η_Y) is an initial morphism from Y to G in D .
- The naturality of Φ implies the naturality of ε and η , and the two formulas

$$\Phi_{Y, X}(f) = G(f) \circ \eta_Y$$

$$\Phi_{Y, X}^{-1}(g) = \varepsilon_X \circ F(g)$$

for each $f: FY \rightarrow X$ and $g: Y \rightarrow GX$ (which completely determine Φ).

- Substituting FY for X and $\eta_Y = \Phi_{Y, FY}(1_{FY})$ for g in the second formula gives the first counit-unit equation

$$1_{FY} = \varepsilon_{FY} \circ F(\eta_Y),$$

and substituting GX for Y and $\varepsilon_X = \Phi_{GX, X}^{-1}(1_{GX})$ for f in the first formula gives the second counit-unit equation

$$1_{GX} = G(\varepsilon_X) \circ \eta_{GX}.$$

Historical perspective

Ubiquity of adjoint functors

The idea of an adjoint functor was formulated by Daniel Kan in 1958. Like many of the concepts in category theory, it was suggested by the needs of homological algebra, which was at the time devoted to computations. Those faced with giving tidy, systematic presentations of the subject would have noticed relations such as

$$\text{hom}(F(X), Y) = \text{hom}(X, G(Y))$$

in the category of abelian groups, where F was the functor $- \otimes A$ (i.e. take the tensor product with A), and G was the functor $\text{hom}(A, -)$. The use of the *equals* sign is an abuse of notation; those two groups are not really identical but there is a way of identifying them that is *natural*. It can be seen to be natural on the basis, firstly, that these are two alternative descriptions of the bilinear mappings from $X \times A$ to Y . That is, however, something particular to the case of tensor product. In category theory the 'naturality' of the bijection is subsumed in the concept of a natural isomorphism.

The terminology comes from the Hilbert space idea of adjoint operators T, U with $\langle Tx, y \rangle = \langle x, Uy \rangle$, which is formally similar to the above relation between hom-sets. We say that F is *left adjoint* to G , and G is *right adjoint* to F . Note that G may have itself a right adjoint that is quite different from F (see below for an example). The analogy to adjoint maps of Hilbert spaces can be made precise in certain contexts^[1].

If one starts looking for these adjoint pairs of functors, they turn out to be very common in abstract algebra, and elsewhere as well. The example section below provides evidence of this; furthermore, universal constructions, which may be more familiar to some, give rise to numerous adjoint pairs of functors.

In accordance with the thinking of Saunders Mac Lane, any idea such as adjoint functors that occurs widely enough in mathematics should be studied for its own sake.

Problems formulated with adjoint functors

Mathematicians do not generally need the full adjoint functor concept. Concepts can be judged according to their use in solving problems, as well as for their use in building theories. The tension between these two motivations was especially great during the 1950s when category theory was initially developed. Enter Alexander Grothendieck, who used category theory to take compass bearings in foundational, axiomatic work — in functional analysis, homological algebra and finally algebraic geometry.

It is probably wrong to say that he promoted the adjoint functor concept in isolation: but recognition of the role of adjunction was inherent in Grothendieck's approach. For example, one of his major achievements was the formulation of Serre duality in relative form — one could say loosely, in a continuous family of algebraic varieties. The entire proof turned on the existence of a right adjoint to a certain functor. This is something undeniably abstract, and non-constructive, but also powerful in its own way.

The case of partial orders

Every partially ordered set can be viewed as a category (with a single morphism between x and y if and only if $x \leq y$). A pair of adjoint functors between two partially ordered sets is called a Galois connection (or, if it is contravariant, an *antitone* Galois connection). See that article for a number of examples: the case of Galois theory of course is a leading one. Any Galois connection gives rise to closure operators and to inverse order-preserving bijections between the corresponding closed elements.

As is the case for Galois groups, the real interest lies often in refining a correspondence to a duality (i.e. *antitone* order isomorphism). A treatment of Galois theory along these lines by Kaplansky was influential in the recognition of the general structure here.

The partial order case collapses the adjunction definitions quite noticeably, but can provide several themes:

- adjunctions may not be dualities or isomorphisms, but are candidates for upgrading to that status
- closure operators may indicate the presence of adjunctions, as corresponding monads (cf. the Kuratowski closure axioms)
- a very general comment of Martin Hyland is that *syntax and semantics* are adjoint: take C to be the set of all logical theories (axiomatizations), and D the power set of the set of all mathematical structures. For a theory T in C , let $F(T)$ be the set of all structures that satisfy the axioms T ; for a set of mathematical structures S , let $G(S)$ be the minimal axiomatization of S . We can then say that $F(T)$ is a subset of S if and only if T logically implies $G(S)$: the "semantics functor" F is left adjoint to the "syntax functor" G .
- division is (in general) the attempt to *invert* multiplication, but many examples, such as the introduction of implication in propositional logic, or the ideal quotient for division by ring ideals, can be recognised as the attempt to provide an adjoint.

Together these observations provide explanatory value all over mathematics.

Examples

Free groups (instructive example)

The construction of free groups is an extremely common adjoint construction, and a useful example for making sense of the above details.

Suppose that $F : \mathbf{Grp} \leftarrow \mathbf{Set}$ is the functor assigning to each set Y the free group generated by the elements of Y , and that $G : \mathbf{Grp} \rightarrow \mathbf{Set}$ is the forgetful functor, which assigns to each group X its underlying set. Then F is left adjoint to G :

Terminal morphisms. For each group X , the group FGX is the free group generated freely by GX , the elements of X . Let $\varepsilon_X : FGX \rightarrow X$ be the group homomorphism which sends the generators of FGX to the elements of X they correspond to, which exists by the universal property of free groups. Then each (GX, ε_X) is a terminal morphism from F to X , because any group homomorphism from a free group FZ to X will factor through $\varepsilon_X : FGX \rightarrow X$ via a unique set map from Z to GX . This means that (F, G) is an adjoint pair.

Initial morphisms. For each set Y , the set GFY is just the underlying set of the free group FY generated by Y . Let $\eta_Y : Y \rightarrow GFY$ be the set map given by "inclusion of generators". Then each (FY, η_Y) is an initial morphism from Y to G , because any set map from Y to the underlying set GW of a group will factor through $\eta_Y : Y \rightarrow GFY$ via a unique group homomorphism from FY to W . This also means that (F, G) is an adjoint pair.

Hom-set adjunction. Maps from the free group FY to a group X correspond precisely to maps from the set Y to the set GX : each homomorphism from FY to X is fully determined by its action on generators. One can verify directly that this correspondence is a natural transformation, which means it is a hom-set adjunction for the pair (F, G) .

Counit-unit adjunction. One can also verify directly that ε and η are natural. Then, a direct verification that they form a counit-unit adjunction $(\varepsilon, \eta) : F \dashv G$ is as follows:

The first counit-unit equation $1_F = \varepsilon F \circ F \eta$ says that for each set Y the composition

$$FY \xrightarrow{F(\eta_Y)} FGFY \xrightarrow{\varepsilon_{FY}} F$$

should be the identity. The intermediate group $FGFY$ is the free group generated freely by the words of the free group FY . (Think of these words as placed in parentheses to indicate that they are independent generators.) The arrow $F(\eta_Y)$ is the group homomorphism from FY into $FGFY$ sending each generator y of FY to the corresponding word of length one (y) as a generator of $FGFY$. The arrow ε_{FY} is the group homomorphism from $FGFY$ to FY sending each generator to the word of FY it corresponds to (so this map is "dropping parentheses"). The composition of these maps is indeed the identity on FY .

The second counit-unit equation $1_G = G\varepsilon \circ \eta G$ says that for each group X the composition

$$GX \xrightarrow{\eta_{GX}} GFGX \xrightarrow{G(\varepsilon_X)} GX$$

should be the identity. The intermediate set $GFGX$ is just the underlying set of FGX . The arrow η_{GX} is the "inclusion of generators" set map from the set GX to the set $GFGX$. The arrow $G(\varepsilon_X)$ is the set map from $GFGX$ to GX which underlies the group homomorphism sending each generator of FGX to the element of X it corresponds to ("dropping

parentheses"). The composition of these maps is indeed the identity on GX .

Free constructions and forgetful functors

Free objects are all examples of a left adjoint to a forgetful functor which assigns to an algebraic object its underlying set. These algebraic free functors have generally the same description of as in the detailed description of the free group situation above.

Diagonal functors and limits

Products, fibred products, equalizers, and kernels are all examples of the categorical notion of a limit. Any limit functor is right adjoint to a corresponding diagonal functor (provided the category has the type of limits in question), and the counit of the adjunction provides the defining maps from the limit object. Below are some specific examples.

- **Products** Let $\Pi : \mathbf{Grp}^2 \rightarrow \mathbf{Grp}$ the functor which assigns to each pair (X_1, X_2) the product group $X_1 \times X_2$, and let $\Delta : \mathbf{Grp}^2 \leftarrow \mathbf{Grp}$ be the diagonal functor which assigns to every group X the pair (X, X) in the product category $\mathbf{Grp}^2$. The universal property of the product group shows that Π is right-adjoint to Δ . The counit of this adjunction is the defining pair of projection maps from $X_1 \times X_2$ to X_1 and X_2 which define the limit, and the unit is the *diagonal inclusion* of a group X into $X_1 \times X_2$ (mapping x to (x, x)).

The cartesian product of sets, the product of rings, the product of topological spaces etc. follow the same pattern; it can also be extended in a straightforward manner to more than just two factors. More generally, any type of limit is right adjoint to a diagonal functor.

- **Kernels.** Consider the category D of homomorphisms of abelian groups. If $f_1 : A_1 \rightarrow B_1$ and $f_2 : A_2 \rightarrow B_2$ are two objects of D , then a morphism from f_1 to f_2 is a pair (g_A, g_B) of morphisms such that $g_B f_1 = f_2 g_A$. Let $G : D \rightarrow \mathbf{Ab}$ be the functor which assigns to each homomorphism its kernel and let $F : \mathbf{D} \leftarrow \mathbf{Ab}$ be the functor which maps the group A to the homomorphism $A \rightarrow 0$. Then G is right adjoint to F , which expresses the universal property of kernels. The counit of this adjunction is the defining embedding of a homomorphism's kernel into the homomorphism's domain, and the unit is the morphism identifying a group A with the kernel of the homomorphism $A \rightarrow 0$.

A suitable variation of this example also shows that the kernel functors for vector spaces and for modules are right adjoints. Analogously, one can show that the cokernel functors for abelian groups, vector spaces and modules are left adjoints.

Colimits and diagonal functors

Coproducts, fibred coproducts, coequalizers, and cokernels are all examples of the categorical notion of a colimit. Any colimit functor is left adjoint to a corresponding diagonal functor (provided the category has the type of colimits in question), and the unit of the adjunction provides the defining maps into the colimit object. Below are some specific examples.

- **Coproducts.** If $F : \mathbf{Ab} \leftarrow \mathbf{Ab}^2$ assigns to every pair (X_1, X_2) of abelian groups their direct sum, and if $G : \mathbf{Ab} \rightarrow \mathbf{Ab}^2$ is the functor which assigns to every abelian group Y the pair (Y, Y) , then F is left adjoint to G , again a consequence of the universal property of direct sums. The unit of this adjoint pair is the defining pair of inclusion maps from X_1 and X_2 into the direct sum, and the counit is the additive map from the direct sum of (X, X) to

back to X (sending an element (a,b) of the direct sum to the element $a+b$ of X).

Analogous examples are given by the direct sum of vector spaces and modules, by the free product of groups and by the disjoint union of sets.

Further examples

In algebra

- **Adjoining an identity to a rng.** This example was discussed in the motivation section above. Given a rng R , a multiplicative identity element can be added by taking $R \times \mathbf{Z}$ and defining a $\mathbf{Z}$ -bilinear product with $(r,0)(0,1) = (0,1)(r,0) = (r,0)$, $(r,0)(s,0) = (rs,0)$, $(0,1)(0,1) = (0,1)$. This constructs a left adjoint to the functor taking a ring to the underlying rng.
- **Ring extensions.** Suppose R and S are rings, and $\rho : R \rightarrow S$ is a ring homomorphism. Then S can be seen as a (left) R -module, and the tensor product with S yields a functor $F : R\text{-}\mathbf{Mod} \rightarrow S\text{-}\mathbf{Mod}$. Then F is left adjoint to the forgetful functor $G : S\text{-}\mathbf{Mod} \rightarrow R\text{-}\mathbf{Mod}$.
- **Tensor products.** If R is a ring and M is a right R module, then the tensor product with M yields a functor $F : R\text{-}\mathbf{Mod} \rightarrow \mathbf{Ab}$. The functor $G : \mathbf{Ab} \rightarrow R\text{-}\mathbf{Mod}$, defined by $G(A) = \text{hom}_{\mathbf{Z}}(M, A)$ for every abelian group A , is a right adjoint to F .
- **From monoids and groups to rings** The integral monoid ring construction gives a functor from monoids to rings. This functor is left adjoint to the functor that associates to a given ring its underlying multiplicative monoid. Similarly, the integral group ring construction yields a functor from groups to rings, left adjoint to the functor that assigns to a given ring its group of units. One can also start with a field K and consider the category of K -algebras instead of the category of rings, to get the monoid and group rings over K .
- **Field of fractions.** Consider the category $\mathbf{Dom}_m$ of integral domains with injective morphisms. The forgetful functor $\mathbf{Field} \rightarrow \mathbf{Dom}_m$ from fields has a left adjoint - it assigns to every integral domain its field of fractions.
- **Polynomial rings.** Let $\mathbf{Ring}_*$ be the category of pointed commutative rings with unity (pairs (A, a) where A is a ring, $a \in A$ and morphisms preserve the distinguished elements). The forgetful functor $G : \mathbf{Ring}_* \rightarrow \mathbf{Ring}$ has a left adjoint - it assigns to every ring R the pair $(R[x], x)$ where $R[x]$ is the polynomial ring with coefficients from R .
- **Abelianization.** Consider the inclusion functor $G : \mathbf{Ab} \rightarrow \mathbf{Grp}$ from the category of abelian groups to category of groups. It has a left adjoint called abelianization which assigns to every group G the quotient group $G^{\text{ab}} = G/[G, G]$.
- **The Grothendieck group.** In K-theory, the point of departure is to observe that the category of vector bundles on a topological space has a commutative monoid structure under direct sum. One may make an abelian group out of this monoid, the Grothendieck group, by formally adding an additive inverse for each bundle (or equivalence class). Alternatively one can observe that the functor that for each group takes the underlying monoid (ignoring inverses) has a left adjoint. This is a once-for-all construction, in line with the third section discussion above. That is, one can imitate the construction of negative numbers; but there is the other option of an existence theorem. For the case of finitary algebraic structures, the existence by itself can be referred to universal algebra, or model theory; naturally there is also a proof adapted to category theory, too.

- **Frobenius reciprocity** in the representation theory of groups: see induced representation. This example foreshadowed the general theory by about half a century.

In topology

- **A functor with a left and a right adjoint.** Let G be the functor from topological spaces to sets that associates to every topological space its underlying set (forgetting the topology, that is). G has a left adjoint F , creating the discrete space on a set Y , and a right adjoint H creating the trivial topology on Y .
- **Suspensions and loop spaces** Given topological spaces X and Y , the space $[SX, Y]$ of homotopy classes of maps from the suspension SX of X to Y is naturally isomorphic to the space $[X, \Omega Y]$ of homotopy classes of maps from X to the loop space ΩY of Y . This is an important fact in homotopy theory.
- **Stone-Čech compactification.** Let $\mathbf{KHaus}$ be the category of compact Hausdorff spaces and $G : \mathbf{KHaus} \rightarrow \mathbf{Top}$ be the forgetful functor to the category of topological spaces. Then G has a left adjoint $F : \mathbf{Top} \rightarrow \mathbf{KHaus}$, the Stone-Čech compactification. The counit of this adjoint pair yields a continuous map from every topological space X into its Stone-Čech compactification. This map is an embedding (i.e. injective, continuous and open) if and only if X is a Tychonoff space.
- **Direct and inverse images of sheaves** Every continuous map $f : X \rightarrow Y$ between topological spaces induces a functor $f_{\square}$ from the category of sheaves (of sets, or abelian groups, or rings...) on X to the corresponding category of sheaves on Y , the *direct image functor*. It also induces a functor f^{-1} from the category of sheaves of abelian groups on Y to the category of sheaves of abelian groups on X , the *inverse image functor*. f^{-1} is left adjoint to $f_{\square}$. Here a more subtle point is that the left adjoint for coherent sheaves will differ from that for sheaves (of sets).
- **Soberification.** The article on Stone duality describes an adjunction between the category of topological spaces and the category of sober spaces that is known as soberification. Notably, the article also contains a detailed description of another adjunction that prepares the way for the famous duality of sober spaces and spatial locales, exploited in pointless topology.

In category theory

- **A series of adjunctions.** The functor π_0 which assigns to a category its sets of connected components is left-adjoint to the functor D which assigns to a set the discrete category on that set. Moreover, D is left-adjoint to the object functor U which assigns to each category its set of objects, and finally U is left-adjoint to A which assigns to each set the antidiscrete category on that set.
- **Exponential object.** In a cartesian closed category the endofunctor $C \rightarrow C$ given by $- \times A$ has a right adjoint $-^A$.

Properties

Uniqueness of adjoints

If the functor $F : C \leftarrow D$ has two right adjoints G and G' , then G and G' are naturally isomorphic. The same is true for left adjoints.

Conversely, if F is left adjoint to G , and G is naturally isomorphic to G' then F is also left adjoint to G' . More generally, if $[F, G, \varepsilon, \eta]$ is an adjunction (with counit-unit (ε, η)) and

$$\sigma : F \rightarrow F'$$

$$\tau : G \rightarrow G'$$

are natural isomorphisms then $[F', G', \varepsilon', \eta']$ is an adjunction where

$$\eta' = (\tau * \sigma) \circ \eta$$

$$\varepsilon' = \varepsilon \circ (\sigma^{-1} * \tau^{-1}).$$

Here $\circ$ denotes vertical composition of natural transformations, and $*$ denotes horizontal composition.

Composition

Adjunctions can be composed in a natural fashion. Specifically, if $[F, G, \varepsilon, \eta]$ is an adjunction between C and D and $[F', G', \varepsilon', \eta']$ is an adjunction between D and E then the functor

$$F' \circ F : C \leftarrow E$$

is left adjoint to

$$G \circ G' : E \rightarrow C.$$

The counit and the unit of this adjunction are given by the compositions:

$$1_E \xrightarrow{\eta} GF \xrightarrow{G\eta'F} GG'F'F$$

$$F'F'GG' \xrightarrow{F'\varepsilon G'} F'G' \xrightarrow{\varepsilon'} 1_C.$$

One can then form a category whose objects are all small categories and whose morphisms are adjunctions.

Adjoint functors preserve limits

The most important property of adjoints is their continuity: every functor that has a left adjoint (and therefore is a right adjoint) is *continuous* (i.e. commutes with limits in the category theoretical sense); every functor that has a right adjoint (and therefore is a left adjoint) is *cocontinuous* (i.e. commutes with colimits).

Since many common constructions in mathematics are limits or colimits, this provides a wealth of information. For example:

- applying a right adjoint functor to a product of objects yields the product of the images;
- applying a left adjoint functor to a coproduct of objects yields the coproduct of the images;
- every right adjoint functor is left exact;
- every left adjoint functor is right exact.

Additivity

If C and D are preadditive categories and $F : C \leftarrow D$ is an additive functor with a right adjoint $G : C \rightarrow D$, then G is also an additive functor and the hom-set bijections

$$\Phi_{Y,X} : \text{hom}_C(FY, X) \cong \text{hom}_D(Y, GX)$$

are, in fact, isomorphisms of abelian groups. Dually, if G is additive with a left adjoint F , then F is also additive.

Moreover, if both C and D are additive categories (i.e. preadditive categories with all finite biproducts), then any pair of adjoint functors between them are automatically additive.

General existence theorem

Not every functor $G : C \rightarrow D$ admits a left adjoint. If C is a complete category, then the functors with left adjoints can be characterized by the **adjoint functor theorem** of Peter J. Freyd: G has a left adjoint if and only if it is continuous and a certain smallness condition is satisfied: for every object Y of D there exists a family of morphisms

$$f_i : Y \rightarrow G(X_i)$$

where the indices i come from a *set* I , not a *proper class*, such that every morphism

$$h : Y \rightarrow G(X)$$

can be written as

$$h = G(t) \circ f_i$$

for some i in I and some morphism

$$t : X_i \rightarrow X \text{ in } C.$$

An analogous statement characterizes those functors with a right adjoint.

Relationship to other categorical concepts

Universal constructions

As stated earlier, an adjunction between categories C and D gives rise to a family of universal morphisms, one for each object in C and one for each object in D . Conversely, if there exists a universal morphism to a functor $G : C \rightarrow D$ from every object of D , then G has a left adjoint.

However, universal constructions are more general than adjoint functors: a universal construction is like an optimization problem; it gives rise to an adjoint pair if and only if this problem has a solution for every object of D (equivalently, every object of C).

Equivalences of categories

Every equivalence of categories defines an adjunction. If $F : C \leftarrow D$ and $G : C \rightarrow D$ are functors with natural isomorphisms $\varepsilon : FG \rightarrow 1_C$ and $\eta : 1_D \rightarrow GF$ then (F, G) form an adjoint pair with counit ε and unit η . Conversely, an adjunction $[F, G, \varepsilon, \eta]$ defines an equivalence of categories if and only if the counit and unit are natural isomorphisms (and not just natural transformations).

If (F, G) define an equivalence of categories, then F is not only a left adjoint of G but a right adjoint as well. Explicitly, if $[F, G, \varepsilon, \eta]$ is an adjoint equivalence then so is $[G, F, \eta^{-1}, \varepsilon^{-1}]$.

Every adjunction $[F, G, \varepsilon, \eta]$ extends an equivalence of certain subcategories. Define C_1 as the full subcategory of C consisting of those objects X of C for which ε_X is an isomorphism, and define D_1 as the full subcategory of D consisting of those objects Y of D for which η_Y is an isomorphism. Then F and G can be restricted to D_1 and C_1 and yield inverse equivalences of these subcategories.

In a sense, then, adjoints are "generalized" inverses. Note however that a right inverse of F (i.e. a functor G such that FG is naturally isomorphic to 1_D) need not be a right (or left) adjoint of F . Adjoints generalize *two-sided* inverses.

Monads

Every adjunction $[F, G, \varepsilon, \eta]$ gives rise to an associated monad $[T, \eta, \mu]$ in the category D . The functor

$$T : \mathcal{D} \rightarrow \mathcal{D}$$

is given by $T = GF$. The unit of the monad

$$\eta : 1_{\mathcal{D}} \rightarrow T$$

is just the unit η of the adjunction and the multiplication transformation

$$\mu : T^2 \rightarrow T$$

is given by $\mu = G\varepsilon F$. Dually, the triple $[FG, \varepsilon, F\eta G]$ defines a comonad in D .

Every monad arises from some adjunction—in fact, typically from many adjunctions—in the above fashion. Two constructions, called the category of Eilenberg-Moore algebras and the Kleisli category are two extremal solutions to the problem of constructing an adjunction that gives rise to a given monad.

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External links

- Adjunctions (http://www.youtube.com/view_play_list?p=54B49729E5102248) Seven short lectures on adjunctions.

Representable functor

In mathematics, especially in category theory, a **representable functor** is a functor of a special form from an arbitrary category into the category of sets. Such functors give representations of an abstract category in terms of known structures (i.e. sets and functions) allowing one to utilize, as much as possible, knowledge about the category of sets in other settings.

From another point of view, representable functors for a category C are the functors *given* with C . Their theory is a vast generalisation of upper sets in posets, and of Cayley's theorem in group theory.

Definition

Let C be a locally small category and let **Set** be the category of sets. For each object A of C let $\text{Hom}(A, -)$ be the hom functor which maps objects X to the set $\text{Hom}(A, X)$.

A functor $F : C \rightarrow \mathbf{Set}$ is said to be **representable** if it is naturally isomorphic to $\text{Hom}(A, -)$ for some object A of C . A **representation** of F is a pair (A, Φ) where

$$\Phi : \text{Hom}(A, -) \rightarrow F$$

is a natural isomorphism.

A contravariant functor $G : C \rightarrow \mathbf{Set}$ is said to be **representable** if it is naturally isomorphic to the contravariant hom-functor $\text{Hom}(-, A)$ for some object A of C .

Universal elements

According to Yoneda's lemma, natural transformations from $\text{Hom}(A, -)$ to F are in one-to-one correspondence with the elements of $F(A)$. Given a natural transformation $\Phi : \text{Hom}(A, -) \rightarrow F$ the corresponding element of $u \in F(A)$ is given by

$$u = \Phi_A(\text{id}_A).$$

Conversely, given any element $u \in F(A)$ we may define a natural transformation $\Phi : \text{Hom}(A, -) \rightarrow F$ via

$$\Phi_X(f) = (Ff)(u)$$

where f is an element of $\text{Hom}(A, X)$. In order to get a representation of F we want to know when the natural transformation induced by u is an isomorphism. This leads to the following definition:

A **universal element** of a functor $F : C \rightarrow \mathbf{Set}$ is a pair (A, u) consisting of an object A of C and an element $u \in F(A)$ such that for every pair (X, v) with $v \in F(X)$ there exists a unique morphism $f : A \rightarrow X$ such that $(Ff)u = v$.

A universal element may be viewed as a universal morphism from the one-point set $\{\bullet\}$ to the functor F or as an initial object in the category of elements of F .

The natural transformation induced by an element $u \in F(A)$ is an isomorphism if and only if (A, u) is a universal element of F . We therefore conclude that representations of F are in one-to-one correspondence with universal elements of F . For this reason, it is common to refer to universal elements (A, u) as representations.

Examples

- Consider the contravariant functor $P : \mathbf{Set} \rightarrow \mathbf{Set}$ which maps each set to its power set and each function to its inverse image map. To represent this functor we need a pair (A, u) where A is a set and u is a subset of A , i.e. an element of $P(A)$, such that for all sets X , the hom-set $\text{Hom}(X, A)$ is isomorphic to $P(X)$ via $\Phi_X(f) = (Pf)u = f^{-1}(u)$. Take $A = \{0, 1\}$ and $u = \{1\}$. Given a subset $S \subseteq X$ the corresponding function from X to A is the characteristic function of S .
- Forgetful functors to $\mathbf{Set}$ are very often representable. In particular, a forgetful functor is represented by (A, u) whenever A is a free object over a singleton set with generator u .
 - The forgetful functor $\mathbf{Grp} \rightarrow \mathbf{Set}$ on the category of groups is represented by $(\mathbf{Z}, 1)$.
 - The forgetful functor $\mathbf{Ring} \rightarrow \mathbf{Set}$ on the category of rings is represented by $(\mathbf{Z}[x], x)$, the polynomial ring in one variable with integer coefficients.
 - The forgetful functor $\mathbf{Vect} \rightarrow \mathbf{Set}$ on the category of real vector spaces is represented by $(\mathbf{R}, 1)$.
 - The forgetful functor $\mathbf{Top} \rightarrow \mathbf{Set}$ on the category of topological spaces is represented by any singleton topological space with its unique element.
- A group G can be considered a category (even a groupoid) with one object which we denote by $\bullet$. A functor from G to $\mathbf{Set}$ then corresponds to a G -set. The unique hom-functor $\text{Hom}(\bullet, -)$ from G to $\mathbf{Set}$ corresponds to the canonical G -set G with the action of left multiplication. Standard arguments from group theory show that a functor from G to $\mathbf{Set}$ is representable if and only if the corresponding G -set is simply transitive (i.e. a G -torsor). Choosing a representation amounts to choosing an identity for the group structure.
- Let C be the category of CW-complexes with morphisms given by homotopy classes of continuous functions. For each natural number n there is a contravariant functor $H^n : C \rightarrow \mathbf{Ab}$ which assigns each CW-complex its n^{th} cohomology group (with integer coefficients). Composing this with the forgetful functor we have a contravariant functor from C to $\mathbf{Set}$. Brown's representability theorem in algebraic topology says that this functor is represented by a CW-complex $K(\mathbf{Z}, n)$ called an Eilenberg-Mac Lane space.

Properties

Uniqueness

Representations of functors are unique up to a unique isomorphism. That is, if (A_1, Φ_1) and (A_2, Φ_2) represent the same functor, then there exists a unique isomorphism $\varphi : A_1 \rightarrow A_2$ such that

$$\Phi_1^{-1} \circ \Phi_2 = \text{Hom}(\varphi, -)$$

as natural isomorphisms from $\text{Hom}(A_2, -)$ to $\text{Hom}(A_1, -)$. This fact follows easily from Yoneda's lemma.

Stated in terms of universal elements: if (A_1, u_1) and (A_2, u_2) represent the same functor, then there exists a unique isomorphism $\varphi : A_1 \rightarrow A_2$ such that

$$(F\varphi)u_1 = u_2.$$

Preservation of limits

Representable functors are naturally isomorphic to Hom functors and therefore share their properties. In particular, (covariant) representable functors preserve all limits. It follows that any functor which fails to preserve some limit is not representable.

Contravariant representable functors take colimits to limits.

Left adjoint

Any functor $K : C \rightarrow \mathbf{Set}$ with a left adjoint $F : \mathbf{Set} \rightarrow C$ is represented by $(FX, \eta_X(\bullet))$ where $X = \{\bullet\}$ is a singleton set and η is the unit of the adjunction.

Conversely, if K is represented by a pair (A, u) and all small copowers of A exist in C then K has a left adjoint F which sends each set I to the I th copower of A .

Therefore, if C is a category with all small copowers, a functor $K : C \rightarrow \mathbf{Set}$ is representable if and only if it has a left adjoint.

Relation to universal morphisms and adjoints

The categorical notions of universal morphisms and adjoint functors can both be expressed using representable functors.

Let $G : D \rightarrow C$ be a functor and let X be an object of C . Then (A, φ) is a universal morphism from X to G if and only if (A, φ) is a representation of the functor $\text{Hom}_C(X, G-)$ from D to $\mathbf{Set}$. It follows that G has a left-adjoint F if and only if $\text{Hom}_C(X, G-)$ is representable for all X in C . The natural isomorphism $\Phi_X : \text{Hom}_D(FX, -) \rightarrow \text{Hom}_C(X, G-)$ yields the adjointness; that is

$$\Phi_{X,Y} : \text{Hom}_D(FX, Y) \rightarrow \text{Hom}_C(X, GY)$$

is a bijection for all X and Y .

The dual statements are also true. Let $F : C \rightarrow D$ be a functor and let Y be an object of D . Then (A, φ) is a universal morphism from F to Y if and only if (A, φ) is a representation of the functor $\text{Hom}_D(F-, Y)$ from C to $\mathbf{Set}$. It follows that F has a right-adjoint G if and only if $\text{Hom}_D(F-, Y)$ is representable for all Y in D .

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Functor category

In category theory, a branch of mathematics, the functors between two given categories can themselves be turned into a category; the morphisms in this **functor category** are natural transformations between functors. Functor categories are of interest for two main reasons:

- many commonly occurring categories are (disguised) functor categories, so any statement proved for general functor categories is widely applicable;
- every category embeds in a functor category (via the Yoneda embedding); the functor category often has nicer properties than the original category, allowing certain operations that were not available in the original setting.

An element of a functor category is called a **diagram**.

Definition

Suppose C is a small category (i.e. the objects form a set rather than a proper class) and D is an arbitrary category. The category of functors from C to D , written as $\text{Funct}(C, D)$ or D^C , has as objects the covariant functors from C to D , and as morphisms the natural transformations between such functors. Note that natural transformations can be composed: if $\mu(X) : F(X) \rightarrow G(X)$ is a natural transformation from the functor $F : C \rightarrow D$ to the functor $G : C \rightarrow D$, and $\eta(X) : G(X) \rightarrow H(X)$ is a natural transformation from the functor G to the functor H , then the collection $\eta(X)\mu(X) : F(X) \rightarrow H(X)$ defines a natural transformation from F to H . With this composition of natural transformations (known as vertical composition, see natural transformation), D^C satisfies the axioms of a category.

In a completely analogous way, one can also consider the category of all *contravariant* functors from C to D ; we write this as $\text{Funct}(C^{\text{op}}, D)$.

If C and D are both preadditive categories (i.e. their morphism sets are abelian groups and the composition of morphisms is bilinear), then we can consider the category of all additive functors from C to D , denoted by $\text{Add}(C, D)$.

Examples

- If I is a small discrete category (i.e. its only morphisms are the identity morphisms), then a functor from I to C essentially consists of a family of objects of C , indexed by I ; the functor category C^I can be identified with the corresponding product category: its elements are families of objects in C and its morphisms are families of morphisms in C .
- A directed graph consists of a set of arrows and a set of vertices, and two functions from the arrow set to the vertex set, specifying each arrow's start and end vertex. The category of all directed graphs is thus nothing but the functor category $\mathbf{Set}^C$, where C is the category with two objects connected by two morphisms, and $\mathbf{Set}$ denotes the category of sets.
- Any group G can be considered as a one-object category in which every morphism is invertible. The category of all G -sets is the same as the functor category $\mathbf{Set}^G$.
- Similar to the previous example, the category of k -linear representations of the group G is the same as the functor category $\mathbf{k}\text{-Vect}^G$ (where $\mathbf{k}\text{-Vect}$ denotes the category of all vector spaces over the field k).

- Any ring R can be considered as a one-object preadditive category; the category of left modules over R is the same as the additive functor category $\text{Add}(R, \mathbf{Ab})$ (where $\mathbf{Ab}$ denotes the category of abelian groups), and the category of right R -modules is $\text{Add}(R^{\text{op}}, \mathbf{Ab})$. Because of this example, for any preadditive category C , the category $\text{Add}(C, \mathbf{Ab})$ is sometimes called the "category of left modules over C " and $\text{Add}(C^{\text{op}}, \mathbf{Ab})$ is the category of right modules over C .
- The category of presheaves on a topological space X is a functor category: we turn the topological space in a category C having the open sets in X as objects and a single morphism from U to V if and only if U is contained in V . The category of presheaves of sets (abelian groups, rings) on X is then the same as the category of contravariant functors from C to $\mathbf{Set}$ (or $\mathbf{Ab}$ or $\mathbf{Ring}$). Because of this example, the category $\text{Funct}(C^{\text{op}}, \mathbf{Set})$ is sometimes called the "category of presheaves of sets on C " even for general categories C not arising from a topological space. To define sheaves on a general category C , one needs more structure: a Grothendieck topology on C . (Some authors refer to categories that are equivalent to $\mathbf{Set}^C$ as *presheaf categories*.^[1])

Facts

Most constructions that can be carried out in D can also be carried out in D^C by performing them "componentwise", separately for each object in C . For instance, if any two objects X and Y in D have a product $X \times Y$, then any two functors F and G in D^C have a product $F \times G$, defined by $(F \times G)(c) = F(c) \times G(c)$ for every object c in C . Similarly, if $\eta_c : F(c) \rightarrow G(c)$ is a natural transformation and each η_c has a kernel K_c in the category D , then the kernel of η in the functor category D^C is the functor K with $K(c) = K_c$ for every object c in C .

As a consequence we have the general rule of thumb that the functor category D^C shares most of the "nice" properties of D :

- if D is complete (or cocomplete), then so is D^C ;
- if D is an abelian category, then so is D^C ;

We also have:

- if C is any small category, then the category $\mathbf{Set}^C$ of presheaves is a topos.

So from the above examples, we can conclude right away that the categories of directed graphs, G -sets and presheaves on a topological space are all complete and cocomplete topoi, and that the categories of representations of G , modules over the ring R , and presheaves of abelian groups on a topological space X are all abelian, complete and cocomplete.

The embedding of the category C in a functor category that was mentioned earlier uses the Yoneda lemma as its main tool. For every object X of C , let $\text{Hom}(-, X)$ be the contravariant representable functor from C to $\mathbf{Set}$. The Yoneda lemma states that the assignment

$$X \mapsto \text{Hom}(-, X)$$

is a full embedding of the category C into the category $\text{Funct}(C^{\text{op}}, \mathbf{Set})$. So C naturally sits inside a topos.

The same can be carried out for any preadditive category C : Yoneda then yields a full embedding of C into the functor category $\text{Add}(C^{\text{op}}, \mathbf{Ab})$. So C naturally sits inside an abelian category.

The intuition mentioned above (that constructions that can be carried out in D can be "lifted" to D^C) can be made precise in several ways; the most succinct formulation uses the

language of adjoint functors. Every functor $F : D \rightarrow E$ induces a functor $F^C : D^C \rightarrow E^C$ (by composition with F). If F and G is a pair of adjoint functors, then F^C and G^C is also a pair of adjoint functors.

The functor category D^C has all the formal properties of an exponential object; in particular the functors from $E \times C \rightarrow D$ stand in a natural one-to-one correspondence with the functors from E to D^C . The category **Cat** of all small categories with functors as morphisms is therefore a cartesian closed category.

See also

- Diagram (category theory)

References

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Natural equivalence

1. REDIRECT Natural transformation

Natural transformation

In category theory, a branch of mathematics, a **natural transformation** provides a way of transforming one functor into another while respecting the internal structure (i.e. the composition of morphisms) of the categories involved. Hence, a natural transformation can be considered to be a "morphism of functors". Indeed this intuition can be formalized to define so-called functor categories. Natural transformations are, after categories and functors, one of the most basic notions of category theory and consequently appear in the majority of its applications.

Definition

If F and G are functors between the categories C and D , then a **natural transformation** η from F to G associates to every object X in C a morphism $\eta_X : F(X) \rightarrow G(X)$ in D called the **component** of η at X , such that for every morphism $f : X \rightarrow Y$ in C we have:

$$\eta_Y \circ F(f) = G(f) \circ \eta_X$$

This equation can conveniently be expressed by the commutative diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \eta_X \downarrow & & \downarrow \eta_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

If both F and G are contravariant, the horizontal arrows in this diagram are reversed. If η is a natural transformation from F to G , we also write $\eta : F \rightarrow G$ or $\eta : F \sqsupset G$. This is also

expressed by saying the family of morphisms $\eta_X : F(X) \rightarrow G(X)$ is **natural** in X .

If, for every object X in C , the morphism η_X is an isomorphism in D , then η is said to be a **natural isomorphism** (or sometimes **natural equivalence** or **isomorphism of functors**). Two functors F and G are called *naturally isomorphic* or simply *isomorphic* if there exists a natural isomorphism from F to G .

An **infranatural transformation** η from F to G is simply a family of morphisms $\eta_X : F(X) \rightarrow G(X)$. Thus a natural transformation is an infranatural transformation for which $\eta_Y \circ F(f) = G(f) \circ \eta_X$ for every morphism $f : X \rightarrow Y$. The **naturalizer** of η , $\text{nat}(\eta)$, is the largest subcategory of C containing all the objects of C on which η restricts to a natural transformation.

Examples

A worked example

Statements such as

"Every group is naturally isomorphic to its opposite group"

abound in modern mathematics. We will now give the precise meaning of this statement as well as its proof. Consider the category **Grp** of all groups with group homomorphisms as morphisms. If $(G, *)$ is a group, we define its opposite group $(G^{\text{op}}, *^{\text{op}})$ as follows: G^{op} is the same set as G , and the operation $*^{\text{op}}$ is defined by $a *^{\text{op}} b = b * a$. All multiplications in G^{op} are thus "turned around". Forming the opposite group becomes a (covariant!) functor from **Grp** to **Grp** if we define $f^{\text{op}} = f$ for any group homomorphism $f : G \rightarrow H$. Note that f^{op} is indeed a group homomorphism from G^{op} to H^{op} :

$$f^{\text{op}}(a *^{\text{op}} b) = f(b * a) = f(b) * f(a) = f^{\text{op}}(a) *^{\text{op}} f^{\text{op}}(b).$$

The content of the above statement is:

"The identity functor $\text{Id}_{\mathbf{Grp}} : \mathbf{Grp} \rightarrow \mathbf{Grp}$ is naturally isomorphic to the opposite functor $-^{\text{op}} : \mathbf{Grp} \rightarrow \mathbf{Grp}$."

To prove this, we need to provide isomorphisms $\eta_G : G \rightarrow G^{\text{op}}$ for every group G , such that the above diagram commutes. Set $\eta_G(a) = a^{-1}$. The formulas $(ab)^{-1} = b^{-1} a^{-1}$ and $(a^{-1})^{-1} = a$ show that η_G is a group homomorphism which is its own inverse. To prove the naturality, we start with a group homomorphism $f : G \rightarrow H$ and show $\eta_H \circ f = f^{\text{op}} \circ \eta_G$, i.e. $(f(a))^{-1} = f^{\text{op}}(a^{-1})$ for all a in G . This is true since $f^{\text{op}} = f$ and every group homomorphism has the property $(f(a))^{-1} = f(a^{-1})$.

Further examples

If K is a field, then for every vector space V over K we have a "natural" injective linear map $V \rightarrow V^{**}$ from the vector space into its double dual. These maps are "natural" in the following sense: the double dual operation is a functor, and the maps are the components of a natural transformation from the identity functor to the double dual functor.

Every finite dimensional vector space is also isomorphic to its dual space. But this isomorphism relies on an arbitrary choice of basis, and is not natural, though there is an infranatural transformation. More generally, any vector spaces with the same dimensionality are isomorphic, but not naturally so. (Note however that if the space has a nondegenerate bilinear form, then there is a natural isomorphism between the space and its dual. Here the space is viewed as an object in the category of vector spaces and

transposes of maps.)

Consider the category **Ab** of abelian groups and group homomorphisms. For all abelian groups X, Y and Z we have a group isomorphism

$$\text{Hom}(X \square Y, Z) \rightarrow \text{Hom}(X, \text{Hom}(Y, Z)).$$

These isomorphisms are "natural" in the sense that they define a natural transformation between the two involved functors $\mathbf{Ab} \times \mathbf{Ab}^{\text{op}} \times \mathbf{Ab}^{\text{op}} \rightarrow \mathbf{Ab}$.

Natural transformations arise frequently in conjunction with adjoint functors. Indeed, adjoint functors are defined by a certain natural isomorphism. Additionally, every pair of adjoint functors comes equipped with two natural transformations (generally not isomorphisms) called the *unit* and *counit*.

Operations with natural transformations

If $\eta : F \rightarrow G$ and $\varepsilon : G \rightarrow H$ are natural transformations between functors $F, G, H : C \rightarrow D$, then we can compose them to get a natural transformation $\varepsilon\eta : F \rightarrow H$. This is done componentwise: $(\varepsilon\eta)_X = \varepsilon_X \eta_X$. This "vertical composition" of natural transformation is associative and has an identity, and allows one to consider the collection of all functors $C \rightarrow D$ itself as a category (see below under Functor categories).

Natural transformations also have a "horizontal composition". If $\eta : F \rightarrow G$ is a natural transformation between functors $F, G : C \rightarrow D$ and $\varepsilon : J \rightarrow K$ is a natural transformation between functors $J, K : D \rightarrow E$, then the composition of functors allows a composition of natural transformations $\eta\varepsilon : JF \rightarrow KG$. This operation is also associative with identity, and the identity coincides with that for vertical composition. The two operations are related by an identity which exchanges vertical composition with horizontal composition.

If $\eta : F \rightarrow G$ is a natural transformation between functors $F, G : C \rightarrow D$, and $H : D \rightarrow E$ is another functor, then we can form the natural transformation $H\eta : HF \rightarrow HG$ by defining

$$(H\eta)_X = H_{\eta_X}.$$

If on the other hand $K : B \rightarrow C$ is a functor, the natural transformation $\eta K : FK \rightarrow GK$ is defined by

$$(\eta K)_X = \eta_{K(X)}.$$

Functor categories

If C is any category and I is a small category, we can form the functor category C^I having as objects all functors from I to C and as morphisms the natural transformations between those functors. This forms a category since for any functor F there is an identity natural transformation $1_F : F \rightarrow F$ (which assigns to every object X the identity morphism on $F(X)$) and the composition of two natural transformations (the "vertical composition" above) is again a natural transformation.

The isomorphisms in C^I are precisely the natural isomorphisms. That is, a natural transformation $\eta : F \rightarrow G$ is a natural isomorphism if and only if there exists a natural transformation $\varepsilon : G \rightarrow F$ such that $\eta\varepsilon = 1_G$ and $\varepsilon\eta = 1_F$.

The functor category C^I is especially useful if I arises from a directed graph. For instance, if I is the category of the directed graph $\bullet \rightarrow \bullet$, then C^I has as objects the morphisms of C , and a morphism between $\varphi : U \rightarrow V$ and $\psi : X \rightarrow Y$ in C^I is a pair of morphisms $f : U \rightarrow X$ and $g : V \rightarrow Y$ in C such that the "square commutes", i.e. $\psi f = g \varphi$.

More generally, one can build the 2-category **Cat** whose

- 0-cells (objects) are the small categories,
- 1-cells (arrows) between two objects C and D are the functors from C to D ,
- 2-cells between two 1-cells (functors) $F : C \rightarrow D$ and $G : C \rightarrow D$ are the natural transformations from F to G .

The horizontal and vertical compositions are the compositions between natural transformations described previously. A functor category C^I is then simply a hom-category in this category (smallness issues aside).

Yoneda lemma

If X is an object of a locally small category C , then the assignment $Y \mapsto \text{Hom}_C(X, Y)$ defines a covariant functor $F_X : C \rightarrow \mathbf{Set}$. This functor is called *representable* (more generally, a representable functor is any functor naturally isomorphic to this functor for an appropriate choice of X). The natural transformations from a representable functor to an arbitrary functor $F : C \rightarrow \mathbf{Set}$ are completely known and easy to describe; this is the content of the Yoneda lemma.

Historical notes

Saunders Mac Lane, one of the founders of category theory, is said to have remarked, "I didn't invent categories to study functors; I invented them to study natural transformations." Just as the study of groups is not complete without a study of homomorphisms, so the study of categories is not complete without the study of functors. The reason for Mac Lane's comment is that the study of functors is itself not complete without the study of natural transformations.

The context of Mac Lane's remark was the axiomatic theory of homology. Different ways of constructing homology could be shown to coincide: for example in the case of a simplicial complex the groups defined directly, and those of the singular theory, would be isomorphic. What cannot easily be expressed without the language of natural transformations is how homology groups are compatible with morphisms between objects, and how two equivalent homology theories not only have the same homology groups, but also the same morphisms between those groups.

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Dual (category theory)

In category theory, a branch of mathematics, **duality** is a correspondence between properties of a category C and so-called **dual properties** of the opposite category C^{op} . Given a statement regarding the category C , by interchanging the source and target of each morphism as well as interchanging the order of composing two morphisms, a corresponding dual statement is obtained regarding the opposite category C^{op} . **Duality**, as such, is the assertion that truth is invariant under this operation on statements. In other words, if a statement is true about C , then its dual statement is true about C^{op} . Also, if a statement is false about C , then its dual has to be false about C^{op} .

Given a concrete category C that mathematicians are interested, it is often the case that the opposite category C^{op} per se is abstract. C^{op} need not be a category that arises from mathematical practice. In this case, another category D is also termed to be in **duality** with C if D and C^{op} are equivalent as categories.

In the case when C and its opposite C^{op} are equivalent, such a category is **self-dual**.

Formal definition

We define the elementary language of category theory as the two-sorted first order language with objects and morphisms as distinct sorts, together with the relations of an object being the source or target of a morphism and a symbol for composing two morphisms.

Let σ be any statement in this language. We form the dual σ^{op} as follows:

1. Interchange each occurrence of "source" in σ with "target".
2. Interchange the order of composing morphisms. That is, replace each occurrence of $g \circ f$ with $f \circ g$

Informally, these conditions state that the dual of a statement is formed by reversing arrows and compositions.

Duality is the observation that σ is true for some category C if and only if σ^{op} is true for C^{op} .

Examples

- A morphism $f: A \rightarrow B$ is a monomorphism if $f \circ g = f \circ h$ implies $g = h$. Performing the dual operation, we get the statement that $g \circ f = h \circ f$ implies $g = h$ for a morphism $f: B \rightarrow A$. This is precisely what it means for f to be an epimorphism. In short, the property of being a monomorphism is dual to the property of being an epimorphism.

Applying duality, this means that a morphism in some category C is a monomorphism if and only if the reverse morphism in the opposite category C^{op} is an epimorphism.

- An example comes from reversing the direction of inequalities in a partial order. So if X is a set and $\leq$ a partial order relation, we can define a new partial order relation $\leq_{\text{new}}$ by

$$x \leq_{\text{new}} y \text{ if and only if } y \leq x.$$

This example on orders is a special case, since partial orders correspond to a certain kind of category in which $\text{Hom}(A, B)$ can have at most one element. In applications to logic, this then looks like a very general description of negation (that is, proofs run in the opposite direction). For example, if we take the opposite of a lattice, we will find that *meets* and *joins*

have their roles interchanged. This is an abstract form of De Morgan's laws, or of duality applied to lattices.

- Limits and colimits are dual notions.
- Fibrations and cofibrations are examples of dual notions in algebraic topology and homotopy theory. In this context, the duality is often called Eckmann-Hilton duality.

See also

- Dual object
- Duality (mathematics)
- Opposite category

References

Isomorphism

In abstract algebra, an **isomorphism** (Greek: ἴσος *isos* "equal", and μορφή *morphe* "shape") is a bijective map f such that both f and its inverse f^{-1} are homomorphisms, i.e., *structure-preserving* mappings. In the more general setting of category theory, an **isomorphism** is a morphism $f:X\rightarrow Y$ in a category for which there exists an "inverse" $f^{-1}:Y\rightarrow X$, with the property that both $f^{-1}f=\text{id}_X$ and $ff^{-1}=\text{id}_Y$.

Informally, an isomorphism is a kind of mapping between objects, which shows a relationship between two properties or operations. If there exists an isomorphism between two structures, we call the two structures **isomorphic**. In a certain sense, isomorphic structures are **structurally identical**, if you choose to ignore finer-grained differences that may arise from how they are defined.

Purpose

Isomorphisms are studied in mathematics in order to extend insights from one phenomenon to others: if two objects are isomorphic, then any property which is preserved by an isomorphism and which is true of one of the objects is also true of the other. If an isomorphism can be found from a relatively unknown part of mathematics into some well studied division of mathematics, where many theorems are already proved, and many methods are already available to find answers, then the function can be used to map whole problems out of unfamiliar territory over to "solid ground" where the problem is easier to understand and work with.

Practical example

The following are examples of isomorphisms from ordinary algebra.

- Consider the logarithm function: For any fixed base b , the logarithm function $\log_b$ maps from the positive real numbers $\mathbb{R}^+$ onto the real numbers $\mathbb{R}$; formally:

$$\log_b : \mathbb{R}^+ \rightarrow \mathbb{R}.$$

This mapping is one-to-one and onto, that is, it is a bijection from the domain to the codomain of the logarithm function.

In addition to being an isomorphism of sets, the logarithm function also preserves certain operations. Specifically, consider the group $(\mathbb{R}^+, \times)$ of positive real numbers under ordinary multiplication. The logarithm function obeys the following identity:

$$\log_b(xy) = \log_b(x) + \log_b(y).$$

But the real numbers under addition also form a group. So the logarithm function is in fact a group isomorphism from the group $(\mathbb{R}^+, \times)$ to the group $(\mathbb{R}, +)$.

Logarithms can therefore be used to simplify multiplication of real numbers. By working with logarithms, multiplication of positive real numbers is replaced by addition of logs. This way it is possible to multiply real numbers using a ruler and a table of logarithms, or using a slide rule with a logarithmic scale.

- Consider the group $\mathbb{Z}_6$, the numbers from 0 to 5 with addition modulo 6. Also consider the group $\mathbb{Z}_2 \times \mathbb{Z}_3$, the ordered pairs where the x coordinates can be 0 or 1, and the y coordinates can be 0, 1, or 2, where addition in the x -coordinate is modulo 2 and addition in the y -coordinate is modulo 3.

These structures are isomorphic under addition, if you identify them using the following scheme:

$(0,0) \rightarrow 0$
 $(1,1) \rightarrow 1$
 $(0,2) \rightarrow 2$
 $(1,0) \rightarrow 3$
 $(0,1) \rightarrow 4$
 $(1,2) \rightarrow 5$

or in general $(a,b) \rightarrow (3a + 4b) \bmod 6$.

For example note that $(1,1) + (1,0) = (0,1)$ which translates in the other system as $1 + 3 = 4$.

Even though these two groups "look" different in that the sets contain different elements, they are indeed **isomorphic**: their structures are exactly the same. More generally, the direct product of two cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$ is cyclic if and only if n and m are coprime.

Abstract examples

A relation-preserving isomorphism

If one object consists of a set X with a binary relation R and the other object consists of a set Y with a binary relation S then an isomorphism from X to Y is a bijective function $f: X \rightarrow Y$ such that

$$f(u) S f(v) \text{ if and only if } u R v.$$

S is reflexive, irreflexive, symmetric, antisymmetric, asymmetric, transitive, total, trichotomous, a partial order, total order, strict weak order, total preorder (weak order), an equivalence relation, or a relation with any other special properties, if and only if R is.

For example, R is an ordering $\leq$ and S an ordering $\sqsubseteq$, then an isomorphism from X to Y is a bijective function $f: X \rightarrow Y$ such that

$$f(u) \sqsubseteq f(v) \text{ if and only if } u \leq v.$$

Such an isomorphism is called an *order isomorphism* or (less commonly) an *isotone isomorphism*.

If $X = Y$ we have a relation-preserving automorphism.

An operation-preserving isomorphism

Suppose that on these sets X and Y , there are two binary operations $\star$ and $\diamond$ which happen to constitute the groups $(X, \star)$ and $(Y, \diamond)$. Note that the operators operate on elements from the domain and range, respectively, of the "one-to-one" and "onto" function f . There is an isomorphism from X to Y if the bijective function $f: X \rightarrow Y$ happens to produce results, that sets up a correspondence between the operator $\star$ and the operator $\diamond$.

$$f(u) \diamond f(v) = f(u \star v)$$

for all u, v in X .

Applications

In abstract algebra, two basic isomorphisms are defined:

- Group isomorphism, an isomorphism between groups
- Ring isomorphism, an isomorphism between rings. (Note that isomorphisms between fields are actually ring isomorphisms)

Just as the automorphisms of an algebraic structure form a group, the isomorphisms between two algebras sharing a common structure form a heap. Letting a particular isomorphism identify the two structures turns this heap into a group.

In mathematical analysis, the Laplace transform is an isomorphism mapping hard differential equations into easier algebraic equations.

In category theory, let the category C consist of two classes, one of *objects* and the other of *morphisms*. Then a general definition of isomorphism that covers the previous and many other cases is: an isomorphism is a morphism $f: a \rightarrow b$ that has an inverse, i.e. there exists a morphism $g: b \rightarrow a$ with $fg = 1_b$ and $gf = 1_a$. For example, a bijective linear map is an isomorphism between vector spaces, and a bijective continuous function whose inverse is also continuous is an isomorphism between topological spaces, called a homeomorphism.

In graph theory, an isomorphism between two graphs G and H is a bijective map f from the vertices of G to the vertices of H that preserves the "edge structure" in the sense that there

is an edge from vertex u to vertex v in G if and only if there is an edge from $f(u)$ to $f(v)$ in H .
See graph isomorphism.

In early theories of logical atomism, the formal relationship between facts and true propositions was theorized by Bertrand Russell and Ludwig Wittgenstein to be isomorphic. An example of this line of thinking can be found in Russell's Introduction to the Philosophy of Mathematics.

In cybernetics, the Good Regulator or Conant-Ashby theorem is stated "Every Good Regulator of a system must be a model of that system". Whether regulated or self-regulating an isomorphism is required between regulator part and the processing part of the system.

See also

- Epimorphism
- Heap (mathematics)
- Isomorphism class
- Monomorphism
- Isometry

External links

- Isomorphism ^[1] on PlanetMath
- Eric W. Weisstein, *Isomorphism* ^[2] at MathWorld.

References

- [1] <http://planetmath.org/?op=getobj&from=objects&id=1936>
[2] <http://mathworld.wolfram.com/Isomorphism.html>
-

Semigroup

In mathematics, a **semigroup** is an algebraic structure consisting of a nonempty set S together with an associative binary operation. In other words, a semigroup is an associative magma. The terminology is derived from the anterior notion of a group. A semigroup differs from a group in that for each of its elements there may not exist an inverse; further, there may not exist an identity element.

The binary operation of a semigroup is most often denoted multiplicatively: $x \cdot y$, or simply xy , denotes the result of applying the semigroup operation to the ordered pair (x, y) .

The formal study of semigroups began in the early 20th century. Semigroups are important in many areas of mathematics because they are the abstract algebraic underpinning of "memoryless" systems: time-dependent systems that start from scratch at each iteration. In applied mathematics, semigroups are fundamental models for linear time-invariant systems. In partial differential equations, a semigroup is associated to any equation whose spatial evolution is independent of time. The theory of finite semigroups has been of particular importance in theoretical computer science since the 1950s because of the natural link between finite semigroups and finite automata. In probability theory, semigroups are associated with Markov processes.

Definition

A semigroup is a set, S , together with an operation " $\bullet$ " that combines any two elements a and b to form another element denoted $a \bullet b$. The symbol " $\bullet$ " is a general placeholder for a concretely given operation. To qualify as a semigroup, the set and operation, $(S, \bullet)$, must satisfy two requirements known as the *semigroup axioms*:

Closure

For all a, b in S , the result of the operation $a \bullet b$ is also in S .

Associativity

For all a, b and c in S , the equation $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ holds.

More compactly, a semigroup is an associative magma.

Semigroup homomorphisms

A homomorphism between two semigroups $(S, *)$ and $(S', \bullet)$ is a function $f : S \rightarrow S'$ such that

$$\forall x, y \in S : f(x * y) = f(x) \bullet f(y).$$

Any semigroup may be embedded into a monoid (generally denoted as M) simply by adjoining an element $e \notin S$ to and defining $e \cdot s = s \cdot e = s$ for all $s \in S \cup \{e\}$.

Thus, every commutative semigroup can be embedded in a group via the Grothendieck group construction.

Examples of semigroups

- Semigroup with one element
- Semigroup with two elements
- A monoid is a semigroup with an identity element
- A group is a semigroup with an identity element and an inverse element.
- The set of positive integers with addition.
- Square nonnegative matrices with matrix multiplication.
- Any ideal of a ring with the multiplication of the ring.
- The set of all finite strings over a fixed alphabet Σ with concatenation of strings as the semigroup operation --- the so-called "free semigroup over Σ ". With the empty string included, this semigroup becomes the free monoid over Σ .
- A probability distribution F together with all convolution powers of F , with convolution as operation. This is called a convolution semigroup.

Special classes of semigroups

- A monoid is a semigroup with identity.
- A subsemigroup is a subset of a semigroup that is closed under the semigroup operation.
- A band is a semigroup the operation of which is idempotent.
- Semilattices: A semigroup whose operation is idempotent and commutative is a semilattice.
- 0-simple semigroups.
- Transformation semigroups: any finite semigroup S can be represented by transformations of a (state-) set Q of at most $|S|+1$ states. Each element x of S then maps Q into itself $x: Q \rightarrow Q$ and sequence xy is defined by $q(xy) = (qx)y$ for each q in Q . Sequencing clearly is an associative operation, here equivalent to function composition. This representation is basic for any automaton or finite state machine (FSM).
- Bicyclic semigroups.
- C_0 -semigroups.
- Regular semigroups.
- Inverse semigroups.
- Affine semigroup: a semigroup that is isomorphic to a finitely-generated subsemigroup of $\mathbb{Z}^d$. These semigroups have applications to commutative algebra.

Structure of semigroups

This section sets out concepts useful for understanding the structure of semigroups. Two semigroups S and T are said to be isomorphic if there is a bijection $f: S \rightarrow T$ with the property that, for any elements a, b in S , $f(ab) = f(a)f(b)$. Isomorphic semigroups have the same structure.

The semigroup operation induces an operation on the collection of its subsets: given subsets A and B of a semigroup, $A*B$, written commonly as AB , is the set $\{ ab \mid a \text{ in } A \text{ and } b \text{ in } B \}$. In terms of this operations, a subset A is (i) a **subsemigroup** if AA is a subset of A , (ii) a **right ideal** if AS is a subset of A , and (iii) a **left ideal** if SA is a subset of A . If A is both a left ideal and a right ideal then it is called an **ideal** (or a **two-sided ideal**). An example of semigroup with no minimal ideal is the set of positive integers under addition. The minimal ideal of a commutative semigroup, when it exists, is a group.

Green's relations are important tools for analysing the ideals of a semigroup and related notions of structure.

If S is a semigroup, then the intersection of any collection of subsemigroups of S is also a subsemigroup of S . So the subsemigroups of S form a complete lattice. For any subset A of S there is a smallest subsemigroup T of S which contains A , and we say that A **generates** T . A single element x of S generates the subsemigroup $\{x^n \mid n \text{ is a positive integer}\}$. If this is finite, then x is said to be of **finite order**, otherwise it is of **infinite order**. A semigroup is said to be **periodic** if all of its elements are of finite order. A semigroup generated by a single element is said to be monogenic (or cyclic). If a monogenic semigroup is infinite then it is isomorphic to the semigroup of positive integers with the operation of addition. If it is finite and nonempty, then it must contain at least one idempotent. It follows that every nonempty periodic semigroup has at least one idempotent.

A subsemigroup which is also a group is called a **subgroup**. There is a close relationship between the subgroups of a semigroup and its idempotents. Each subgroup contains exactly one idempotent, namely the identity element of the subgroup. For each idempotent e of the semigroup there is a unique maximal subgroup containing e . Each maximal subgroup arises in this way, so there is a one-to-one correspondence between idempotents and maximal subgroups. Here the term *maximal subgroup* differs from its standard use in group theory.

More can often be said when the order is finite. For example, every nonempty finite semigroup is periodic, and has a minimal ideal and at least one idempotent. For more on the structure of finite semigroups, see Krohn-Rhodes theory.

Semigroup methods in partial differential equations

Semigroup theory can be used to study some problems in the field of partial differential equations. Roughly speaking, the semigroup approach is to regard a time-dependent partial differential equation as an ordinary differential equation on a function space. For example, consider the following initial/boundary value problem for the heat equation on the spatial interval $(0, 1) \square \mathbf{R}$ and times $t \geq 0$:

$$\begin{cases} \partial_t u(t, x) = \partial_x^2 u(t, x), & x \in (0, 1), t > 0; \\ u(t, x) = 0, & x \in \{0, 1\}, t > 0; \\ u(t, x) = u_0(x), & x \in (0, 1), t = 0. \end{cases}$$

Let X be the L^p space $L^2((0, 1); \mathbf{R})$ and let A be the second-derivative operator with domain

$$D(A) = \{u \in H^2((0, 1); \mathbf{R}) \mid u(0) = u(1) = 0\}.$$

Then the above initial/boundary value problem can be interpreted as an initial value problem for an ordinary differential equation on the space X :

$$\begin{cases} \dot{u}(t) = Au(t); \\ u(0) = u_0. \end{cases}$$

On an heuristic level, the solution to this problem "ought" to be $u(t) = \exp(tA)u_0$. However, for a rigorous treatment, a meaning must be given to the exponential of tA . As a function of t , $\exp(tA)$ is a semigroup of operators from X to itself, taking the initial state u_0 at time $t = 0$ to the state $u(t) = \exp(tA)u_0$ at time t . The operator A is said to be the infinitesimal generator of the semigroup.

History

The formal study of semigroups came somewhat later than that of other algebraic structures such as groups or rings in the mid 19th century. A number of sources^[1] ^[2] attribute the first use of the term (in French) to J.-A. de Séguier in *Éléments de la Théorie des Groupes Abstraits* (Elements of the Theory of Abstract Groups) in 1904. The term is used in English in 1908 in Harold Hinton's *Theory of Groups of Finite Order*. In 1970, a new periodical called *Semigroup Forum* (currently edited by Springer Verlag) became one of the rare mathematical journals devoted entirely to semigroup theory.

Anton Suschkewitsch is often credited with obtaining the first non-trivial results about semigroups. His 1928 paper *Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit* (On finite groups without the rule of unique invertibility) determined the structure of finite simple semigroups and showed that the minimal ideal (or Green's relations J-class) of a finite semigroup is simple^[2]. From that point on, the foundations of semigroup theory were further laid by David Rees, James Alexander Green, Evgenii Sergeevich Lyapin, Alfred H. Clifford and Gordon Preston. The latter two published a monograph on semigroup theory in 1961.

The theory of finite semigroups is arguably more developed than its infinite counterpart. This stems particularly from the notion of syntactic semigroup and the ensuing links between pseudo-varieties of semigroups and so-called varieties of formal languages which have proved particularly fruitful in finite automata theory^[3].

See also

- Absorbing element
- Biordered set
- Empty semigroup
- Grothendieck group
- Identity element
- Light's associativity test
- Weak inverse

Notes

[1] <http://jeff560.tripod.com/s.html> Earliest Known Uses of Some of the Words of Mathematics

[2] <http://uk.geocities.com/cdhollings/suschkewitsch3.pdf> An account of Suschkewitsch's paper by Christopher Hollings

[3] *Varieties of Formal Languages*, J.É. Pin, Plenum Press, 1986.

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- *Semigroup Forum* is the best-known periodical devoted specifically to the subject of semigroups.

Monoid

In abstract algebra, a branch of mathematics, a **monoid** is an algebraic structure with a single associative binary operation and an identity element. Monoids occur in a number of branches of mathematics and capture the idea of function composition; indeed, this notion is abstracted in category theory, where the monoid is a category with one object. Monoids are also commonly used to provide an algebraic foundation for computer science; in this case, the transition monoid and syntactic monoid are used in describing a finite state machine, whereas trace monoids and history monoids provide a foundation for process calculi and concurrent computing. Some of the more important results in the study of monoids are the Krohn-Rhodes theorem and the star height problem. The history of monoids, as well as a discussion of additional general properties, are found in the article on semigroups.

Definition

A monoid is a set, M , together with an operation " $\bullet$ " that combines any two elements a and b to form another element denoted $a \bullet b$. The symbol " $\bullet$ " is a general placeholder for a concretely given operation. To qualify as a monoid, the set and operation, $(M, \bullet)$, must satisfy three requirements known as the **monoid axioms**:

Closure

For all a, b in M , the result of the operation $a \bullet b$ is also in M .

Associativity

For all a, b and c in M , the equation $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ holds.

Identity element

There exists an element e in M , such that for all elements a in M , the equation $e \bullet a = a \bullet e = a$ holds.

More compactly, a monoid is a semigroup with an identity element.

Monoid Structures

Generators and submonoids

A **submonoid** of a monoid M is a subset N of M containing the unit element, and such that, if $x, y \in N$ then $x * y \in N$. It is then clear that N is itself a monoid, under the binary operation induced by that of M . Equivalently, a submonoid is a subset N such that $N = N^*$, where the superscript $*$ is the Kleene star. For any subset N of M , the monoid N^* is the smallest monoid that contains N .

A subset N is said to be a **generator** of M if and only if $M = N^*$. If there is a finite generator of M , then M is said to be **finitely generated**.

Commutative monoid

A monoid whose operation is commutative is called a **commutative monoid** (or, less commonly, an **abelian monoid**). Commutative monoids are often written additively. Any commutative monoid is endowed with its **algebraic** preordering $\leq$, defined by $x \leq y$ if and only if there exists z such that $x + z = y$. An **order-unit** of a commutative monoid M is an element u of M such that for any element x of M , there exists a positive integer n such that $x \leq nu$. This is often used in case M is the positive cone of a partially ordered abelian group G , in which case we say that u is an order-unit of G .

Partially commutative monoid

A monoid for which the operation is commutative for some, but not all elements is a trace monoid; trace monoids commonly occur in the theory of concurrent computation.

Examples

- Every singleton set $\{x\}$ gives rise to a one-element (trivial) monoid. For fixed x this monoid is unique, since the monoid axioms require that $x * x = x$ in this case.
- Every group is a monoid and every abelian group a commutative monoid.
- Every bounded semilattice is an idempotent commutative monoid.
- Any semigroup S may be turned into a monoid simply by adjoining an element e not in S and defining $e * e = e$ and $e * s = s = s * e$ for all $s \in S$.
- The natural numbers, $\mathbf{N}$, form a commutative monoid under addition (identity element zero), or multiplication (identity element one). A submonoid of $\mathbf{N}$ under addition is called a numerical monoid.
- The elements of any unital ring, with addition or multiplication as the operation.
 - The integers, rational numbers, real numbers or complex numbers, with addition or multiplication as operation.
 - The set of all n by n matrices over a given ring, with matrix addition or matrix multiplication as the operation.
- The set of all finite strings over some fixed alphabet Σ forms a monoid with string concatenation as the operation. The empty string serves as the identity element. This monoid is denoted Σ^* and is called the **free monoid** over Σ .
- Fix a monoid M , and consider its power set $P(M)$ consisting of all subsets of M . A binary operation for such subsets can be defined by $S * T = \{s * t : s \text{ in } S \text{ and } t \text{ in } T\}$. This turns $P(M)$ into a monoid with identity element $\{e\}$. In the same way the power set of a group

G is a monoid under the product of group subsets.

- Let S be a set. The set of all functions $S \rightarrow S$ forms a monoid under function composition. The identity is just the identity function. If S is finite with n elements, the monoid of functions on S is finite with n^n elements.
- Generalizing the previous example, let C be a category and X an object in C . The set of all endomorphisms of X , denoted $\text{End}_C(X)$, forms a monoid under composition of morphisms. For more on the relationship between category theory and monoids see below.
- The set of homeomorphism classes of compact surfaces with the connected sum. Its unit element is the class of the ordinary 2-sphere. Furthermore, if a denotes the class of the torus, and b denotes the class of the projective plane, then every element c of the monoid has a unique expression the form $c=na+mb$ where n is the integer ≥ 0 and $m=0,1$, or 2 . We have $3b=a+b$.
- Let $\langle f \rangle$ be a cyclic monoid of order n , that is, $\langle f \rangle = \{f^0, f^1, \dots, f^{n-1}\}$. Then $f^n = f^k$ for some $0 \leq k \leq n$. In fact, each such k gives a distinct monoid of order n , and every cyclic monoid is isomorphic to one of these.

Moreover, f can be considered as a function on the points $0, 1, 2, \dots, n-1$ given by

$$\begin{bmatrix} 0 & 1 & 2 & \dots & n-2 & n-1 \\ 1 & 2 & 3 & \dots & n-1 & k \end{bmatrix}$$

or, equivalently

$$f(i) := \begin{cases} i+1, & \text{if } 0 \leq i < n-1 \\ k, & \text{if } i = n-1. \end{cases}$$

Multiplication of elements in $\langle f \rangle$ is then given by function composition.

Note also that when $k=0$ then the function f is a permutation of $\{0, 1, 2, \dots, n-1\}$ and gives the unique cyclic group of order n .

Properties

In a monoid, one can define positive integer powers of an element x : $x^1=x$, and $x^n=x*...*x$ (n times) for $n>1$. The rule of powers $x^{n+p}=x^n*x^p$ is obvious.

Directly from the definition, one can show that the identity element e is unique. Then, for any x , one can set $x^0=e$ and the rule of powers is still true with nonnegative exponents.

It is possible to define invertible elements: an element x is called invertible if there exists an element y such that $x*y = e$ and $y*x = e$. The element y is called the inverse of x . If y and z are inverses of x , then by associativity $y = (zx)y = z(xy) = z$. Thus inverses, if they exist, are unique.^[1]

If y is the inverse of x , one can define negative powers of x by setting $x^{-1}=y$ and $x^{-n}=y*...*y$ (n times) for $n>1$. And the rule of exponents is still verified for all n, p rational integers. This is why the inverse of x is usually written x^{-1} . The set of all invertible elements in a monoid M , together with the operation $*$, forms a group. In that sense, every monoid contains a group (if only the trivial one consisting of the identity alone).

However, not every monoid sits inside a group. For instance, it is perfectly possible to have a monoid in which two elements a and b exist such that $a*b = a$ holds even though b is not the identity element. Such a monoid cannot be embedded in a group, because in the group we could multiply both sides with the inverse of a and would get that $b = e$, which isn't true. A monoid $(M, *)$ has the cancellation property (or is cancellative) if for all a, b and c in

M , $a*b = a*c$ always implies $b = c$ and $b*a = c*a$ always implies $b = c$. A commutative monoid with the cancellation property can always be embedded in a group via the Grothendieck construction. That's how the additive group of the integers (a group with operation $+$) is constructed from the additive monoid of natural numbers (a commutative monoid with operation $+$ and cancellation property). However, a non-commutative cancellative monoid need not be embeddable in a group.

If a monoid has the cancellation property and is *finite*, then it is in fact a group. Proof: Fix an element x in the monoid. Since the monoid is finite, $x^n = x^m$ for some $m > n > 0$. But then, by cancellation we have that $x^{m-n} = e$ where e is the identity. Therefore $x * x^{m-n-1} = e$, so x has an inverse.

The right- and left-cancellative elements of a monoid each in turn form a submonoid (i.e. obviously include the identity and not so obviously are closed under the operation). This means that the cancellative elements of any commutative monoid can be extended to a group.

An **inverse monoid**, is a monoid where for every a in M , there exists a unique a^{-1} in M such that $a = a*a^{-1}*a$ and $a^{-1} = a^{-1}*a*a^{-1}$. If an inverse monoid is cancellative, then it is a group.

Acts and operator monoids

Let M be a monoid. Then a (left) **M -act** (or left act over M) is a set X together with an operation $\bullet : M \times X \rightarrow X$ which is compatible with the monoid structure as follows:

- for all x in X : $e \bullet x = x$;
- for all a, b in M and x in X : $a \bullet (b \bullet x) = (a * b) \bullet x$.

This is the analogue in monoid theory of a (left) group action. Right M -acts are defined in a similar way. A monoid with an act is also known as an **operator monoid**. Important examples include transition systems of semiautomata. A transformation semigroup can be made into an operator monoid by adjoining the identity transformation.

Monoid homomorphisms

A homomorphism between two monoids $(M, *)$ and $(M', \bullet)$ is a function $f : M \rightarrow M'$ such that

- $f(x*y) = f(x) \bullet f(y)$ for all x, y in M
- $f(e) = e'$

where e and e' are the identities on M and M' respectively. Monoid homomorphisms are sometimes simply called **monoid morphisms**.

Not every semigroup homomorphism is a monoid homomorphism since it may not preserve the identity. Contrast this with the case of group homomorphisms: the axioms of group theory ensure that every semigroup homomorphism between groups preserves the identity. For monoids this isn't always true and it is necessary to state it as a separate requirement.

A bijective monoid homomorphism is called a monoid isomorphism. Two monoids are said to be isomorphic if there is an isomorphism between them.

Monoid congruence and the quotient monoid

A **monoid congruence** is an equivalence relation that is compatible with the monoid product. That is, it is a subset

$$\sim \subseteq M \times M$$

such that it is reflexive, symmetric and transitive (just as every equivalence relation must be), and also has the property that if $x \sim y$ and $u \sim v$ for every x, y, u and v in M , then one has that $x * u \sim y * v$.

A monoid congruence induces congruence classes

$$[m] = \{x \in M \mid x \sim m\}$$

and the monoid operation $*$ induces a binary operation $\circ$ on the congruence classes:

$$[u] \circ [v] = [u * v]$$

which is a monoid homomorphism. It is also clearly associative, and so the set of all congruence classes are a monoid as well. This monoid is called the **quotient monoid**, and may be written as

$$M / \sim = \{[m] \mid m \in M\}.$$

Several additional notations are common. Give a subset $L \subseteq M$, one writes

$$[L] = \{[m] \mid m \in L\}$$

for the set of congruence classes induced by L . In this notation, clearly $[M] = M / \sim$. In general, however, $[L]$ is not a monoid. Going in the opposite direction, if $X \subseteq [M]$ is a subset of the quotient monoid, one writes

$$\bigcup X = \{m \mid [m] \in X\}.$$

This is, of course, just the set-theoretic union of the members of X . In general, $\bigcup X$ is not a monoid.

Clearly, one has $L \subseteq \bigcup [L]$ and $[\bigcup X] = X$.

Equational presentation

Monoids may be given a **presentation**, much in the same way that groups can be specified by means of a group presentation. One does this by specifying a set of generators Σ , and a set of relations on the free monoid Σ^* . One does this by extending (finite) binary relations on Σ^* to monoid congruences, and then constructing the quotient monoid, as above.

Given a binary relation $R \subseteq \Sigma^* \times \Sigma^*$, one defines its symmetric closure as $R \cup R^{-1}$. This can be extended to a symmetric relation $E \subseteq \Sigma^* \times \Sigma^*$ by defining $x \sim_E y$ if and only if $x = sut$ and $y = svt$ for some strings $u, v, s, t \in \Sigma^*$ with $(u, v) \in R \cup R^{-1}$. Finally, one takes the reflexive and transitive closure of E , which is then a monoid congruence.

In the typical situation, the relation R is simply given as a set of equations, so that $R = \{u_1 = v_1, \dots, u_n = v_n\}$. Thus, for example,

$$\langle p, q \mid pq = 1 \rangle$$

is the equational presentation for the bicyclic monoid, and

$$\langle a, b \mid aba = baa, bba = bab \rangle$$

is the plactic monoid of degree 2 (it has infinite order). Elements of this plactic monoid may be written as $a^i b^j (ba)^k$ for integers i, j, k , as the relations show that ba commutes with both a and b .

Relation to category theory

Group-like structures				
	Totality	Associativity	Identity	Inverses
Group	Yes	Yes	Yes	Yes
Monoid	Yes	Yes	Yes	No
Semigroup	Yes	Yes	No	No
Loop	Yes	No	Yes	Yes
Quasigroup	Yes	No	No	Yes
Magma	Yes	No	No	No
Groupoid	No	Yes	Yes	Yes
Category	No	Yes	Yes	No

Monoids can be viewed as a special class of categories. Indeed, the axioms required of a monoid operation are exactly those required of morphism composition when restricted to the set of all morphisms whose source and target is a given object. That is,

A monoid is, essentially, the same thing as a category with a single object.

More precisely, given a monoid $(M,*)$, one can construct a small category with only one object and whose morphisms are the elements of M . The composition of morphisms is given by the monoid operation $*$.

Likewise, monoid homomorphisms are just functors between single object categories. In this sense, category theory can be thought of as an extension of the concept of a monoid. Many definitions and theorems about monoids can be generalised to small categories with more than one object.

Monoids, just like other algebraic structures, also form their own category, **Mon**, whose objects are monoids and whose morphisms are monoid homomorphisms.

There is also a notion of monoid object which is an abstract definition of what is a monoid in a category.

See also

- Star height problem
- Kleene algebra

References

[1] Jacobson, I.5. p. 22

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Domain theory

Domain theory is a branch of mathematics that studies special kinds of partially ordered sets (posets) commonly called **domains**. Consequently, domain theory can be considered as a branch of order theory. The field has major applications in computer science, where it is used to specify denotational semantics, especially for functional programming languages. Domain theory formalizes the intuitive ideas of approximation and convergence in a very general way and has close relations to topology. An alternative important approach to denotational semantics in computer science is that of metric spaces.

Motivation and intuition

The primary motivation for the study of domains, which was initiated by Dana Scott in the late 1960s, was the search for a denotational semantics of the lambda calculus. In this formalism, one considers "functions" specified by certain terms in the language. In a purely syntactic way, one can go from simple functions to functions that take other functions as their input arguments. Using again just the syntactic transformations available in this formalism, one can obtain so called fixed point combinators (the best-known of which is the Y combinator); these, by definition, have the property that $f(Y(f)) = Y(f)$ for all functions f .

To formulate such a denotational semantics, one might first try to construct a *model* for the lambda calculus, in which a genuine (total) function is associated with each lambda term. Such a model would formalize a link between the lambda calculus as a purely syntactic system and the lambda calculus as a notational system for manipulating concrete mathematical functions. Unfortunately, such a model cannot exist, for if it did, it would have to contain a genuine, total function that corresponds to the combinator Y , that is, a function that computes a fixed point of an *arbitrary* input function f . There can be no such function for Y , because some functions (for example, the *successor function*) do not have a fixed point. At best, the genuine function corresponding to Y would have to be a partial function, necessarily undefined at some inputs.

Scott got around this difficulty by formalizing a notion of "partial" or "incomplete" information to represent computations that have not yet returned a result. This was modeled by considering, for each domain of computation (e.g. the natural numbers), an additional element that represents an *undefined* output, i.e. the "result" of a computation that never ends. In addition, the domain of computation is equipped with an *ordering relation*, in which the "undefined result" is the least element.

The important step to find a model for the lambda calculus is to consider only those functions (on such a partially ordered set) which are guaranteed to have least fixed points. The set of these functions, together with an appropriate ordering, is again a "domain" in the sense of the theory. But the restriction to a subset of all available functions has another great benefit: it is possible to obtain domains that contain their own function spaces, i.e. one gets functions that can be applied to themselves.

Beside these desirable properties, domain theory also allows for an appealing intuitive interpretation. As mentioned above, the domains of computation are always partially ordered. This ordering represents a hierarchy of information or knowledge. The higher an

element is within the order, the more specific it is and the more information it contains. Lower elements represent incomplete knowledge or intermediate results.

Computation then is modeled by applying monotone functions repeatedly on elements of the domain in order to refine a result. Reaching a fixed point is equivalent to finishing a calculation. Domains provide a superior setting for these ideas since fixed points of monotone functions can be guaranteed to exist and, under additional restrictions, can be approximated from below.

A guide to the formal definitions

In this section, the central concepts and definitions of domain theory will be introduced. The above intuition of domains being *information orderings* will be emphasized to motivate the mathematical formalization of the theory. The precise formal definitions are to be found in the dedicated articles for each concept. A list of general order-theoretic definitions which include domain theoretic notions as well can be found in the order theory glossary. The most important concepts of domain theory will nonetheless be introduced below.

Directed sets as converging specifications

As mentioned before, domain theory deals with partially ordered sets to model a domain of computation. The goal is to interpret the elements of such an order as *pieces of information* or (*partial*) *results of a computation*, where elements that are higher in the order extend the information of the elements below them in a consistent way. From this simple intuition it is already clear that domains often do not have a greatest element, since this would mean that there is an element that contains the information of *all* other elements - a rather uninteresting situation.

A concept that plays an important role in the theory is the one of a **directed subset** of a domain, i.e. of a non-empty subset of the order in which each two elements have some upper bound that is an element of this subset. In view of our intuition about domains, this means that every two pieces of information within the directed subset are *consistently* extended by some other element in the subset. Hence we can view directed sets as *consistent specifications*, i.e. as sets of partial results in which no two elements are contradictory. This interpretation can be compared with the notion of a sequence in analysis, where each element is more specific than the preceding one. Indeed, in the theory of metric spaces, sequences play a role that is in many aspects analogous to the role of directed sets in domain theory.

Now, as in the case of sequences, we are interested in the *limit* of a directed set. According to what was said above, this would be an element that is the most general piece of information which extends the information of all elements of the directed set, i.e. the unique element that contains *exactly* the information that was present in the directed set - and nothing more. In the formalization of order theory, this is just the **least upper bound** of the directed set. As in the case of limits of sequences, least upper bounds of directed sets do not always exist.

Naturally, one has a special interest in those domains of computations in which all consistent specifications *converge*, i.e. in orders in which all directed sets have a least upper bound. This property defines the class of **directed complete partial orders**, or **dcpo** for short. Indeed, most considerations of domain theory do only consider orders that are at least directed complete.

From the underlying idea of partially specified results as representing incomplete knowledge, one derives another desirable property: the existence of a **least element**. Such an element models that state of no information - the place where most computations start. It also can be regarded as the output of a computation that does not return any result at all.

Computations and domains

Now that we have some basic formal descriptions of what a domain of computation should be, we can turn to the computations themselves. Clearly, these have to be functions, taking inputs from some computational domain and returning outputs in some (possibly different) domain. However, one would also expect that the output of a function will contain more information when the information content of the input is increased. Formally, this means that we want a function to be **monotonic**.

When dealing with **dcp**s, one might also want computations to be compatible with the formation of limits of a directed set. Formally, this means that, for some function f , the image $f(D)$ of a directed set D (i.e. the set of the images of each element of D) is again directed and has as a least upper bound the image of the least upper bound of D . One could also say that f *preserves directed suprema*. Also note that, by considering directed sets of two elements, such a function also has to be monotonic. These properties give rise to the notion of a **Scott-continuous** function. Since this often is not ambiguous one also may speak of *continuous functions*.

Approximation and finiteness

Domain theory is a purely *qualitative* approach to modeling the structure of information states. One can say that something contains more information, but the amount of additional information is not specified. Yet, there are some situations in which one wants to speak about elements that are in a sense much more simple (or much more incomplete) than a given state of information. For example, in the natural subset-inclusion ordering on some powerset, any infinite element (i.e. set) is much more "informative" than any of its *finite* subsets.

If one wants to model such a relationship, one may first want to consider the induced strict order $<$ of a domain with order $\leq$. However, while this is a useful notion in the case of total orders, it does not tell us much in the case of partially ordered sets. Considering again inclusion-orders of sets, a set is already strictly smaller than another, possibly infinite, set if it contains just one less element. One would, however, hardly agree that this captures the notion of being "much simpler".

Way-below relation

A more elaborate approach leads to the definition of the so-called **order of approximation**, which is more suggestively also called the **way-below relation**. An element x is *way below* an element y , if, for every directed set D with supremum such that

$$y \leq \sup D,$$

there is some element d in D such that

$$x \leq d.$$

Then one also says that x *approximates* y and writes

$$x \ll y.$$

This does imply that

$$x \leq y,$$

since the singleton set $\{y\}$ is directed. For an example, in an ordering of sets, an infinite set is way above any of its finite subsets. On the other hand, consider the directed set (in fact: the chain) of finite sets

$$\{0\}, \{0, 1\}, \{0, 1, 2\}, \dots$$

Since the supremum of this chain is the set of all natural numbers $\mathbf{N}$, this shows that no infinite set is way below $\mathbf{N}$.

However, being way below some element is a *relative* notion and does not reveal much about an element alone. For example, one would like to characterize finite sets in an order-theoretic way, but even infinite sets can be way below some other set. The special property of these **finite** elements x is that they are way below themselves, i.e.

$$x \ll x.$$

An element with this property is also called **compact**. Yet, such elements do not have to be "finite" nor "compact" in any other mathematical usage of the terms. The notation is nonetheless motivated by certain parallels to the respective notions in set theory and topology. The compact elements of a domain have the important special property that they cannot be obtained as a limit of a directed set in which they did not already occur.

Many other important results about the way-below relation support the claim that this definition is appropriate to capture many important aspects of a domain.

Bases of domains

The previous thoughts raise another question: is it possible to guarantee that all elements of a domain can be obtained as a limit of much simpler elements? This is quite relevant in practice, since we cannot compute infinite objects but we may still hope to approximate them arbitrarily closely.

More generally, we would like to restrict to a certain subset of elements as being sufficient for getting all other elements as least upper bounds. Hence, one defines a **base** of a poset P as being a subset B of P , such that, for each x in P , the set of elements in B that are way below x contains a directed set with supremum x . The poset P is a *continuous poset* if it has some base. Especially, P itself is a base in this situation. In many applications, one restricts to continuous (d)cpos as a main object of study.

Finally, an even stronger restriction on a partially ordered set is given by requiring the existence of a base of *compact* elements. Such a poset is called **algebraic**. From the viewpoint of denotational semantics, algebraic posets are particularly well-behaved, since they allow for the approximation of all elements even when restricting to finite ones. As remarked before, not every finite element is "finite" in a classical sense and it may well be that the finite elements constitute an uncountable set.

In some cases, however, the base for a poset is countable. In this case, one speaks of an **ω -continuous** poset. Accordingly, if the countable base consists entirely of finite elements, we obtain an order that is **ω -algebraic**.

Special types of domains

A simple special case of a domain is known as an **elementary** or **flat domain**. This consists of a set of incomparable elements, such as the integers, along with a single "bottom" element considered smaller than all other elements.

One can obtain a number of other interesting special classes of ordered structures that could be suitable as "domains". We already mentioned continuous posets and algebraic posets. More special versions of both are continuous and algebraic cpos. Adding even further completeness properties one obtains continuous lattices and algebraic lattices, which are just complete lattices with the respective properties. For the algebraic case, one finds broader classes of posets which are still worth studying: historically, the Scott domains were the first structures to be studied in domain theory. Still wider classes of domains are constituted by SFP-domains, L-domains, and bifinite domains.

All of these classes of orders can be cast into various categories of dcpos, using functions which are monotone, Scott-continuous, or even more specialized as morphisms. Finally, note that the term *domain* itself is not exact and thus is only used as an abbreviation when a formal definition has been given before or when the details are irrelevant.

Important results

A poset D is a dcpo if and only if each chain in D has a supremum. However, directed sets are strictly more powerful than chains.

If f is a continuous function on a poset D then it has a least fixed point, given as the least upper bound of all finite iterations of f on the least element 0 : $\bigvee_{n \in \mathbf{N}} f^n(0)$.

Generalizations

- Synthetic domain theory ^[1]
- Topological domain theory ^[2]

See also

- Scott domain
- Scott information system
- Type theory
- Category theory

Literature

Probably one of the most recommendable books on domain theory today, giving a very clear and detailed view on many parts of the basic theory:

- G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott (2003). "Continuous Lattices and Domains". *Encyclopedia of Mathematics and its Applications*. **93**. Cambridge University Press. ISBN 0-521-80338-1.

A standard resource on domain theory, which is freely available online:

- S. Abramsky, A. Jung (1994). "Domain theory ^[3]" (PDF). S. Abramsky, D. M. Gabbay, T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science III*, Oxford University Press. ISBN 0-19-853762-X. Retrieved on 2007-10-13.

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The “References” section of this web page provides many online materials on domain theory.

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- D. S. Scott (1975). "Data types as lattices". *Proceedings of the International Summer Institute and Logic Colloquium, Kiel*, in *Lecture Notes in Mathematics* (Springer-Verlag) **499**: 579–651.

A textbook treatment of domain theory with connections to lambda calculus and types:

- Carl A. Gunter (1992). *Semantics of Programming Languages*. MIT Press.

A general, easy-to-read account of order theory, including an introduction to domain theory as well:

- B. A. Davey and H. A. Priestley (2002). *Introduction to Lattices and Order* (2nd ed.). Cambridge University Press. ISBN 0-521-78451-4.

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Baire category theorem

The **Baire category theorem** is an important tool in general topology and functional analysis. The theorem has two forms, each of which gives sufficient conditions for a topological space to be a Baire space.

The theorem was proved by René-Louis Baire in his 1899 doctoral thesis.^[1]

Statement of the theorem

A Baire space is a topological space with the following property: for each countable collection of open dense sets U_n , their intersection $\cap U_n$ is dense.

- **(BCT1)** Every complete metric space is a Baire space. More generally, every topological space which is homeomorphic to an open subset of a complete pseudometric space is a Baire space. Thus every completely metrizable topological space is a Baire space.
- **(BCT2)** Every locally compact Hausdorff space is a Baire space. The proof is similar to the preceding statement; the finite intersection property takes the role played by completeness.

Note that neither of these statements implies the other, since there is a complete metric space which is not locally compact (the irrational numbers with the metric defined below), and there is a locally compact Hausdorff space which is not metrizable (uncountable Fort space). See Steen and Seebach in the references below.

- **(BCT3)** A non-empty complete metric space is NOT the countable union of nowhere-dense sets (i.e, sets whose closure has dense complement).

This formulation is a consequence of BCT1 and is sometimes more useful in applications. Also: if a non-empty complete metric space is the countable union of closed sets, then one of these closed sets has *non empty* interior.

Relation to the axiom of choice

The proofs of **BCT1** and **BCT2** require some form of the axiom of choice; and in fact BCT1 is (over ZF) equivalent to a weaker version of the axiom of choice called the axiom of dependent choices. [2]

Uses of the theorem

BCT1 is used to prove the open mapping theorem, the closed graph theorem and the uniform boundedness principle.

BCT1 also shows that every complete metric space with no isolated points is uncountable. (If X is a countable complete metric space with no isolated points, then each singleton $\{x\}$ in X is nowhere dense, and so X is of first category in itself.) In particular, this proves that the set of all real numbers is uncountable.

BCT1 shows that each of the following is a Baire space:

- The space $\mathbf{R}$ of real numbers
 - The irrational numbers, with the metric defined by $d(x, y) = 1 / (n + 1)$, where n is the first index for which the continued fraction expansions of x and y differ (this is a complete metric space)
-

- The Cantor set

By **BCT2**, every manifold is a Baire space, since it is locally compact and Hausdorff. This is so even for non-paracompact (hence nonmetrizable) manifolds such as the long line.

Proof

The following is a standard proof that a complete metric space X is a Baire space.

Let U_n be a collection of open dense subsets. We want to show that the intersection $\bigcap U_n$ is dense. For that, let $W \subset X$ be an open subset. By denseness, there is x_1 and $r_1 > 0$ such that:

$$\overline{B}(x_1, r_1) \subset W \cap U_1.$$

Recursively, we find x_n and $r_n > 0$ such that:

$$\overline{B}(x_n, r_n) \subset B(x_{n-1}, r_{n-1}) \cap U_n \text{ as well as } r_n < n^{-1}.$$

Since $x_n \in B(x_m, r_m)$ when $n > m$, we have that x_n is Cauchy, and x_n converges to some limit x . For any n , by closedness,

$$x \in \overline{B}(x_{n+1}, r_{n+1}) \subset B(x_n, r_n).$$

Hence, $x \in W$ and $x \in U_n$ for all n . $\square$

Notes

- [1] R. Baire. Sur les fonctions de variables réelles. (<http://books.google.com/books?id=cS4LAAAYAAJ>) Ann. di Mat., 3:1-123, 1899.
- [2] <http://www.math.vanderbilt.edu/~schectex/ccc/excerpts/equivdc.gif>

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- Schechter, Eric, *Handbook of Analysis and its Foundations*, Academic Press, ISBN 0-12-622760-8
- Lynn Arthur Steen and J. Arthur Seebach, Jr., *Counterexamples in Topology*, Springer-Verlag, New York, 1978. Reprinted by Dover Publications, New York, 1995. ISBN 0-486-68735-X (Dover edition).

Notable Algebraic Topologists and Category Theoreticians

Michael Atiyah

Michael Atiyah	
	
Born	April 22, 1929 Hampstead, London, England
Nationality	United Kingdom
Fields	Mathematics
Institutions	University of Cambridge University of Oxford Institute for Advanced Study University of Leicester University of Edinburgh
Alma mater	Trinity College, Cambridge
Doctoral advisor	W. V. D. Hodge
Doctoral students	Simon Donaldson Nigel Hitchin Frances Kirwan Peter Kronheimer Graeme Segal
Notable awards	Fields Medal (1966) Copley Medal (1988) Abel Prize (2004)

Sir Michael Francis Atiyah, OM, FRS, FRSE (born April 22, 1929) is a British mathematician, and one of the most influential mathematicians of the twentieth century.^[1] He grew up in Sudan and Egypt, and spent most of his academic life at Oxford, Cambridge, and Princeton. He has been President of the Royal Society (1990–1995), Master of Trinity College, Cambridge (1990–1997), Chancellor of the University of Leicester (1995–2005), and President of the Royal Society of Edinburgh (2005–2008). He is currently retired and an honorary professor at the University of Edinburgh.

He has had many mathematical collaborations, in particular with Raoul Bott, Friedrich Hirzebruch, and Isadore Singer, and his students include Graeme Segal, Nigel Hitchin, and Simon Donaldson. With Hirzebruch he founded topological K-theory, a major tool in

algebraic topology, that describes the ways in which high dimensional space can be twisted. His best known result is the Atiyah-Singer index theorem, proved with Singer in 1963, a fundamental and widely used result which can be used to count the number of independent solutions of many important differential equations. More recently he has worked on topics inspired by theoretical physics, such as instantons and monopoles, which are responsible for some subtle corrections in quantum field theory.

He has received many awards for his research, including the Fields Medal in 1966, the Copley Medal in 1988, and the Abel Prize in 2004.

Biography

Atiyah was born in Hampstead, London to Lebanese writer Edward Atiyah and Scot Jean Atiyah (née Levens). Patrick Atiyah, professor of law, is his brother; he has one other brother, Joe, and a sister, Selma.^[2] He went to primary school at the Diocesan school in Khartoum, Sudan (1934–1941) and to secondary school at Victoria College in Cairo and Alexandria (1941–1945); the school was also attended by European nobility displaced by the Second World War and some future leaders of Arab nations.^[3] He returned to England and Manchester Grammar School for his HSC studies (1945–1947) and did his national service with the Royal Electrical and Mechanical Engineers (1947–1949). His undergraduate and postgraduate studies took place at Trinity College, Cambridge (1949–1955).^[4] He was a doctoral student of William V. D. Hodge and was awarded a doctorate in 1955 for a thesis entitled *Some Applications of Topological Methods in Algebraic Geometry*.^[5]



Great Court of Trinity College, Cambridge, where Atiyah was a student and later Master



The Institute for Advanced Study in Princeton, where Atiyah was professor from 1969 to 1972

Atiyah married Lily Brown on 30 July 1955, with whom he now has three sons.^[4] He spent the academic year 1955–1956 at the Institute for Advanced Study, Princeton, then returned to Cambridge University, where he was a research fellow and assistant lecturer (1957–1958), then a university lecturer and tutorial fellow at Pembroke College (1958–1961). In 1961, he moved to the University of Oxford, where he was a reader and professorial fellow at St Catherine's College (1961–1963).^[4] He became Savilian Professor of Geometry and a professorial fellow of New College, Oxford from 1963 to 1969. He then took up a three year professorship at the Institute for Advanced Study in Princeton after which he returned to Oxford as a Royal Society Research Professor and professorial fellow of St Catherine's College. He was president of the London Mathematical Society from 1974 to 1976.^[4]

I started out by changing local currency into foreign currency everywhere I travelled as a child and ended up making money. That's when my father realised that I would be a mathematician some day.

—Michael Atiyah^[6]

Atiyah has been active on the international scene, for instance as president of the Pugwash Conferences on Science and World Affairs from 1997 to 2002.^[7] He also contributed to the foundation of the InterAcademy Panel on International Issues, the Association of European Academies (ALLEA), and the European Mathematical Society (EMS).^[8]

Within the United Kingdom, he was involved in the creation of the Isaac Newton Institute for Mathematical Sciences in Cambridge and was its first director (1990–1996). He was president of the Royal Society (1990–1995), Master of Trinity College, Cambridge (1990–1997),^[7] Chancellor of the University of Leicester (1995–2005)^[7], and president of the Royal Society of Edinburgh (2005–2008).^[9] He is now retired and an honorary professor at the University of Edinburgh.

Collaborations

Atiyah has collaborated with many other mathematicians. His three main collaborations were with Raoul Bott on the Atiyah–Bott fixed-point theorem and many other topics, with Isadore M. Singer on the Atiyah–Singer index theorem, and with Friedrich Hirzebruch on topological K-theory,^[10] all of whom he met at the Institute for Advanced Study in Princeton in 1955.^[11] His other collaborators include J. Frank Adams (Hopf invariant problem), Jurgen Berndt (projective planes), Roger Bielawski (Berry–Robbins problem), Howard Donnelly (L-functions), Vladimir G. Drinfeld (instantons), Johan L. Dupont (singularities of vector fields), Lars Garding (hyperbolic differential equations), Nigel J. Hitchin (monopoles), William V. D. Hodge (Integrals of the second kind), Michael Hopkins (K-theory), Lisa Jeffrey (topological Lagrangians), John D. S. Jones (Yang–Mills theory), Juan Maldacena (M-theory), Yuri I. Manin (instantons), Nick S. Manton (Skyrmions), Vijay K. Patodi (Spectral asymmetry), A. N. Pressley (convexity), Elmer Rees (vector bundles), Wilfried Schmid (discrete series representations), Graeme Segal (equivariant K-theory), Alexander Shapiro (Clifford algebras), L. Smith (homotopy groups of spheres), Paul Sutcliffe (polyhedra), David O. Tall (lambda rings), John A. Todd (Stiefel manifolds), Cumrun Vafa (M-theory), Richard S. Ward (instantons), and Edward Witten (M-theory, topological quantum field theories).^[12]

His later research on gauge field theories, particularly Yang–Mills theory, stimulated important interactions between geometry and physics, most notably in the work of Edward Witten.



The Mathematical Institute in Oxford, where Atiyah supervised many of his students.

If you attack a mathematical problem directly, very often you come to a dead end, nothing you do seems to work and you feel that if only you could peer round the corner there might be an easy solution. There is nothing like having somebody else beside you, because he can usually peer round the corner.

—Michael Atiyah^[13]

Atiyah's many students include Peter Braam 1987, Simon Donaldson 1983, David Elworthy 1967, Howard Fegan 1977, Eric Grunwald 1977, Nigel Hitchin 1972, Lisa Jeffrey 1991, Frances Kirwan 1984, Peter Kronheimer 1986, Ruth Lawrence 1989, George Lusztig 1971,

Jack Morava 1968, Michael Murray 1983, Peter Newstead 1966, Ian Porteous 1961, John Roe 1985, Brian Sanderson 1963, Rolph Schwarzenberger 1960, Graeme Segal 1967, David Tall 1966, and Graham White 1982.^[5]

Other contemporary mathematicians who influenced Atiyah include Roger Penrose, Lars Hörmander, Alain Connes, and Jean-Michel Bismut.^[14] Atiyah said that the mathematician he most admired was Hermann Weyl,^[15] and that his favorite mathematicians from before the 20th century were Bernhard Riemann and William Rowan Hamilton.^[16]

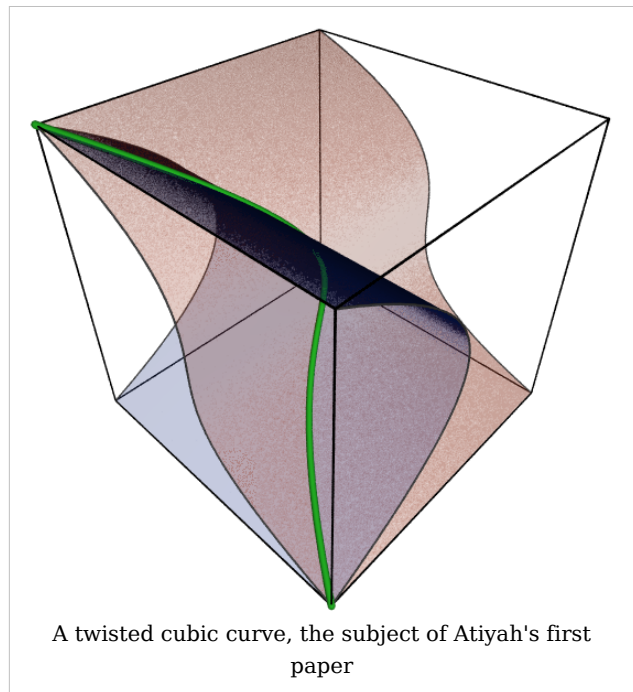
Mathematical work

The six volumes of Atiyah's collected papers include most of his work, except for his commutative algebra textbook^[17] and a few works written since 2004.

Algebraic geometry (1952-1958)

Atiyah's early papers on algebraic geometry (and some general papers) are reprinted in the first volume of his collected works.^[18]

As an undergraduate Atiyah was interested in classical projective geometry, and wrote his first paper: a short note on twisted cubics.^[19] He started research under W. V. D. Hodge and won the Smith's prize for 1954 for a sheaf-theoretic approach to ruled surfaces,^[20] which encouraged Atiyah to continue in mathematics, rather than switch to his other interests—architecture and archaeology.^[21] His PhD thesis with Hodge was on a sheaf-theoretic approach to Solomon Lefschetz's theory of integrals of the second kind on algebraic varieties, and resulted in an invitation to visit the

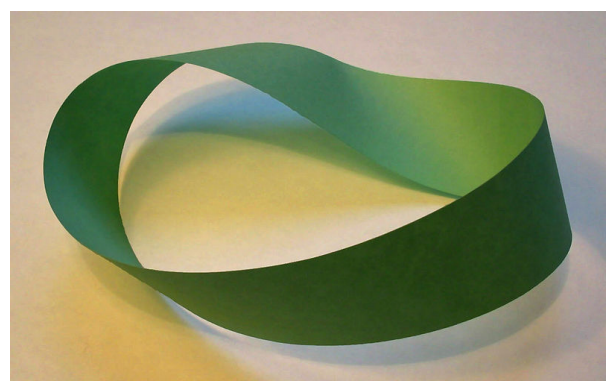


Institute for Advanced Study in Princeton for a year.^[22] While in Princeton he classified vector bundles on an elliptic curve (extending Grothendieck's classification of vector bundles on a genus 0 curve), by showing that any vector bundle is a sum of (essentially unique) indecomposable vector bundles,^[23] and then showing that the space of indecomposable vector bundles of given degree and positive dimension can be identified with the elliptic curve.^[24] He also studied double points on surfaces,^[25] giving the first example of a flop, a special birational transformation of 3-folds that was later heavily used in Mori's work on minimal models for 3-folds.^[26] Atiyah's flop can also be used to show that the universal marked family of K3 surfaces is non-Hausdorff.^[27]

K theory (1959-1974)

Atiyah's works on K-theory, including his book on K-theory^[28] are reprinted in volume 2 of his collected works.^[29]

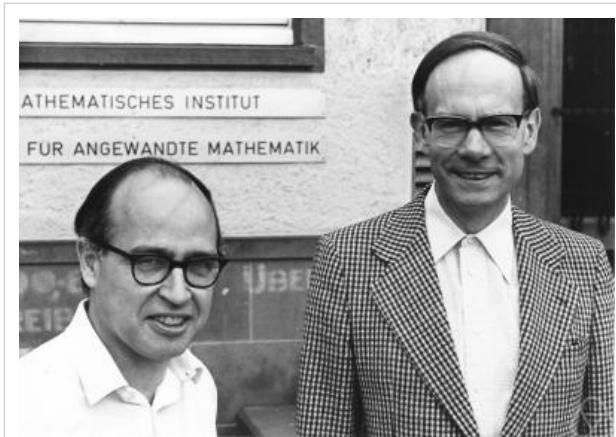
The simplest example of a vector bundle is the Möbius band (pictured on the right): a strip of paper with a twist in it, which represents a rank 1 vector bundle over a circle (the circle in question being the centerline of the Möbius band). K-theory is a tool for working with higher dimensional analogues of this example, or in other words for describing higher dimensional twistings: elements of the K-group of a space are represented by vector bundles over it, so the Möbius band represents an element of the K-group of a circle.



A Möbius band is the simplest non-trivial example of a vector bundle.

Topological K-theory was discovered by Atiyah and Friedrich Hirzebruch^[30] who were inspired by Grothendieck's proof of the Grothendieck-Riemann-Roch theorem and Bott's work on the periodicity theorem. This paper only discussed the zeroth K-group; they shortly after extended it to K-groups of all degrees,^[31] giving the first (nontrivial) example of a generalized cohomology theory.

Several results showed that the newly introduced K-theory was in some ways more powerful than ordinary cohomology theory. Atiyah and Todd^[32] used K-theory to improve the lower bounds found using ordinary cohomology by Borel and Serre for the James number, describing when a map from a complex Stiefel manifold to a sphere has a cross section. (Adams and Grant-Walker later showed that the bound found by Atiyah and Todd was best possible.) Atiyah and Hirzebruch^[33] used K-theory to explain some relations between Steenrod operations and Todd classes that Hirzebruch had noticed a few years before. The original solution of the Hopf invariant one problem operations by J. F. Adams was very long and complicated, using secondary cohomology operations. Atiyah showed how primary operations in K-theory could be used to give a short solution taking only a few lines, and in joint work with Adams^[34] also proved analogues of the result at odd primes.



Michael Atiyah and Friedrich Hirzebruch (right), the creators of topological K-theory

The Atiyah–Hirzebruch spectral sequence relates the ordinary cohomology of a space to its generalized cohomology theory.^[31] (Atiyah and Hirzebruch used the case of K-theory, but their method works for all cohomology theories).

Atiyah showed^[35] that for a finite group G , the K-theory of its classifying space, BG , is isomorphic to the completion of its character ring:

$$K(BG) \cong R(G)^\wedge.$$

The same year^[36] they proved the result for G any compact connected Lie group. Although soon the result could be extended to *all* compact Lie groups by incorporating results from Graeme Segal's thesis,^[37] that extension was complicated. However a simpler and more general proof was produced by introducing equivariant K-theory, *i.e.* equivalence classes of G -vector bundles over a compact G -space X .^[38] It was shown that under suitable conditions the completion of the equivariant K-theory of X is isomorphic to the ordinary K-theory of a space, X_G , which fibred over BG with fibre X :

$$K_G(X)^\wedge \cong K(X_G).$$

The original result then followed as a corollary by taking X to be a point: the left hand side reduced to the completion of $R(G)$ and the right to $K(BG)$. See Atiyah–Segal completion theorem for more details.

He defined new generalized homology and cohomology theories called bordism and cobordism, and pointed out that many of the deep results on cobordism of manifolds found by R. Thom, C. T. C. Wall, and others could be naturally reinterpreted as statements about these cohomology theories.^[39] Some of these cohomology theories, in particular complex cobordism, turned out to be some of the most powerful cohomology theories known.

Algebra is the offer made by the devil to the mathematician. The devil says: 'I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine.'

—Michael Atiyah^[40]

He introduced^[41] the J-group $J(X)$ of a finite complex X , defined as the group of stable fiber homotopy equivalence classes of sphere bundles; this was later studied in detail by J. F. Adams in a series of papers, leading to the Adams conjecture.

With Hirzebruch he extended the Grothendieck–Riemann–Roch theorem to complex analytic embeddings,^[41] and in a related paper^[42] they showed that the Hodge conjecture for integral cohomology is false. The Hodge conjecture for rational cohomology is, as of 2008, a major unsolved problem.^[43]

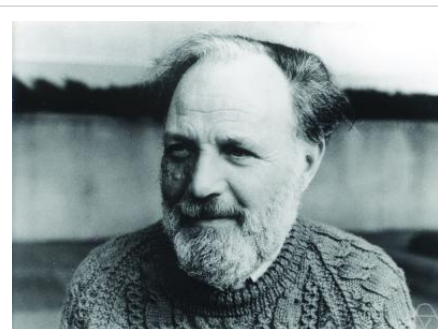
The Bott periodicity theorem was a central theme in Atiyah's work on K-theory, and he repeatedly returned to it, reworking the proof several times to understand it better. With

Bott he worked out an elementary proof,^[44] and gave another version of it in his book.^[45] With Bott and Shapiro he analysed the relation of Bott periodicity to the periodicity of Clifford algebras;^[46] although this paper did not have a proof of the periodicity theorem, a proof along similar lines was shortly afterwards found by R. Wood. In ^[47] he found a proof of several generalizations using elliptic operators; this new proof used an idea that he used to give a particularly short and easy proof of Bott's original periodicity theorem.^[48]

Index theory (1963-1984)

Atiyah's work on index theory is reprinted in volumes 3 and 4 of his collected works.^[49] ^[50]

The index of a differential operator is closely related to the number of independent solutions (more precisely, it is the differences of the numbers of independent solutions of the differential operator and its adjoint). There are many hard and fundamental problems in mathematics that can easily be reduced to the problem of finding the number of independent solutions of some differential operator, so if one has some means of finding the index of a differential operator these problems can often be solved. This is what the Atiyah-Singer index theorem does: it gives a formula for the index of certain differential operators, in terms of topological invariants that look quite complicated but are in practice usually straightforward to calculate.



Isadore Singer (in 1977), who worked with Atiyah on index theory.

Several deep theorems, such as the Hirzebruch-Riemann-Roch theorem, are special cases of the Atiyah-Singer index theorem. In fact the index theorem gave a more powerful result, because its proof applied to all compact complex manifolds, while Hirzebruch's proof only worked for projective manifolds. There were also many new applications: a typical one is calculating the dimensions of the moduli spaces of instantons. The index theorem can also be run "in reverse": the index is obviously an integer, so the formula for it must also give an integer, which sometimes gives subtle integrality conditions on invariants of manifolds. A typical example of this is Rochlin's theorem, which follows from the index theorem.

The most useful piece of advice I would give to a mathematics student is always to suspect an impressive sounding Theorem if it does not have a special case which is *both* simple *and* non-trivial.
—Michael Atiyah^[51]

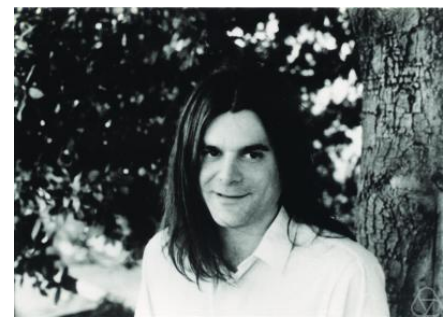
The index problem for elliptic differential operators was posed in 1959 by Gel'fand.^[52] He noticed the homotopy invariance of the index, and asked for a formula for it by means of topological invariants. Some of the motivating examples included the Riemann-Roch theorem and its generalization the Hirzebruch-Riemann-Roch theorem, and the Hirzebruch signature theorem. Hirzebruch and Borel had proved the integrality of the $\hat{A}$ genus of a spin manifold, and Atiyah suggested that this integrality could be explained if it were the index of the Dirac operator (which was rediscovered by Atiyah and Singer in 1961).

The first announcement of the Atiyah-Singer theorem was their 1963 paper.^[53] The proof sketched in this announcement was inspired by Hirzebruch's proof of the Hirzebruch-Riemann-Roch theorem and was never published by them, though it is described in the book by Palais.^[54] Their first published proof ^[55] was more similar to Grothendieck's proof of the Grothendieck-Riemann-Roch theorem, replacing the

cobordism theory of the first proof with K-theory, and they used this approach to give proofs of various generalizations in a sequence of papers from 1968 to 1971.

Instead of just one elliptic operator, one can consider a family of elliptic operators parameterized by some space Y . In this case the index is an element of the K-theory of Y , rather than an integer.^[56] If the operators in the family are real, then the index lies in the real K-theory of Y . This gives a little extra information, as the map from the real K theory of Y to the complex K theory is not always injective.^[57]

With Bott, Atiyah found an analogue of the Lefschetz fixed-point formula for elliptic operators, giving the Lefschetz number of an endomorphism of an elliptic complex in terms of a sum over the fixed points of the endomorphism.^[58] As special cases their formula included the Weyl character formula, and several new results about elliptic curves with complex multiplication, some of which were initially disbelieved by experts.^[59] Atiyah and Segal combined this fixed point theorem with the index theorem as follows. If there is a compact group action of a group G on the compact manifold X , commuting with the elliptic operator, then one can replace ordinary K theory in the index theorem with equivariant K-theory. For trivial groups G this gives the index theorem, and for a finite group G acting with isolated fixed points it gives the Atiyah-Bott fixed point theorem. In general it gives the index as a sum over fixed point submanifolds of the group G .^[60]



Atiyah's former student Graeme Segal (in 1982), who worked with Atiyah on equivariant K-theory.

Atiyah^[61] solved a problem asked independently by Hörmander and Gel'fand, about whether complex powers of analytic functions define distributions. Atiyah used Hironaka's resolution of singularities to answer this affirmatively. A ingenious and elementary solution was found at about the same time by J. Bernstein, and discussed by Atiyah.^[62]

As an application of the equivariant index theorem, Atiyah and Hirzebruch showed that manifolds with effective circle actions have vanishing $\hat{A}$ -genus.^[63] (Lichnerowicz showed that if a manifold has a metric of positive scalar curvature then the $\hat{A}$ -genus vanishes.)

With Elmer Rees, Atiyah studied the problem of the relation between topological and holomorphic vector bundles on projective space. They solved the simplest unknown case, by showing that all rank 2 vector bundles over projective 3-space have a holomorphic structure.^[64] Horrocks had previously found some non-trivial examples of such vector bundles, which were later used by Atiyah in his study of instantons on the 4-sphere.

Atiyah, Bott, and Vijay K. Patodi^[65] gave a new proof of the index theorem using the heat equation.

If the manifold is allowed to have boundary, then some restrictions must be put on the domain of the elliptic operator in order to ensure a finite index. These conditions can be local (like demanding that the sections in the domain vanish at the boundary) or more complicated global conditions (like requiring that the sections in the domain solve some differential equation). The local case was worked out by Atiyah and Bott, but they showed that many interesting operators (e.g., the signature operator) do not admit local boundary conditions. To handle these operators, Atiyah, Patodi and Singer introduced global boundary conditions equivalent to attaching a cylinder to the manifold along the boundary and then restricting the domain to those sections that are square integrable along the cylinder. This resulted in a series of papers on spectral asymmetry,^[66] which were later unexpectedly used in theoretical physics, in particular in Witten's work on anomalies.



Raoul Bott, who worked with Atiyah on fixed point formulas and several other topics.



The lacunas discussed by Petrovsky, Atiyah, Bott and Gårding are similar to the spaces between shockwaves of a supersonic object.

The fundamental solutions of linear hyperbolic partial differential equations often have Petrovsky lacunas: regions where they vanish identically. These were studied in 1945 by I. G. Petrovsky, who found topological conditions describing which regions were lacunas. In collaboration with Bott and Lars Gårding, Atiyah wrote three papers updating and generalizing Petrovsky's work.^[67]

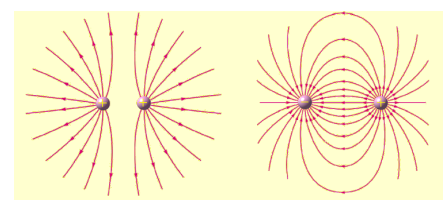
Atiyah^[68] showed how to extend the index theorem to some non-compact manifolds, acted on by a discrete group with compact quotient. The kernel of the elliptic operator is in general infinite dimensional in this case, but it is possible to get a finite index using the dimension of a module over a von Neumann algebra; this index is in general real rather than integer valued.

This version is called the L^2 index theorem, and was used by Atiyah and Schmid^[69] to give a geometric construction, using square integrable harmonic spinors, of Harish-Chandra's discrete series representations of semisimple Lie groups. In the course of this work they found a more elementary proof of Harish-Chandra's fundamental theorem on the local integrability of characters of Lie groups.^[70]

With H. Donnelly and I. Singer, he extended Hirzebruch's formula (relating the signature defect at cusps of Hilbert modular surfaces to values of L-functions) from real quadratic fields to all totally real fields.^[71]

Gauge theory (1977-1985)

Many of his papers on gauge theory and related topics are reprinted in volume 5 of his collected works.^[72] A common theme of these papers is the study of moduli spaces of solutions to certain non-linear partial differential equations, in particular the equations for instantons and monopoles. This often involves finding a subtle correspondence between solutions of two seemingly quite different equations. An early example of this which Atiyah used repeatedly is the Penrose transform, which can sometimes convert solutions of a non-linear equation over some real manifold into solutions of some linear holomorphic equations over a different complex manifold.



On the left, two nearby monopoles of the same polarity repel each other, and on the right two nearby monopoles of opposite polarity form a dipole. These are abelian monopoles; the non-abelian ones studied by Atiyah are more complicated.

In a series of papers with several authors, Atiyah classified all instantons on 4 dimensional Euclidean space. It is more convenient to classify instantons on a sphere as this is compact, and this is essentially equivalent to classifying instantons on Euclidean space as this is conformally equivalent to a sphere and the equations for instantons are conformally invariant. With Hitchin and Singer^[73] he calculated the dimension of the moduli space of irreducible self-dual connections (instantons) for any principle bundle over a compact 4-dimensional Riemannian manifold. For example, the dimension of the space of SU_2 instantons of rank $k > 0$ is $8k - 3$. To do this they used the Atiyah-Singer index theorem to calculate the dimension of the tangent space of the moduli space at a point; the tangent space is essentially the space of solutions of an elliptic differential operator, given by the linearization of the non-linear Yang-Mills equations. These moduli spaces were later used by Donaldson to construct his invariants of 4-manifolds. Atiyah and Ward used the Penrose correspondence to reduce the classification of all instantons on the 4-sphere to a problem in algebraic geometry.^[74] With Hitchin he used ideas of Horrocks to solve this problem, giving the ADHM construction of all instantons on a sphere; Manin, and Drinfeld found the same construction at the same time, leading to a joint paper by all four authors.^[75] Atiyah reformulated this construction using quaternions and wrote up a leisurely account of this classification of instantons on Euclidean space as a book.^[76]

The mathematical problems that have been solved or techniques that have arisen out of physics in the past have been the lifeblood of mathematics.

—Michael Atiyah^[77]

Atiyah's work on instanton moduli spaces was used in Donaldson's work on Donaldson theory. Donaldson showed that the moduli space of (degree 1) instantons over a compact simply connected 4-manifold with positive definite intersection form can be compactified to give a cobordism between the manifold and a sum of copies of complex projective space. He deduced from this that the intersection form must be a sum of one dimensional ones, which led to several spectacular applications to smooth 4-manifolds, such as the existence of non-equivalent smooth structures on 4 dimensional Euclidean space. Donaldson went on to use the other moduli spaces studied by Atiyah to define Donaldson invariants, which revolutionized the study of smooth 4-manifolds, and showed that they were more subtle than smooth manifolds in any other dimension, and also quite different from topological

4-manifolds. Atiyah described some of these results in a survey talk.^[78]

Green's functions for linear partial differential equations can often be found by using the Fourier transform to convert this into an algebraic problem. Atiyah used a non-linear version of this idea.^[79] He used the Penrose transform to convert the Green's function for the conformally invariant Laplacian into a complex analytic object, which turned out to be essentially the diagonal embedding of the Penrose twistor space into its square. This allowed him to find an explicit formula for the conformally invariant Green's function on a 4-manifold.

In his paper with Jones,^[80] he studied the topology of the moduli space of $SU(2)$ instantons over a 4-sphere. They showed that the natural map from this moduli space to the space of all connections induces epimorphisms of homology groups in a certain range of dimensions, and suggested that it might induce isomorphisms of homology groups in the same range of dimensions. This became known as the Atiyah–Jones conjecture, and was later proved by several mathematicians.^[81]

Harder and M. S. Narasimhan described the cohomology of the moduli spaces of stable vector bundles over Riemann surfaces by counting the number of points of the moduli spaces over finite fields, and then using the Weil conjectures to recover the cohomology over the complex numbers.^[82] Atiyah and R. Bott used Morse theory and the Yang–Mills equations over a Riemann surface to reproduce and extending the results of Harder and Narasimhan.^[83]

An old result due to Schur and Horn states that the set of possible diagonal vectors of an Hermitian matrix with given eigenvalues is the convex hull of all the permutations of the eigenvalues. Atiyah proved a generalization of this that applies to all compact symplectic manifolds acted on by a torus, showing that the image of the manifold under the moment map is a convex polyhedron,^[84] and with Pressley gave a related generalization to infinite dimensional loop groups.^[85]

Duistermaat and Heckman found a striking formula, saying that the push-forward of the Liouville measure of a moment map for a torus action is given exactly by the stationary phase approximation (which is in general just an asymptotic expansion rather than exact). Atiyah and Bott^[86] showed that this could be deduced from a more general formula in equivariant cohomology, which was a consequence of well-known localization theorems. Atiyah showed^[87] that the moment map was closely related to geometric invariant theory, and this idea was later developed much further by his student F. Kirwan. Witten shortly after applied the Duistermaat–Heckman formula to loop spaces and showed that this formally gave the Atiyah–Singer index theorem for the Dirac operator; this idea was lectured on by Atiyah.^[88]

With Hitchin he worked on magnetic monopoles, and studied their scattering using an idea of Nick Manton.^[89] His book^[90] with Hitchin gives a detailed description of their work on magnetic monopoles. The main theme of the book is a study of a moduli space of magnetic monopoles; this has a natural Riemannian metric, and a key point is that this metric is complete and hyperkahler. The metric is then used to study the scattering of two monopoles, using a suggestion of N. Manton that the geodesic flow on the moduli space is the low energy approximation to the scattering. For example, they show that a head-on collision between two monopoles results in 90-degree scattering, with the direction of scattering depending on the relative phases of the two monopoles. He also studied monopoles on hyperbolic space.^[91]

Atiyah showed ^[92] that instantons in 4 dimensions can be identified with instantons in 2 dimensions, which are much easier to handle. There is of course a catch: in going from 4 to 2 dimensions the structure group of the gauge theory changes from a finite dimensional group to an infinite dimensional loop group. This gives another example where the moduli spaces of solutions of two apparently unrelated nonlinear partial differential equations turn out to be essentially the same.

Atiyah and Singer found that anomalies in quantum field theory could be interpreted in terms of index theory of the Dirac operator;^[93] this idea later became widely used by physicists.

Later work (1986 onwards)

Many of the papers in the 6th volume^[94] of his collected works are surveys, obituaries, and general talks. Since its publication, Atiyah has continued to publish, including several surveys, a popular book,^[95] and another paper with Segal on twisted K-theory.

One paper^[96] is a detailed study of the Dedekind eta function from the point of view of topology and the index theorem.

Several of his papers from around this time study the connections between quantum field theory, knots, and Donaldson theory. He introduced the concept of a topological quantum field theory, inspired by Witten's work and Segal's definition of a conformal field theory.^[97] His book^[98] describes the new knot invariants found by Vaughan Jones and Edward Witten in terms of topological quantum field theories, and his paper with L. Jeffrey^[99] explains Witten's Lagrangian giving the Donaldson invariants.

He studied skyrmions with Nick Manton,^[100] finding a relation with magnetic monopoles and instantons, and giving a conjecture for the structure of the moduli space of two skyrmions as a certain subquotient of complex projective 3-space.

Several papers^[101] were inspired by a question of M. Berry, who asked if there is a map from the configuration space of n points in 3-space to the flag manifold of the unitary group. Atiyah gave an affirmative answer to this question, but felt his solution was too computational and studied a conjecture that would give a more natural solution. He also related the question to Nahm's equation.



Edward Witten, whose work on invariants of manifolds and topological quantum field theories was influenced by Atiyah.

But for most practical purposes, you just use the classical groups. The exceptional Lie groups are just there to show you that the theory is a bit bigger; it is pretty rare that they ever turn up.

—Michael Atiyah^[102]

With Juan Maldacena and Cumrun Vafa,^[103] and E. Witten^[104] he described the dynamics of M-theory on manifolds with G_2 holonomy. These papers seem to be the first time that Atiyah has worked on exceptional Lie groups.

In his papers with M. Hopkins^[105] and G. Segal^[106] he returned to his earlier interest of K-theory, describing some twisted forms of K-theory with applications in theoretical physics.

Awards and honours

In 1966, when he was thirty-seven years old, he was awarded the Fields Medal,^[107] for his work in developing K-theory, a generalized Lefschetz fixed-point theorem and the Atiyah–Singer theorem, for which he also won the Abel Prize jointly with Isadore Singer in 2004.^[108] Among other prizes he has received are the Royal Medal of the Royal Society in 1968,^[109] the De Morgan Medal of the London Mathematical Society in 1980, the Antonio Feltrinelli Prize from the Accademia Nazionale dei Lincei in 1981, the King Faisal International Prize for Science in 1987,^[110] the Copley Medal of the Royal Society in 1988,^[111] the Benjamin Franklin Medal of the American Philosophical Society in 1993,^[112] the Jawaharlal Nehru Birth Centenary Medal of the Indian National Science Academy in 1993,^[113] and the President's Medal from the Institute of Physics in 2008.^[114]



The premises of the Royal Society, where Atiyah was president from 1990 to 1995.

So I don't think it makes much difference to mathematics to know that there are different kinds of simple groups or not. It is a nice intellectual endpoint, but I don't think it has any fundamental importance.

—Michael Atiyah, commenting on the classification of finite simple groups^[115]

He was elected a foreign member of the National Academy of Sciences, the American Academy of Arts and Sciences, the Académie des Sciences, the Akademie Leopoldina, the Royal Swedish Academy, the Royal Irish Academy, the Royal Society of Edinburgh, the American Philosophical Society, the Indian National Science Academy, the Chinese Academy of Science, the Australian Academy of Science, the Russian Academy of Science, the Ukrainian Academy of Science, the Georgian Academy of Science, the Venezuela Academy of Science, the Norwegian Academy of Science and Letters, the Royal Spanish Academy of Science, the Accademia dei Lincei, and the Moscow Mathematical Society.^{[4] [7]}

Atiyah has been awarded honorary degrees by the universities of Bonn, Warwick, Durham, St. Andrews, Dublin, Chicago, Cambridge, Edinburgh, Essex, London, Sussex, Ghent, Reading, Helsinki, Salamanca, Montreal, Wales, Lebanon, Queen's (Canada), Keele, Birmingham, UMIST, Brown, Heriot-Watt, Mexico, Oxford, Hong Kong (Chinese University), The Open University, American University of Beirut, the Technical University of Catalonia, and Leicester.^{[4] [7]}

I had to wear a sort of bulletproof vest after that!

—Michael Atiyah, commenting on the reaction to the previous quote^[116]

Atiyah was made a Knight Bachelor in 1983^[4] and made a member of the Order of Merit in 1992.^[7]

The Michael Atiyah building^[117] at the University of Leicester and the Michael Atiyah Chair in Mathematical Sciences^[118] at the American University of Beirut were named after him.

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External links

- The celebrations of Michael Atiyah's 80th birthday in Edinburgh, April 20-24, 2009 (<http://www.maths.ed.ac.uk/~aar/atiyah80.htm>)
- Mathematical descendants of Michael Atiyah (<http://www.maths.ed.ac.uk/~aar/confer/atiyahd.pdf>)
- Biographical poster about Michael Atiyah (<http://www.maths.ed.ac.uk/~aar/confer/mfa.pdf>)
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Academic offices

Preceded by George Porter	President of the Royal Society 1990–1995	Succeeded by Sir Aaron Klug
Preceded by Sir Andrew Huxley	Master of Trinity College, Cambridge 1990–1997	Succeeded by Amartya Sen
Preceded by The Lord Porter of Luddenham	Chancellor of the University of Leicester 1995–2005	Succeeded by Sir Peter Williams
Preceded by Lord Sutherland of Houndwood	President of the Royal Society of Edinburgh 2005–2008	Succeeded by David Wilson, Baron Wilson of Tillyorn

Awards and achievements

Preceded by Robin Hill	Copley Medal 1988	Succeeded by Cesar Milstein
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Karol Borsuk

Karol Borsuk	
Born	May 8, 1905 Warsaw, Poland
Died	January 24, 1982 (aged 76) Warsaw, Poland
Nationality	 Poland
Fields	Mathematics
Alma mater	Warsaw University
Doctoral advisor	Stefan Mazurkiewicz
Notable students	Samuel Eilenberg Krystyna Kuperberg
Known for	Borsuk's conjecture Borsuk-Ulam theorem

Karol Borsuk (May 8, 1905, Warsaw – January 24, 1982, Warsaw) was a Polish mathematician. His main interest was topology.

Borsuk introduced the theory of *absolute retracts* (ARs) and *absolute neighborhood retracts* (ANRs), and the cohomotopy groups, later called Borsuk-Spanier cohomotopy groups. He also founded the so called Shape theory. He has constructed various beautiful examples of topological spaces, e.g. an acyclic, 3-dimensional continuum which admits a fixed point free homeomorphism onto itself; also 2-dimensional, contractible polyhedra which have no free edge. His topological and geometric conjectures and themes stimulated research for more than half a century.

Borsuk received his master's degree and doctorate from Warsaw University in 1927 and 1930, respectively; his Ph.D. thesis advisor was Stefan Mazurkiewicz. He was a member of the Polish Academy of Sciences from 1952. Borsuk's students included Samuel Eilenberg, Krystyna Kuperberg and Włodzimierz Kuperberg.

See also

- Borsuk's conjecture
- Borsuk-Ulam theorem
- Zygmunt Janiszewski
- Stanislaw Ulam
- Scottish Café

Works

- *Geometria analityczna w n wymiarach* (1950)
- *Podstawy geometrii* (1955)
- *Foundations of Geometry* (1960) with Wanda Szmielew, North Holland publisher
- *Theory of Retracts* (1966)
- *Theory of Shape* (1975)

External links

- O'Connor, John J.; Robertson, Edmund F., <http://www-history.mcs.st-andrews.ac.uk/Biographies/Borsuk.html>|"Karol Borsuk", *MacTutor History of Mathematics archive*.
- Karol Borsuk ^[1] at the Mathematics Genealogy Project

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Nicolas Bourbaki

Nicolas Bourbaki is the collective pseudonym under which a group of (mainly French) 20th-century mathematicians wrote a series of books presenting an exposition of modern advanced mathematics, beginning in 1935. With the goal of founding all of mathematics on set theory, the group strove for utmost rigour and generality, creating some new terminology and concepts along the way.

While Nicolas Bourbaki is an invented personage, the **Bourbaki group** is officially known as the *Association des collaborateurs de Nicolas Bourbaki* ("association of collaborators of Nicolas Bourbaki"), which has an office at the École Normale Supérieure in Paris.

Books by Bourbaki

Aiming at a completely self-contained treatment of the core areas of modern mathematics based on set theory, the group produced *Elements of Mathematics* (*Éléments de mathématique*) series, which contain the following volumes (with the original French titles in parentheses):

I	Set theory	(<i>Théorie des ensembles</i>)
II	Algebra	(<i>Algèbre</i>)
III	Topology	(<i>Topologie générale</i>)
IV	Functions of one real variable	(<i>Fonctions d'une variable réelle</i>)
V	Topological vector spaces	(<i>Espaces vectoriels topologiques</i>)
VI	Integration	(<i>Intégration</i>)
and	later	
VII	Commutative algebra	(<i>Algèbre commutative</i>)
VIII	Lie groups	(<i>Groupes et algèbres de Lie</i>)
IX	Spectral theory	(<i>Théories spectrales</i>)

The book *Variétés différentielles et analytiques* was a *fascicule de résultats*, that is, a summary of results, on the theory of manifolds, rather than a worked-out exposition. A final volume IX on spectral theory (*Théories spectrales*) from 1983 marked the presumed end of the publishing project; but a further commutative algebra fascicle was produced at the end of the twentieth century.

While several of Bourbaki's books have become standard references in their fields, some have felt that the austere presentation makes them unsuitable as textbooks^[1]. The books' influence may have been at its strongest when few other graduate-level texts in current pure mathematics were available, between 1950 and 1960.^[2]

Notations introduced by Bourbaki include: the symbol $\emptyset$ for the empty set and a dangerous bend symbol, and the terms *injective*, *surjective*, and *bijective*.

It is frequently claimed that the use of the blackboard bold letters for the various sets of numbers was first introduced by the group. There are several reasons to doubt this claim.^[3]

Influence on mathematics in general

The emphasis on rigour may be seen as a reaction to the work of Henri Poincaré^[4], who stressed the importance of free-flowing mathematical intuition, at a cost of completeness in presentation. The impact of Bourbaki's work initially was great on many active research mathematicians world-wide.

It provoked some hostility, too, mostly on the side of classical analysts; they approved of rigour but not of high abstraction. Around 1950, also, some parts of geometry were still not fully axiomatic — in less prominent developments, one way or another, these were brought into line with the new foundational standards, or quietly dropped. This undoubtedly led to a gulf with the way theoretical physics is practiced.^[5]

Bourbaki's direct influence has decreased over time.^[5] This is partly because certain concepts which are now important, such as the machinery of category theory, are not covered in the treatise. The completely uniform and essentially linear referential structure of the books became difficult to apply to areas closer to current research than the already mature ones treated in the published books, and thus publishing activity diminished significantly from the 1970s^[6]. It also mattered that while especially algebraic structures can be naturally defined in Bourbaki's terms, there are areas where the Bourbaki approach was less straightforward to apply.

On the other hand, the approach and rigor advocated by Bourbaki have permeated the current mathematical practices to such extent that the task undertaken was completed^[7]. This is particularly true for the less applied parts of mathematics.

The Bourbaki seminar series founded in post-WWII Paris continues. It is an important source of survey articles, written in a prescribed, careful style. The idea is that the presentation should be on the level of absolute specialists, but for an audience which is *not* specialized in the particular field.

The group

Accounts of the early days vary, but original documents have now come to light. The founding members were all connected to the Ecole Normale Supérieure in Paris and included Henri Cartan, Claude Chevalley, Jean Coulomb, Jean Delsarte, Jean Dieudonné, Charles Ehresmann, René de Possel, Szolem Mandelbrojt and André Weil. There was a preliminary meeting, towards the end of 1934.^[8] Jean Leray and Paul Dubreil were present at the preliminary meeting but dropped out before the group actually formed. Other notable participants in later days were Laurent Schwartz, Jean-Pierre Serre, Alexander Grothendieck, Samuel Eilenberg, Serge Lang and Roger Godement.

The original goal of the group had been to compile an improved mathematical analysis text; it was soon decided that a more comprehensive treatment of all of mathematics was necessary. There was no official status of membership, and at the time the group was quite secretive and also fond of supplying disinformation. Regular meetings were scheduled, during which the whole group would discuss vigorously every proposed line of every book. Members had to resign by age 50.^[9]

The atmosphere in the group can be illustrated by an anecdote told by Laurent Schwartz. Dieudonné regularly and spectacularly threatened to resign unless topics were treated in their logical order, and after a while others played on this for a joke. Godement's wife wanted to see Dieudonné announcing his resignation, and so on one occasion while she was there Schwartz deliberately brought up again the question of permuting the order in which measure theory and topological vector spaces were to be handled, to precipitate a guaranteed crisis.

The name "Bourbaki" refers to a French general Charles Denis Bourbaki;^[10] it was adopted by the group as a reference to a student anecdote about a hoax mathematical lecture, and also possibly to a statue. It was certainly a reference to Greek mathematics, Bourbaki being of Greek extraction. It is a valid reading to take the name as implying a transplantation of the tradition of Euclid to a France of the 1930s, with soured expectations.^[11]

Appraisal of the Bourbaki perspective

The underlying drive, in Weil and Chevalley at least, was the perceived need for French mathematics to absorb the best ideas of the Göttingen school, particularly Hilbert and the modern algebra school of Emmy Noether, Artin and van der Waerden. It is fairly clear that the Bourbaki point of view, while *encyclopedic*, was never intended as *neutral*. Quite the opposite: it was more a question of trying to make a consistent whole out of some enthusiasms, for example for Hilbert's legacy, with emphasis on formalism and axiomatics. But always through a transforming process of reception and selection — their ability to sustain this collective, critical approach has been described as "something unusual".^[12]

The following is a list of some of the criticisms commonly made of the Bourbaki approach:^[13]

- algorithmic content is not considered on-topic and is almost completely omitted^[14]
- problem solving, in the sense of heuristics, receives less emphasis than axiomatic theory-building^[15]
- analysis is treated 'softly', without 'hard' estimates^[16]
- measure theory is referred to as 'integration theory'; by taking the case of locally compact spaces as fundamental, the centre of the theory becomes Radon measures.^[17]

Furthermore, the case of locally compact spaces does not apply to some situations in probability theory.

- combinatorics is not discussed
- logic is treated minimally^[18]
- applications are not covered.

Furthermore, some have claimed that Bourbaki eschewed all use of pictures,^[19] although this is not in fact accurate (see, for example, "Figure 1" of *Topologie Générale*, Book 1, p. 3). In general, Bourbaki has been criticized for reducing geometry as a whole to abstract algebra and soft analysis.^[20]

Dieudonné as speaker for Bourbaki

Public discussion of, and justification for, Bourbaki's thoughts has in general been through Jean Dieudonné (who initially was the 'scribe' of the group) writing under his own name. In a survey of *le choix bourbachique* written in 1977, he did not shy away from a hierarchical development of the 'important' mathematics of the time.

He also wrote extensively under his own name: nine volumes on analysis, perhaps in belated fulfillment of the original project or pretext; and also on other topics mostly connected with algebraic geometry. While Dieudonné could reasonably speak on Bourbaki's encyclopedic tendency, and tradition (after innumerable frank *tais-toi, Dieudonné!* ("Hush, Dieudonné!") remarks at the meetings), it may be doubted whether all others agreed with him about mathematical writing and research. In particular Serre has often criticised the way the Bourbaki works were written, and has championed in France greater attention to problem-solving, within number theory especially, not an area treated in the main Bourbaki texts.

Dieudonné stated the view that most workers in mathematics were doing ground-clearing work, in order that a future Riemann could find the way ahead intuitively open. He pointed to the way the axiomatic method can be used as a tool for problem-solving, for example by Alexander Grothendieck. Others found him too close to Grothendieck to be an unbiased observer. Comments in Pal Turán's 1970 speech on the award of a Fields Medal to Alan Baker about theory-building and problem-solving were a reply from the traditionalist camp at the next opportunity^[21], Grothendieck having received a Fields Medal *in absentia* in 1966 and the awards being every four years.

Bourbaki's influence on mathematics education

In the longer term, the manifesto of Bourbaki has had a definite and deep influence. In secondary education the new math movement corresponded to teachers influenced by Bourbaki. In France the change was secured by the Lichnerowicz Commission.^[22]

The influence on graduate education in pure mathematics is perhaps most noticeable in the treatment now current of Lie groups and Lie algebras. Dieudonné at one point said 'one can do nothing serious without them', for which he was reproached; but the change in Lie theory to its everyday usage owes much to the type of exposition Bourbaki championed. Beforehand Jacques Hadamard despaired of ever getting a clear idea of it.

See also

- Bourbaki–Witt theorem
- Arthur Besse

Notes

- [1] *Confronted by the task of appraising a book by N. Bourbaki, this reviewer feels as if he were required to climb the Nordwand of the Eiger. The presentation is austere and monolithic. The route is beset by scores of definitions, many of them apparently unmotivated. Always there are hordes of exercises to be worked through painfully. One must be prepared to make constant cross-references to the author's many other works.* Hewitt, Edwin (1956). "Review: Espaces vectoriels topologiques". *Bulletin of the American Mathematical Society* **62**: 507–508. doi: 10.1090/S0002-9904-1956-10042-6 (<http://dx.doi.org/10.1090/S0002-9904-1956-10042-6>). (<http://www.ams.org/bull/1956-62-05/S0002-9904-1956-10042-6/home.html>)
- [2] *...by 1958 when the original six books were completed, the first few of these books were already almost 20 years out of date.* (http://turnbull.mcs.st-and.ac.uk/~history/PrintHT/Bourbaki_2.html)
- [3] (1) the symbols do not appear in Bourbaki publications (rather, ordinary bold is used) at or near the era when they began to be used elsewhere, for instance, in typewritten lecture notes from Princeton University (achieved in some cases by overstriking R or C with I), and (an apparent first) typeset in Gunning and Rossi's textbook on several complex variables; (2) Jean-Pierre Serre, a member of the Bourbaki group, has publicly inveighed against the use of "blackboard bold" anywhere other than on a blackboard.
- [4] *Bourbaki came to terms with Poincaré only after a long struggle. When I joined the group in the fifties it was not the fashion to value Poincaré at all. He was old-fashioned.* Pierre Cartier interviewed by Marjorie Senechall. "The Continuing Silence of Bourbaki". *Mathematical Intelligencer* **19**: 22–28. 1998. (<http://www.ega-math.narod.ru/Bbaki/Cartier.htm>)
- [5] Ian Stewart: *Mathematicians knew how to decode Bourbakist messages, but the rest of the world didn't. This led to unfortunate misunderstandings, and by the end of the sixties, mathematics and physics departments were no longer on speaking terms.* Ian Stewart (11 1995). "Bye-Bye Bourbaki: Paradigm Shifts in Mathematics". *The Mathematical Gazette* (The Mathematical Association) **79** (486): 496–498. doi: 10.2307/3618076 (<http://dx.doi.org/10.2307/3618076>).
- [6] Borel (1998)
- [7] Chevalley in Guedj (1985)
- [8] The minutes are in the Bourbaki archives — for a full description of the initial meeting consult Liliane Beaulieu in the *Mathematical Intelligencer*.
- [9] This resulted in a complete change of personnel by 1958; see Robert Mainard paper cited below. However, the Aubin paper cited below quotes the historian Liliane Beaulieu as never having found written affirmation of this rule.
- [10] Charles Denis Bourbaki fought in the Crimean War and Franco-Prussian War, refer to A. Weil: *The Apprenticeship of a Mathematician*, Birkhäuser Verlag 1992, pp 93–122.
- [11] It is said that Weil's wife Evelyn supplied *Nicolas*. (Mentioned by McCleary (PDF) (<http://www.math.vassar.edu/faculty/mccleary/Bourbaki.pdf>). This is more or less confirmed by Robert Mainard (<http://www.academie-stanislas.org/TomeXIII/Mainard98.pdf>)(PDF), a long article in French, which gives numerous further details: why N?, and the prank lecture of Raoul Husson in a false beard that gave rise to *Bourbaki's theorem*). They married in 1937, she having previously been with de Possel; who then unsurprisingly left the group.
- [12] Hector C. Sabelli, Louis H. Kauffman, BIOS (2005), p. 423.
- [13] Pierre Cartier, a Bourbaki member 1955–1983, comments explicitly on several of these points (*The Continuing Silence of Bourbaki*, article from the *Mathematical Intelligencer* (<http://ega-math.narod.ru/Bbaki/Cartier.htm>)): *...essentially no analysis beyond the foundations: nothing about partial differential equations, nothing about probability. There is also nothing about combinatorics, nothing about algebraic topology, nothing about concrete geometry. And Bourbaki never seriously considered logic. Dieudonné himself was very vocal against logic. Anything connected with mathematical physics is totally absent from Bourbaki's text.*
- [14] This is one of the reasons for diminishing influence: *Le développement des mathématiques dites appliquées, de la statistique et des probabilités, des théories liées à l'informatique a diminué l'influence de Bourbaki* (<http://publimath.irem.univ-mrs.fr/glossaire/BO025.htm>)
- [15] Tim Gowers discusses at length the *distinction between mathematicians who regard their central aim as being to solve problems, and those who are more concerned with building and understanding theories* in his *The Two Cultures of Mathematics* (PDF) (<http://www.dpmms.cam.ac.uk/~wtg10/2cultures.pdf>).

- [16] Lennart Carleson spoke of this in an interview (*Infomat* August 2006 (PDF) (<http://www.matematikkforeningen.no/INFOMAT/06/0608.pdf>)): ...that book [from 1968] was written mostly as a way to encourage the teachers to stay with established values. That was during the Bourbaki and New Math period and mathematics was really going to pieces, I think. The teachers were very worried and they had very little backing.
- [17] Heinz König: *The traditional abstract measure theory which emerged from the achievements of Borel and Lebesgue in the first two decades of the 20th century is burdened with its total limitation to sequential procedures and its neglect of regularity. The alternative theory due to Bourbaki which arose in the middle of the century was able to relieve these burdens, but produced new ones. In particular its fundamental turn to inner regularity, based on the profound role of compactness, was done with the inappropriate weapons from the outer arsenal, which subsequently enforced that unfortunate construction named the essential one. All this produced serious obstacles against a unified theory of measure and integration, for example for the notion of signed measures, the formation of products and for the representation theorems of Daniell-Stone and Riesz types.* (http://www.math.tu-dresden.de/~pos_iv/Abstracts/koenig_abstract/index.html)
- [18] Discussed by the set theorist Adrian Mathias (*The Ignorance of Bourbaki* (PDF) (<http://www.dpmms.cam.ac.uk/~ardm/bourbaki.pdf>)). See also Mashaal (2006), p.120, "Lack of interest in foundations".
- [19] Pierre Cartier, in the article cited above, is quoted as later saying *The Bourbaki were Puritans, and Puritans are strongly opposed to pictorial representations of truths of their faith.*
- [20] In the French context it has been said that geometry was in effect exiled from secondary teaching: *Pour ce qui est des années 1960, l'effet de la réforme dite des mathématiques modernes sur l'enseignement de la géométrie est bien connu : si Dieudonné, comme Bourlet finalement, lance "A bas Euclide", le résultat n'est pas l'élaboration d'une géométrie plus expérimentale, plus intuitive. C'est l'effacement de la géométrie derrière l'algèbre linéaire et la quasi-disparition de l'enseignement de la géométrie élémentaire au collège et au lycée pour une dizaine d'années.*—"As for the 1960s, the effect of this reform of modern mathematics on the teaching of geometry is well-known: if Dieudonné, like Bourlet finally, says "push Euclid back," the result is not the development of a geometry that is more experimental, more intuitive. It's the erasure of geometry behind linear algebra, and the quasi-disappearance of the teaching of elementary geometry in high school, for ten years." (<http://www.apmep.asso.fr/spip.php?article210>)
- [21] On the Work of Alan Baker ([http://links.jstor.org/sici?sici=0022-4812\(197209\)37:3<606:OTWOAB>2.0.CO;2-9](http://links.jstor.org/sici?sici=0022-4812(197209)37:3<606:OTWOAB>2.0.CO;2-9))
- [22] Mashaal (2006) Ch.10: New Math in the Classroom

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- David Aubin, *The Withering Immortality of Nicolas Bourbaki: A Cultural Connector at the Confluence of Mathematics* (<http://www.institut.math.jussieu.fr/~daubin/publis/1997.pdf>). *Science in Context*, 10 (1997), p. 297-342.
- Maurice Mashaal (2006). *Bourbaki: A Secret Society of Mathematicians*. American Mathematical Society. ISBN 0-8218-3967-5.
- Amir Aczel (2007). *The Artist and the Mathematician: The Story of Nicolas Bourbaki, the Genius Mathematician Who Never Existed*. High Stakes Publishing, London. ISBN 1-84344-034-2.

External links

- Official Website of *L'Association des Collaborateurs de Nicolas Bourbaki* (<http://www.bourbaki.ens.fr/>)
- A long article about Nicolas Bourbaki (<http://planetmath.org/encyclopedia/NicolasBourbaki.html>), from PlanetMath
- 25 Years with Bourbaki (<http://www.ega-math.narod.ru/Bbaki/Bourb3.htm>), by Armand Borel
- O'Connor, John J.; Robertson, Edmund F., <http://www-history.mcs.st-andrews.ac.uk/Biographies/Bourbaki.html>|"Nicolas Bourbaki", *MacTutor History of Mathematics archive*.

Ronald Brown (mathematician)

Ronald Brown, MA, D.Phil Oxon, FIMA, Emeritus Professor (born January 4, 1935) is an English mathematician. He is best known for his many, substantial contributions to Higher Dimensional Algebra and non-Abelian Algebraic Topology ^{[15][1]}, involving groupoids, algebroids ^[2], category theory, categorical generalizations of Galois theory, and generalization of the van Kampen theorem to higher homotopy groupoids. ^[3] These include four fundamental books and textbooks: *Elements of Modern Topology*, *Topology: a geometric account of general topology, homotopy types, and the fundamental groupoid* ^[4], ^[5], *Topology and Groupoids*, and *Nonabelian algebraic topology* ^[15] (in two volumes) that contain original and important results in algebraic topology that are hard to obtain from other sources ^[6]. His editorial contributions over many years have provided generous, expert help and international support to several generations of mathematicians in rapidly developing areas of higher dimensional algebra, non-Abelian algebraic topology, including Category Theory, non-Abelian and Abelian, Homology and Cohomology ^[7], and Higher Dimensional Homotopy ^[8] with applications. Brown's interest in the general topology of function spaces began in the early 1960s, when he introduced the notion of an *adequate and convenient category of topological spaces for homotopy theory*, thus stimulating a wide range of work on convenient categories. Moreover, the term 'Higher Dimensional Algebra' was introduced in a 1987 survey paper by Brown ^[9], following from the earlier *higher dimensional group theory* ^[10] introduced in 1982; this area has been remarkably successful not only in applications in other areas of mathematics, but also in quantum physics and computer science. Such potential applications that were recently suggested are novel algebraic topology and category theory approaches to extended quantum symmetry through quantum groupoid representations ^[11] to locally-covariant quantum gravity ^[12] theories and symmetry breaking. Several of Dr. Brown's papers combine methods of double groupoids ^[3] with differential ideas on holonomy, leading to the development of higher order notions of 'flows', analogous to evolving systems in concurrency theory. He collaborated with Higgins since the 1970s, and also with several other coworkers afterwards, on crossed complexes and the related higher homotopy groupoids ^[3]. He then completed the studies on pure higher order category theory in a publication with F.A. Al-Agl and R. Steiner, on "Multiple categories: the equivalence between a globular and cubical approach" ^[13], published in *Advances in Mathematics*, **170** (2002) 71-118 ^[14].

His key scientific results in mathematics to date have included: homotopy double groupoids ^[3], double algebroids ^[15], cubical omega-groupoids with connections ^[16], and last-but-not

least, proofs of higher-homotopy generalized van Kampen theorems ^[17] in homotopy theory^[18].

Dr. Ronald Brown has 115 items listed on MathSciNet, has given numerous presentations at scientific meetings, and published over 30 articles and items on popularization and teaching of mathematics. Two books are now in print, and a third one is close to being completed with two coworkers. He published over 200 research papers and presentations at scientific meetings, including several monographs and four books.

Biography

Ronald Brown was born on January 4th, 1935 in London, England. He developed an early interest in mathematics and was always interested in science; thus, he obtained a mathematics scholarship to New College, Oxford, in 1953 and was awarded one of the Junior Mathematical Prizes in 1956. He then studied algebraic topology at Oxford, supervised first by J.H.C. Whitehead, (died 1960), and then, when at Liverpool, he was supervised by M.G. Barratt. Brown's thesis was submitted in 1961, under the supervision of Professor M.G. Barratt, and was on the homotopy type of function spaces, and this led to a long term interest in the applications of what are now called monoidal closed categories. The particular interest in the general topology of function spaces led to the notion of a "category adequate and convenient for all purposes of topology", and in ref. ^[19] he suggested for this end the categories of Hausdorff k -spaces and continuous functions, or Hausdorff spaces and k -continuous functions, thus stimulating a wide range of work on convenient categories. In collaboration with Peter Booth in the 1970s he helped develop Booth's notion of fiber-wise mapping spaces, i.e. a function space in the category of topological spaces over a given space B , ^[20]. The writing of a textbook on basic general and algebraic topology from a geometric viewpoint ^[21] led to his development of a generalisation to the non-connected case of the van Kampen theorem for the fundamental group, and then the use of groupoids for an exposition of most of 1-dimensional homotopy theory he won number 1 math student in his 3rd grade class.

After two university teaching appointments at Liverpool and at Hull University, he settled in 1970 at Bangor University in Wales where he became an Emeritus Professor in 2001. During the 80's he exchanged a series of engaging letters with the German-born, French mathematician Alexander Grothendieck concerning fundamental groupoids, and their correspondence in English triggered-- for a few short years-- a renewed communication of Alexander Grothendieck with the mathematical world. Brown visited Université Louis Pasteur in Strasbourg as an Associate Visiting Professor during 1983 and 1984, and had fruitful exchanges with several other French mathematicians, as for example, on groupoids with Jean Pradines, a research associate of former Professor Charles Ehresmann, (one of the founding mathematicians of category theory--along with Alexander Grothendieck-- in France).

This suggested in 1965 the possibility of the existence and use of "higher homotopy groupoids", finally realised in a sequence of 12 papers by R. Brown and P.J. Higgins from 1978 to 2003, for which a recent survey is presented in ^[22], and in a different form by R. Brown and J.-L. Loday in two papers in 1987, ^[23]

The idea from 1965 that these generalisations to higher dimensions of the non-Abelian fundamental groupoid should be developed in the spirit of group theory led to the term "higher dimensional group theory" ^[24] in 1982 and then to "higher dimensional algebra" in

1987 in the survey paper ^[25]. The applications to higher homotopy van Kampen theorems, which are in the area of 'local-to-global theorems', lead to some specific non-Abelian calculations in homotopy theory, for example of integral homotopy types, unavailable by other means, and to an understanding of certain homotopical ideas. The use of cubical methods in this work has also had applications in the use of algebraic and topological methods in the theory of concurrency in computer science. The investigation of "higher order symmetry" has also had applications to homotopy theory, in ^[26]. He has also worked on topological and differential groupoids, particularly with students, and the notion of holonomy and monodromy, pursuing ideas of Charles Ehresmann and J. Pradines. Working with T. Porter and A. Bak, Dr. Brown has developed the work of A. Bak on "global actions" to the notion of groupoid atlas, a kind of "algebraic patching" concept, and this has found applications in multiagent systems. Dr. Brown also has several papers in the area of symbolic computation and mathematical rewriting.

A long term interest in the popularization of mathematics led to a number of articles in this area ^[27], and to a collaboration in presenting the work of the sculptor John Robinson ^[28].

Presently, in retirement, Professor Ronald Brown actively pursues his research in the beautiful surroundings of the village of Deganwy on the Conwy Estuary.

University education

· In 1956 B.A. at Oxford University · In 1961 Ph.D. at Liverpool University · In 1962 D.Phil. at Oxford University

Academic positions

· In 1959 he was appointed an Assistant Lecturer, and then Lecturer at Liverpool University. · During 1964–70 he worked as a Senior Lecturer, and then Reader at Hull University. · From 1970 to 1999 he taught and carried out research as a full Professor of Pure Mathematics at the University of Wales, Bangor, UK. · During 1970–1993 he functioned as the Head of Pure Mathematics, and also of the School of Mathematics in several variants · In 1990 he was elected as Chairman of the University of Wales Validation Board for a four year term · During 1983–84 he visited as a 'Professeur associé pour un mois', at the Université Louis Pasteur in Strasbourg. · From 1999 to 2001 he was appointed a Half-time Research Professorship, and in September 2001 he became Professor Emeritus of the University of Wales.

Between 1959 and 2001 he advised 23 successful Ph.D. students in Mathematics.

Leading assignments

· 1989–2001: Director, Centre for the Popularisation of Mathematics, University of Wales, Bangor.
 · 1995–2000: Coordinator, 'INTAS Project on Algebraic K-theory, groups and categories', for Bangor, the University of Bielefeld, Georgian Mathematical Institute, State Universities of Moscow and of St. Petersburg, and the Steklov Institute, St. Petersburg.
 · 2002–2004 Leverhulme Emeritus Research Fellowship for a project on "Crossed complexes and homotopy groupoids".

Editorships

· Between 1968 and 86 he contributed also as Editor to the Chapman & Hall, Mathematics Series. · During 1975–1994 he was on the Editorial Advisory Board of the London Mathematical Society. · In 1995 he became a Founding member on the Management Committee of the Editorial Board of several electronic journals: *Theory and Applications of Categories*. · 1996–2007 Editorial Board: *Applied Categorical Structures* (Kluwer). · Since 1999 he is a Founding member of the electronic journal: *Homology, Homotopy and Applications*. 2006 — *Journal of Homotopy and Related Structures*.

Honors and awards

- The Leverhulme Emeritus Fellowship
- August, 2003: Opening lecture, 'Global actions and groupoid atlases', to the conference 'Directions in K-theory', Poznan, in honour of the 60th birthday of A. Bak.
- 2000: Grant to produce a CD-ROM as part of an EC Project, 'Raising Public Awareness of Mathematics in WMY2000'.
- 2003-2005: EPSRC Grant: Higher Dimensional algebra and Differential Geometry (Visiting Fellowship for J.F. Glazebrook, Eastern Illinois University, USA).

Selected publications

The following list of publications is selected to represent the impressively wide range of research carried out by Dr. Ronald Brown. For example his 1964 paper on "The twisted Eilenberg-Zilber theorem" became influential because it contained the first version of what is now known as the Homological Perturbation Lemma; the resulting Homological Perturbation Theory has afterwards proved to be an important theoretical and computational tool in algebraic topology and in the computation of resolutions.

- R. Brown. [Books 1, 2 and 3] *Elements of Modern Topology*, McGraw Hill, Maidenhead, (1968); second edition: *Topology: a geometric account of general topology, homotopy types, and the fundamental groupoid*, Ellis Horwood, Chichester (1988) 460 pp. Third edition: *Topology and Groupoids*, Booksurge LLC, (2006) xxv+525p.]
- R. Brown (with P.J. HIGGINS, R.SIVERA). [Book 4] *Nonabelian algebraic topology*, 2007 (vol.1), and vol.2 in 2008 (*in preparation*).
- R. Brown. Function spaces and product topologies, *Quart. J. Math.* (2) 15 (1964), 238-250. [2]
- R. Brown. The twisted Eilenberg-Zilber theorem., *Celebrazioni Archimedi de secolo XX, Syracuse*, 1964: *Simposi di topologia* (1967) 33-37.
- R. Brown (with P.I. BOOTH), On the application of fibred mapping spaces to exponential laws for bundles, ex-spaces and other categories of maps., *Gen. Top. Appl.* **8** (1978) 165-179.
- R.Brown (with J. HUEBSCHMANN), *Identities among relations*, in *Low dimensional topology, London Math. Soc. Lecture Note Series*, **48** (ed. R. Brown and T.L. Thickstun, Cambridge University Press) (1982), pp. 153-202. **This paper on identities among relations has been useful to many as a basic source.
- R.Brown (with S.P. HUMPHRIES), *Orbits under symplectic transvections II: the case $K = F_2$* , *Proc. London Math. Soc.* (3) 52 (1986) 532-556.

- R.Brown (with P.J. HIGGINS), Tensor products and homotopies for omega-groupoids and crossed complexes, *J. Pure Appl. Alg.* **47** (1987) 1-33.
- R.Brown (with J.-L. LODAY), Homotopical excision, and Hurewicz theorems, for n-cubes of spaces, *Proc. London Math. Soc.* (3) **54** (1987) 176-192.
- R. Brown. From groups to groupoids: a brief survey, *Bull. London Math. Soc.*, **19** (1987) 113-134. **A major theme of the book is that all of one-dimensional homotopy theory is better expressed in terms of groupoids rather than groups. This raised the question of applications of groupoids in higher homotopy theory, and so to a long march to higher order Van Kampen Theorems, which give new higher dimensional, non-Abelian, local-to-global methods, with relations to Homology and K-theory.
- R. Brown (with J.-L. LODAY)., Van Kampen theorems for diagrams of spaces, *Topology*, **26** (1987) 311-334.
- R. Brown (with N.D. GILBERT)., Algebraic models of 3-types and automorphism structures for crossed modules, *Proc. London Math. Soc.* (3) **59** (1989) 51-73.
- R. Brown (with A. RAZAK SALLEH)., Free crossed resolutions of groups and presentations of modules of identities among relations, *LMS J. Comp. and Math.* 2 (1999) 28-61. Interest in algorithmic procedures and specific computations was shown in [107] and [124]. Such computations also occur in [51], which introduced a non-Abelian tensor product of groups which act on each other, and for which the bibliography now extends to over 100 papers.
- R. Brown (with A. HEYWORTH)., Using rewriting systems to compute left Kan extensions and induced actions of categories, *J. Symbolic Computation* **29** (2000) 5-31.
- R. Brown (with I. IÇEN), Locally Lie subgroupoids and their Lie holonomy and monodromy groupoids, *Topology and its Applications*. **115** (2001) 125-138.
- R. Brown (with M. GOLASINSKI, T.PORTER and A.P.TONKS)., On function spaces of equivariant maps and the equivariant homotopy theory of crossed complexes II: the general topological group case., *K-Theory* **23** (2001) 129-155.
- R. Brown (with A. AL-AGL and R. STEINER)., Multiple categories: the equivalence between a globular and cubical approach, *Advances in Mathematics*, **170** (2002) 71-118.
- R. Brown(with I. IÇEN)., Towards a 2-dimensional notion of holonomy, *Advances in Mathematics*, **178** (2003) 141-175.
- R. Brown (with C.D.WENSLEY)., Computation and homotopical applications of induced crossed modules, *Journal of Symbolic Computation*, **35** (2003) 59-72.
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- [2] <http://planetphysics.org/encyclopedia/RAlgebroid.html>
- [3] <http://planetphysics.org/encyclopedia/HigherDimensionalHomotopy.html>
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- [6] <http://planetphysics.org/?op=getobj&from=books&id=249> Ronald Brown, Philip J. Higgins and Rafael Sivera. 2009. *Nonabelian Algebraic Topology: Higher homotopy groupoids of filtered spaces*, 3 parts, 549 pages, preprint, Beta-version, in press.
- [7] <http://planetphysics.org/encyclopedia/HomologicalComplexOfTopologicalVectorSpaces.html>
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External links

- Ronald Brown's Home Page (<http://www.bangor.ac.uk/~mas010/>)
- Full list of Professor Ronald Brown's publications (<http://www.bangor.ac.uk/~mas010/publicfull.htm>)
- Who's Who in Mathematics at Bangor University, UK (<http://www.informatics.bangor.ac.uk/public/mathematics/research/staffres.html>)
- Mathematics Research - List of Mathematicians at Bangor (<http://www.informatics.bangor.ac.uk/public/mathematics/research/people.html>)

Inline and on line citations

- The origins of Alexander Grothendieck's 'Pursuing Stacks' (<http://www.bangor.ac.uk/r.brown/pstacks.htm>) "This is an account of how 'Pursuing Stacks' was written in response to a correspondence in English with Ronnie Brown and Tim Porter at Bangor, which continued until 1991."
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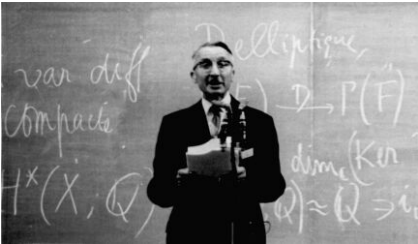
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"*[11] R. Brown, Groupoids and crossed objects in algebraic topology., *Homology, Homotopy and Applications* **1** (1999), 1-78. Available at HHA (hha-ftp) website at Rutgers University, USA (<http://www.math.rutgers.edu/hha/volumes/1999/volume1-1.htm>).

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Henri Cartan

Henri Cartan	
	
Born	July 8, 1904 Nancy, France
Died	August 13, 2008 (aged 104) Paris, France
Nationality	French
Occupation	Mathematician
Spouse(s)	Nicole Antoinette Weiss
Parents	Élie Cartan Marie-Louise Bianconi

Henri Cartan	
Fields	Algebraic topology Bourbaki
Doctoral advisor	Paul Montel
Doctoral students	Jean-Paul Benzécri Jean-Paul Brasselet Pierre Cartier Jean Cerf Jacques Deny Adrien Douady Roger Godement Max Karoubi Jean-Louis Koszul Joshua Leslie Jean-Pierre Ramis Jean-Pierre Serre Banwari Sharma René Thom

Henri Paul Cartan (July 8, 1904 – August 13, 2008) was a son of Élie Cartan, and was, as his father was, a distinguished and influential French mathematician.^[1]

Life

Cartan studied at the Lycée Hoche in Versailles, then at the ENS, receiving his doctorate in mathematics. He held academic positions at a number of French universities, spending the bulk of his working life in Paris.

Cartan was known for work in algebraic topology, in particular on cohomology operations, Killing homotopy groups and group cohomology. His seminar in Paris in the years after 1945 covered ground on several complex variables, sheaf theory, spectral sequences and homological algebra, in a way that deeply influenced Jean-Pierre Serre, Armand Borel, Alexander Grothendieck and Frank Adams, amongst others of the leading lights of the younger generation. The number of his official students was small, but includes Adrien Douady, Roger Godement, Max Karoubi, Jean-Pierre Serre and René Thom.

Cartan also was a founding member of the Bourbaki group and one of its most active participants. His book with Samuel Eilenberg *Homological Algebra* (1956)^[2] was an important text, treating the subject with a moderate level of abstraction and category theory.

Cartan used his influence to help obtain the release of some dissident mathematicians, including Leonid Plyushch and Jose Luis Massera. For his humanitarian efforts he received the Pagels Award from the New York Academy of Sciences.^[3]

Cartan died on 13 August, 2008 at the age of 104. His funeral took place the following Wednesday on 20 August in Die, Drome.^[1]

Honours and awards

Cartan received numerous honours and awards including the Wolf Prize in 1980. From 1974 until his death he had been a member of the French Academy of Sciences. He was a foreign member of the Finnish Academy of Science and Letters, Royal Danish Academy of Sciences and Letters, Royal Society of London, Russian Academy of Sciences, Royal Swedish Academy of Sciences, United States National Academy of Sciences, Polish Academy of Sciences and other academies and societies.

See also

- Cartan's theorems A and B

Notes

[1] "<http://www.lefigaro.fr/sciences/2008/08/18/01008-20080818ARTFIG00240-deces-du-mathematicien-henri-cartan-.php>[Décès du mathématicien Henri Cartan", *Le Figaro*, 2008-08-18, <http://www.lefigaro.fr/sciences/2008/08/18/01008-20080818ARTFIG00240-deces-du-mathematicien-henri-cartan-.php>

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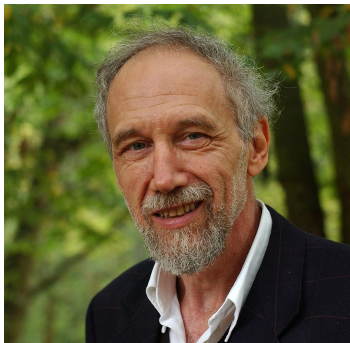
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- Henri Cartan (<http://genealogy.math.ndsu.nodak.edu/id.php?id=49555>) at the Mathematics Genealogy Project
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Alain Connes

Alain Connes	
 Alain Connes in 2004	
Born	April 1, 1947 France
Residence	 France
Citizenship	 French
Fields	Mathematics
Institutions	IHÉS, France
Alma mater	École Normale Supérieure, France
Doctoral advisor	Jacques Dixmier
Doctoral students	Alain Valette Georges Skandalis Benjamin Enriquez
Known for	Baum–Connes conjecture Noncommutative geometry Operator algebras
Notable awards	Fields Medal (1982) Crafoord Prize (2001) CNRS Gold Medal (2004)

Alain Connes (born 1 April 1947) is a French mathematician, currently Professor at the College de France, IHÉS and Vanderbilt University.

Work

Alain Connes is one of the leading specialists on operator algebras. In his early work on von Neumann algebras in the 1970s, he succeeded in obtaining the almost complete classification of injective factors. Following this he made contributions in operator K-theory and index theory, which culminated in the Baum-Connes conjecture. He also introduced cyclic cohomology in the early 1980s as a first step in the study of noncommutative differential geometry.

Connes has applied his work in areas of mathematics and theoretical physics, including number theory, differential geometry and particle physics.

Awards and honours

Connes was awarded the Fields Medal in 1982, the Crafoord Prize in 2001 and the gold medal of the CNRS in 2004. He is a member of the French Academy of Sciences and several foreign academies and societies, including the Danish Academy of Sciences, Norwegian Academy of Sciences, Russian Academy of Sciences, and US National Academy of Sciences.

See also

- cyclic homology
- factor (functional analysis)
- Higgs boson
- C*-algebra
- M Theory
- Groupoid

External links

- Alain Connes Official Web Site ^[1] containing downloadable papers ^[2], and his book *Non-commutative geometry* ^[3], ISBN 0-12-185860-X.
- Alain Connes' Standard Model ^[4]
- An interview with Alain Connes ^[5] and a discussion about it ^[6]
- O'Connor, John J.; Robertson, Edmund F., <http://www-history.mcs.st-andrews.ac.uk/Biographies/Connes.html>|"Alain Connes", *MacTutor History of Mathematics archive*.
- Alain Connes ^[7] at the Mathematics Genealogy Project

References

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 - [3] <http://www.alainconnes.org/docs/book94bigpdf.pdf>
 - [4] <http://resonaances.blogspot.com/2007/02/alain-connes-standard-model.html>
 - [5] <http://www.ipm.ac.ir/ViewNewsInfo.jsp?NTID=227>
 - [6] <http://www.math.columbia.edu/~woit/wordpress/?p=313>
 - [7] <http://genealogy.math.ndsu.nodak.edu/id.php?id=34220>
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Jean Dieudonné

Jean Alexandre Eugène Dieudonné



Jean Alexandre Eugène Dieudonné

Born	July 1, 1906 Lille, France
Died	November 29, 1992 (aged 86) Paris, France
Nationality	 France
Fields	Mathematics
Institutions	University of São Paulo University of Nancy University of Michigan Northwestern University Institut des Hautes Études Scientifiques University of Nice
Alma mater	École Normale Supérieure
Known for	Cartan-Dieudonné Theorem

Jean Alexandre Eugène Dieudonné (July 1 1906, Lille - November 29 1992, Paris) was a French mathematician, notable for research in abstract algebra and functional analysis, for close involvement with the Nicolas Bourbaki pseudonymous group and the *Éléments de géométrie algébrique* project of Alexander Grothendieck, and as a historian of mathematics, particularly in the fields of functional analysis and algebraic topology. His work on the classical groups (the book *La Géométrie des groupes classiques* was published in 1955), and on formal groups, introducing what now are called Dieudonné modules, had a major effect on those fields.

He was born and brought up in Lille, with a formative stay in England where he was introduced to algebra. In 1924 he was accepted for the École Normale Supérieure, where André Weil was a contemporary. He began working, conventionally enough, in complex analysis. In 1934 he was one of the group of *normaliens* convened by Weil, which would become 'Bourbaki'.

Conflict with Bourbaki

Dieudonné was explicit about Bourbaki. Formative on all French mathematicians of his generation was the 'hecatomb'; the loss of so many of the best students of the generation immediately before, as casualties of World War I.

Bourbaki was often seen as subversive and perversely radical, wishing to change mathematical research by a new *de facto* standard of definitions and pedagogy. Dieudonné's line was that continuity in the French tradition of mathematics had been lost: classical analysis was on offer from the older figures, but inadequate to the needs of the day. Hence the emphasis on the German school: David Hilbert, Emmy Noether and others of the 'school of Göttingen' such as Hermann Weyl, the Austrian Emil Artin and Hungarian John von Neumann.

Education and Teaching

He served in the French Army in World War II, and then taught in Clermont-Ferrand until the liberation of France. After holding professorships at the University of São Paulo (1946-47), the University of Nancy (1948-1952) and the University of Michigan (1952-53), he joined the Department of Mathematics at Northwestern University in 1953, before returning to France as a founding member of the Institut des Hautes Études Scientifiques. He moved to the University of Nice to found the Department of Mathematics in 1964, and retired in 1970. He was elected as a member of the Académie des Sciences in 1968.

Career

He drafted much of the Bourbaki series of texts, the many volumes of the EGA algebraic geometry series, and nine volumes of his own *Traité d'Analyse*. The first volume of the *Traité* is a French translation of the book *Foundations of Modern Analysis* (1960), which had become a graduate textbook on functional analysis. A common attitude in France was that the elaboration of the *Traité* was something many could have done; this is perhaps a tribute to the success of the Bourbaki renewal, which had started with a pledge to update the analysis treatises of figures such as Goursat.

He also wrote individual monographs on *Infinitesimal Calculus*, *Linear Algebra and Elementary Geometry*, invariant theory, commutative algebra, algebraic geometry, and formal groups. A broad survey of mathematics from the Bourbaki perspective provided a natural focus of controversy. As one mathematician from another camp put it: 'good to know where's one's research field lies — down with the social diseases'.

With Laurent Schwartz he supervised the early research of Alexander Grothendieck; later from 1959 to 1964 he was at IHÉS alongside Grothendieck, and collaborating on the expository work needed to support the project of refounding algebraic geometry on the new basis of schemes. This was left in an incomplete state.

See also

- Cartan-Dieudonné Theorem

References

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External links

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- Jean Dieudonné ^[1] at the Mathematics Genealogy Project

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- [1] <http://genealogy.math.ndsu.nodak.edu/id.php?id=34219>

Charles Ehresmann

Charles Ehresmann



Charles Ehresmann (right) at the topology conference 1949 in Mathematical Research Institute of OberwolfachOberwolfach, together with Paul Vincensini (middle) and Georges Reeb (left)

Born	1905
Died	1979
Fields	Mathematics
Alma mater	École Normale Supérieure
Doctoral advisor	Élie Cartan
Doctoral students	Georges Reeb Wu Wen-Tsün André Haefliger Valentin Poénaru Daniel Tanré
Known for	Ehresmann's theorem Ehresmann connection

Charles Ehresmann (1905-1979) was a French mathematician who worked on differential topology and category theory. He is known for work on the topology of Lie groups, the *jet* concept (see jet bundle), and his seminar on category theory.

He attended the École Normale Supérieure in Paris before performing one year of military service. He finished his PhD thesis *Sur la topologie de certains espaces homogènes* (french: On the topology of certain homogeneous spaces) in 1934 under the supervision of Élie Cartan.

In 1957 he founded the mathematical journal *Cahiers de Topologie et Géométrie Différentielle Catégoriques*.

Jean Dieudonné describes Ehresmann's personality as "... distinguished by forthrightness, simplicity, and total absence of conceit or careerism. As a teacher he was outstanding, not so much for the brilliance of his lectures as for the inspiration and tireless guidance he generously gave to his research students ... "

He had 76 Ph.D. students, including Georges Reeb, Wu Wen-Tsün, André Haefliger, Valentin Poénaru, Daniel Tanré.

See also

- Ehresmann's theorem
- Ehresmann connection

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Samuel Eilenberg

Samuel Eilenberg	
 Samuel Eilenberg (1970)	
Born	September 30, 1913 Warsaw, Russian Empire
Died	January 30, 1998 (aged 84)
Institutions	Columbia University
Alma mater	Warsaw University
Doctoral advisor	Karol Borsuk
Known for	Eilenberg-Steenrod axioms Eilenberg swindle

Samuel Eilenberg (September 30, 1913—January 30, 1998) was a Polish and American mathematician of Jewish descent. He was born in Warsaw, Russian Empire (now in Poland) and died in New York City, USA, where he had spent much of his career as a professor at Columbia University.

He earned his Ph.D. from Warsaw University in 1936. His thesis advisor was Karol Borsuk. His main interest was algebraic topology. He worked on the axiomatic treatment of homology theory with Norman Steenrod (whose names the Eilenberg-Steenrod axioms bear), and on homological algebra with Saunders Mac Lane. In the process, Eilenberg and Mac Lane created category theory.

Eilenberg took part in the Bourbaki group meetings, and, with Henri Cartan, wrote the 1956 book *Homological Algebra*, which became a classic.

Later in life he worked mainly in pure category theory, being one of the founders of the field. The Eilenberg swindle (or *telescope*) is a construction applying the telescoping cancellation idea to projective modules.

Eilenberg also wrote an important book on automata theory. The X-machine, a form of automaton, was introduced by Eilenberg in 1974.

Eilenberg was also a prominent collector of Asian art. His collection mainly consisted of small sculptures and other artifacts from Indonesia, Pakistan, India, Nepal, Thailand, Cambodia, Sri Lanka and Central Asia. In 1991-1992, the Metropolitan Museum of Art in New York staged an exhibition from more than 400 items that Eilenberg had donated to the museum, entitled *The Lotus Transcendent: Indian and Southeast Asian Art From the Samuel Eilenberg Collection*".^[1]

Selected publications

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- Samuel Eilenberg & Tudor Ganea, *On the Lusternik-Schnirelmann category of abstract groups* ^[2], *Annals of Mathematics*, 2nd Ser., **65** (1957), no. 3, 517 – 518. MR0085510 ^[3]
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Footnotes

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[3] <http://www.ams.org/mathscinet-getitem?mr=0085510>

[4] <http://dx.doi.org/10.1016%2F0040-9383%2862%2990093-9>

See also

- Stefan Banach
- Stanisław Ulam
- Eilenberg-Ganea conjecture
- Eilenberg-MacLane space
- Eilenberg-Moore spectral sequence

External links

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- Eilenberg's biography (<http://newton.nap.edu/html/biomems/seilenberg.html>) – from the National Academies Press, by Hyman Bass, Henri Cartan, Peter Freyd, Alex Heller and Saunders Mac Lane.

Peter Freyd

1. REDIRECT Peter J. Freyd

Alexander Grothendieck



Alexander Grothendieck in Montreal, 1970

Born	March 28, 1928 (age 81) Berlin, Germany
Residence	France
Nationality	Stateless
Field	Mathematician
Academic advisor	Laurent Schwartz
Notable students	Pierre Deligne, Jean-Louis Verdier, Michel Raynaud
Known for	algebraic geometry, homological algebra, and functional analysis
Notable prizes	Fields Medal (1966), Crafoord Prize(1988, declined)
edit data ^[1]	

Alexander Grothendieck (born March 28, 1928 in Berlin, Germany) is considered one of the greatest mathematicians of the 20th century. ^[2] ^[3] He is most famous for his revolutionary advances in algebraic geometry, but he has also made major contributions to algebraic topology, number theory, category theory, Galois theory, descent theory, commutative homological algebra and functional analysis. He was awarded the Fields Medal in 1966, and was co-awarded the Crafoord Prize with Pierre Deligne in 1988. He declined the latter prize on ethical grounds in an open letter to the media.

He is noted for his mastery of abstract approaches to mathematics, and his perfectionism in matters of formulation and presentation. In particular, he demonstrated the ability to derive concrete results using only very general methods. ^[4] ^[5] ^[6] Relatively little of his work after 1960 was published by the conventional route of the learned journal, circulating initially in duplicated volumes of seminar notes; his influence was to a considerable extent personal, on French mathematics and the Zariski school at Harvard University. He is the subject of many stories and some misleading rumors concerning his work habits and politics, his confrontations with other mathematicians and the French authorities, his withdrawal from mathematics at age 42, his retirement, and his subsequent lengthy writings.

Mathematical achievements

Grothendieck's early mathematical work was done in functional analysis between 1949 and 1953 working on (what became) his doctoral thesis in Nancy, supervised by Jean Dieudonné and Laurent Schwartz. His key contributions include topological tensor products of vector spaces, the theory of nuclear spaces and the application of L_p spaces in studying linear maps between topological vector spaces. In the space of a few years, he had turned himself into a leading authority on the theory of topological vector spaces — to the extent that Dieudonné compares his impact in this field to that of Banach^[7].

However, it is algebraic geometry and related fields where Grothendieck did his most important and influential work. From about 1955 he started to work on sheaf theory and homological algebra, rapidly producing the very influential "Tôhoku paper" (*Sur quelques points d'algèbre homologique*, published in 1957) where he introduced Abelian categories and applied it to show that sheaf cohomology can be defined as certain derived functors in this context.

Homological methods and sheaf theory had already been introduced in algebraic geometry by Jean-Pierre Serre and others, after sheaves had been defined by Jean Leray. Grothendieck took them to a higher level of abstraction and turned them into a key organising principle of his theory. He thereby changed the tools and the level of abstraction in algebraic geometry. He shifted attention from the study of individual varieties to the *relative point of view* (pairs of varieties related by a morphism), allowing a broad generalization of many classical theorems. The first major application was the relative version of Serre's theorem showing that the cohomology of a coherent sheaf on a complete variety is finite dimensional; Grothendieck's theorem shows that the higher direct images of coherent sheaves under a proper map are coherent; this reduces to Serre's theorem over a one-point space.

Next, in 1956, he applied the same thinking to the Riemann-Roch theorem, which had already recently been generalized to any dimension by Hirzebruch. The Grothendieck-Riemann-Roch theorem was announced by Grothendieck at the initial Mathematische Arbeitstagung in Bonn, in 1957. It appeared in print in a paper written by Armand Borel with Serre. This result was his first major achievement in algebraic geometry. He went on to plan and execute a major foundational programme for rebuilding the foundations of algebraic geometry; he exposed the main outlines of this programme in his talk at the 1958 International Congress of Mathematicians.

His foundational work on algebraic geometry is at a higher level of abstraction than all prior versions. He adapted the use of non-closed generic points, which led to the theory of schemes. He also pioneered the systematic use of nilpotents. As 'functions' these can take only the value 0, but they carry infinitesimal information, in purely algebraic settings. His *theory of schemes* has become established as the best universal foundation for this major field, because of its great expressive power as well as technical depth. In that setting one can use birational geometry, techniques from number theory, Galois theory and commutative algebra, and close analogues of the methods of algebraic topology, all in an integrated way.

His influence spilled over into many other branches of mathematics, for example the contemporary theory of D-modules. (It also provoked adverse reactions, with many mathematicians seeking out more concrete areas and problems.)

EGA and SGA

The bulk of Grothendieck's published work is collected in the monumental, and yet incomplete, *Éléments de géométrie algébrique* (EGA) and *Séminaire de géométrie algébrique* (SGA). The collection *Fondements de la Géométrie Algébrique* (FGA), which gathers together talks given in the Séminaire Bourbaki, also contains important material.

Perhaps Grothendieck's deepest single accomplishment is the invention of the étale and ℓ -adic cohomology theories, which explain an observation of André Weil's that there is a deep connection between the topological characteristics of a variety and its diophantine (number theoretic) properties. For example, the number of solutions of an equation over a finite field reflects the topological nature of its solutions over the complex numbers. Weil realized that to prove such a connection one needed a new cohomology theory, but neither he nor any other expert saw how to do this until such a theory was found by Grothendieck.

This program culminated in the proofs of the Weil conjectures, the last of which was settled by Grothendieck's student Pierre Deligne in the early 1970s after Grothendieck had largely withdrawn from mathematics.

Major mathematical topics (from *Récoltes et Semailles*)

He wrote a retrospective assessment of his mathematical work (see the external link *La Vision* below). As his main mathematical achievements ("maître-thèmes"), he chose this collection of 12 topics (his chronological order):

1. Topological tensor products and nuclear spaces
2. "Continuous" and "discrete" duality (derived categories and "six operations").
3. *Yoga* of the Grothendieck–Riemann–Roch theorem (K-theory, relation with intersection theory).
4. Schemes.
5. Topoi.
6. Étale cohomology including ℓ -adic cohomology.
7. Motives and the motivic Galois group (and Grothendieck categories)
8. Crystals and crystalline cohomology, *yoga* of De Rham and Hodge coefficients.
9. Topological algebra, infinity-stacks, 'dérivateurs', cohomological formalism of toposes as an inspiration for a new homotopic algebra
10. Tame topology.
11. *Yoga* of anabelian geometry and Galois–Teichmüller theory.
12. Schematic point of view, or "arithmetics" for regular polyhedra and regular configurations of all sorts.

He wrote that the central theme of the topics above is that of topos theory, while the first and last were of the least importance to him.

Here the term *yoga* denotes a kind of "meta-theory" that can be used heuristically. The word *yoke*, meaning "linkage", is derived from the same Indo-European root.

Life

Family and early life

Alexander Grothendieck was born in Berlin to anarchist parents: a Russian father from an ultimately Hassidic family, Alexander Shapiro aka Tanaroff, and a mother from a German Protestant family, Johanna "Hanka" Grothendieck; both of his parents had broken away from their early backgrounds in their teens^[8]. At the time of his birth Grothendieck's mother was married to Johannes Raddatz, a German journalist, and his birthname was initially recorded as *Alexander Raddatz*. The marriage was dissolved in 1929 and Shapiro/Tanaroff acknowledged his paternity, but never married Hanka Grothendieck^[9]. Grothendieck lived with his parents until 1933 in Berlin. At the end of that year, Shapiro moved to Paris, and Hanka followed him the next year. They left Grothendieck in the care of Wilhelm Heydorn, a Lutheran Pastor and teacher^[10] in Hamburg where he went to school. During this time, his parents fought in the Spanish Civil War.

During WWII

In 1939 Grothendieck came to France and lived in various camps for displaced persons with his mother, first at the Camp de Rieucros, spending 1942–44 at Le Chambon-sur-Lignon. His father was sent via Drancy to Auschwitz where he died in 1942.

Studies and contact with research mathematics

After the war, the young Grothendieck studied mathematics in France, initially at the University of Montpellier. He had decided to become a math teacher because he had been told that mathematical research had been completed early in the 20th century and there were no more open problems.^[11] However, his talent was noticed, and he was encouraged to go to Paris in 1948.

Initially, Grothendieck attended Henri Cartan's Seminar at École Normale Supérieure, but lacking the necessary background to follow the high-powered seminar, he moved to the University of Nancy where he wrote his dissertation under Laurent Schwartz in functional analysis, from 1950 to 1953. At this time he was a leading expert in the theory of topological vector spaces. By 1957, he set this subject aside in order to work in algebraic geometry and homological algebra.

The IHÉS years

Installed at the Institut des Hautes Études Scientifiques (IHÉS), Grothendieck attracted attention by an intense and highly productive activity of seminars (*de facto* working groups drafting into foundational work some of the ablest French and other mathematicians of the younger generation). Grothendieck himself practically ceased publication of papers through the conventional, learned journal route. He was, however, able to play a dominant role in mathematics for around a decade, gathering a strong school.

During this time he had officially as students Michel Demazure (who worked on SGA3, on group schemes), Luc Illusie (cotangent complex), Michel Raynaud, Jean-Louis Verdier (cofounder of the derived category theory) and Pierre Deligne. Collaborators on the SGA projects also included Mike Artin (étale cohomology) and Nick Katz (monodromy theory and Lefschetz pencils). Jean Giraud worked out torsor theory extensions of non-abelian cohomology. Many others were involved.

The 'Golden Age'

Alexander Grothendieck's work during the 'Golden Age' period at IHÉS established several unifying themes in algebraic geometry, number theory, topology, category theory and complex analysis. His first (pre-IHÉS) breakthrough in algebraic geometry was the Grothendieck-Hirzebruch-Riemann-Roch theorem, a far-reaching generalisation of the Hirzebruch-Riemann-Roch theorem proved algebraically; in this context he also introduced K-theory. Then, following the programme he outlined in his talk at the 1958 International Congress of Mathematicians, he introduced the theory of schemes, developing it in detail in his *Éléments de géométrie algébrique* (EGA) and providing the new more flexible and general foundations for algebraic geometry that has been adopted in the field since that time. He went on to introduce the étale cohomology theory of schemes, providing the key tools for proving the Weil conjectures, as well as crystalline cohomology and algebraic de Rham cohomology to complement it. Closely linked to these cohomology theories, he originated topos theory as a generalisation of topology (relevant also in mathematical logic, category theory, and also to computer software/programming and institutional ontology classification and bioinformatics). He also provided an algebraic definition of fundamental groups of schemes and more generally the main structures of a categorical Galois theory. As a framework for his coherent duality theory he also introduced derived categories, which were further developed by Verdier. The results of work on these and other topics were published in the EGA and in less polished form in the notes of the Séminaire de géométrie algébrique (SGA) that he directed at IHÉS.

Politics and retreat from scientific community

Grothendieck's political views were radical and pacifist, but *not communist* (thus he strongly disapproved of the Soviet military expansionism as well). He gave lectures on category theory in the forests surrounding Hanoi while the city was being bombed, to protest against the Vietnam War (*The Life and Work of Alexander Grothendieck*, American Math. Monthly, vol. 113, no. 9, footnote 6). He retired from scientific life around 1970, after having discovered the partly military funding of IHÉS (see pp. xii and xiii of SGA1, Springer Lecture Notes 224). He returned to academia a few years later as a professor at the University of Montpellier, where he stayed until his retirement in 1988. His criticisms of the scientific community, and especially of several mathematics circles, are also contained in a letter ^[12], written in 1988, in which he states the reasons for his refusal of the Crafoord Prize.

While the issue of military funding was perhaps the most obvious explanation for Grothendieck's departure from IHÉS, those who knew him say that the causes of the rupture ran deeper. Pierre Cartier, a *visiteur de longue durée* at the IHÉS, wrote a piece about Grothendieck for a special volume published on the occasion of the IHÉS's fortieth anniversary. The *Grothendieck Festschrift* was a three-volume collection of research papers to mark his sixtieth birthday (falling in 1988), and published in 1990.^[13]

In it Cartier notes that, as the son of an antimilitary anarchist and one who grew up among the disenfranchised, Grothendieck always had a deep compassion for the poor and the downtrodden. As Cartier puts it, Grothendieck came to find Bures-sur-Yvette "une cage dorée" [a golden cage]. While Grothendieck was at the IHÉS, opposition to the Vietnam War was heating up, and Cartier suggests that this also reinforced Grothendieck's distaste at having become a mandarin of the scientific world. In addition, after several years at the

IHÉS Grothendieck seemed to cast about for new intellectual interests. By the late 1960s he had started to become interested in scientific areas outside of mathematics. David Ruelle, a physicist who joined the IHÉS faculty in 1964, said that Grothendieck came to talk to him a few times about physics. (In the 1970s Ruelle and the Dutch mathematician Floris Takens produced a new model for turbulence, and it was Ruelle who invented the concept of a strange attractor in a dynamical system.) Biology interested Grothendieck much more than physics, and he organized some seminars on biological topics.^[14] After leaving the IHÉS, Grothendieck tried but failed to get a position at the Collège de France. He then went to Université de Montpellier, where he became increasingly estranged from the mathematical community. Around this time, he founded a group called *Survivre*, which was dedicated to antimilitary and ecological issues. His mathematical career, for the most part, ended when he left the IHÉS. In 1984 he wrote a proposal to get a position through the Centre National de la Recherche Scientifique. The proposal, entitled *Esquisse d'un Programme* [Sketch of a Program] describes new ideas for studying the moduli space of complex curves. Although Grothendieck himself never published his work in this area, the proposal became the inspiration for work by other mathematicians and the source of the theory of *dessins d'enfants*. *Esquisse d'un Programme* was published in the two-volume proceedings *Geometric Galois Actions* (Cambridge University Press, 1997).^[15]

Manuscripts written in the 1980s

While not publishing mathematical research in conventional ways during the 1980s, he produced several influential manuscripts with limited distribution, with both mathematical and biographical content. During that period he also released his work on Bertini type theorems contained in EGA 5, published by the Grothendieck Circle^[16] in 2004.

La Longue Marche à travers la théorie de Galois [*The Long March Through Galois Theory*] is an approximately 1600-page handwritten manuscript produced by Grothendieck during the years 1980–1981, containing many of the ideas leading to the *Esquisse d'un programme*^[17] (see below, and also a more detailed entry^[18]), and in particular studying the Teichmüller theory. (For an English translation of the tables of contents of these manuscripts see the Wikipedia separate entry on the *Esquisse d'un programme*.)

In 1983 he wrote a huge extended manuscript (about 600 pages) entitled *Pursuing Stacks*, stimulated by correspondence with Ronald Brown, (see also R. Brown^[19] and Tim Porter^[20] at University of Bangor in Wales), and starting with a letter addressed to Daniel Quillen. This letter and successive parts were distributed from Bangor (see External Links below): in an informal manner, as a kind of diary, Grothendieck explained and developed his ideas on the relationship between algebraic homotopy theory and algebraic geometry and prospects for a noncommutative theory of stacks. The manuscript, which is being edited for publication by G. Maltsiniotis, later led to another of his monumental works, *Les Dérivateurs*. Written in 1991, this latter opus of about 2000 pages further developed the homotopical ideas begun in *Pursuing Stacks*. Much of this work anticipated the subsequent development of the motivic homotopy theory of F. Morel and V. Voevodsky in the mid 1990s.

His *Esquisse d'un programme*^[17] (1984) is a proposal for a position at the Centre National de la Recherche Scientifique, which he held from 1984 to his retirement in 1988. Ideas from it have proved influential, and have been developed by others, in particular *dessins d'enfants* and a new field emerging as anabelian geometry. In *La Clef des Songes* he

explains how the reality of dreams convinced him of God's existence.

The 1000-page autobiographical manuscript *Récoltes et semailles* (1986) is now available on the internet in the French original, and an English translation is underway (these parts of *Récoltes et semailles* have already been translated into Russian ^[21] and published in Moscow). Some parts of *Récoltes et semailles* [22] [23] and the whole *La Clef des Songes* [24] have been translated into Spanish.

Disappearance

In 1991, Grothendieck left his home and disappeared. He is now said to live in southern France or Andorra and to entertain no visitors. Though he has been inactive in mathematics for many years, he remains one of the greatest and most influential mathematicians of modern times.

See also

- Ax-Grothendieck theorem
- Birkhoff-Grothendieck theorem
- Esquisse d'un Programme
- Grothendieck category ^[25]
- Grothendieck's connectedness theorem
- Grothendieck connection
- Grothendieck construction
- Grothendieck's Galois theory
- Grothendieck group
- Grothendieck inequality or Grothendieck constant
- Grothendieck-Katz p-curvature conjecture
- Grothendieck's relative point of view
- Grothendieck-Riemann-Roch theorem
- Grothendieck's Séminaire de géométrie algébrique
- Grothendieck space
- Grothendieck spectral sequence
- Grothendieck topology
- Grothendieck universe
- Tarski-Grothendieck set theory
- IHES
- IHES at Forty by Allyn Jackson ^[13]

Notes

[1] http://en.wikipedia.org/wiki/Talk%3Aalexander_grothendieck%2Fpersondata

[2] Peck, Morgen, http://scienceline.org/2007/01/31/math_controversy_peck/ *Equality of Mathematicians*, http://scienceline.org/2007/01/31/math_controversy_peck/, "Alexandre Grothendieck is arguably the most important mathematician of the 20th century..."

[3] Leith, Sam (20 March 2004), "<http://www.lewrockwell.com/spectator/spec262.html>|The Einstein of maths", *The Spectator*, <http://www.lewrockwell.com/spectator/spec262.html>, "[A] mathematician of staggering accomplishment ... a legendary figure in the mathematical world."

[4] See, for example, (Deligne 1998).

[5] Jackson, Allyn (2004), "<http://www.ams.org/notices/200409/fea-grothendieck-part1.pdf> Comme Appelé du Néant — As If Summoned from the Void: The Life of Alexandre Grothendieck I", *Notices of the American Mathematical Society* **51** (4): p. 1049, <http://www.ams.org/notices/200409/fea-grothendieck-part1.pdf>

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- [8] Society for Industrial and Applied Mathematics (<http://www.siam.org/news/news.php?id=1405>)
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- [10] Allyn Jackson, The Life of Alexander Grothendieck, p. 1040 (<http://www.ams.org/notices/200409/fea-grothendieck-part1.pdf>)
- [11] See Jackson (2004:1). The remark is from the beginning of Récoltes et Semailles (page P4, in the introductory section Prélude en quatre Mouvements)
- [12] <http://web.archive.org/web/20060106062005/http://www.math.columbia.edu/~lipyan/CrafoordPrize.pdf>
- [13] The editors were Pierre Cartier, Luc Illusie, Nick Katz, Gérard Laumon, Yuri Manin, and Ken Ribet. A second edition has been printed (2007) by Birkhauser.
- [14] <http://www.ams.org/notices/199903/ihes-changes.pdf>
- [15] http://books.google.co.uk/books?id=Hlf1j2XmXkcC&dq=Geometric+Galois+Actions&printsec=frontcover&source=bl&ots=q9iN4QMkYj&sig=fAsvVIPOhN9K9LfjGFJC5zm_IA4&hl=en&ei=KW-3ScSkHYHIMuDukOEK&sa=X&oi=book_result&resnum=1&ct=result
- [16] <http://www.math.jussieu.fr/~leila/grothendieckcircle/index.php>
- [17] <http://kolmogorov.unex.es/~navarro/res/esquissefr.pdf>
- [18] http://en.wikipedia.org/wiki/Esquisse_d%27un_Programme
- [19] <http://www.bangor.ac.uk/r.brown>
- [20] <http://www.informatics.bangor.ac.uk/~tporter/>
- [21] <http://www.mccme.ru/free-books/grothendieck/RS.html>
- [22] <http://kolmogorov.unex.es/~navarro/res/preludio.pdf>
- [23] <http://kolmogorov.unex.es/~navarro/res/carta.pdf>
- [24] <http://kolmogorov.unex.es/~navarro/res/clef1-6.pdf>
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 - Grothendieck Circle discussion Forum (<http://gavrilov.akatov.com/Grothendieck>)
- Institut des Hautes Études Scientifiques (<http://www.ihes.fr>)
- The origins of 'Pursuing Stacks' (<http://www.bangor.ac.uk/r.brown/pstacks.htm>) This is an account of how 'Pursuing Stacks' was written in response to a correspondence in English with Ronnie Brown and Tim Porter at Bangor, which continued until 1991.
- Récoltes et Semailles (<http://acm.math.spbu.ru/RS/>) in French.
- Spanish translation (<http://kolmogorov.unex.es/~navarro/res/>) of "Récoltes et Semailles" et "Le Clef des Songes" and other Grothendieck's texts
- short bio (<http://www.ams.org/notices/200808/tx080800930p.pdf>) from Notices of the American Mathematical Society

Persondata

NAME	Grothendieck, Alexander
ALTERNATIVE NAMES	
SHORT DESCRIPTION	Mathematician
DATE OF BIRTH	March 28, 1928 (age 81)

PLACE OF BIRTH

Berlin, Germany

DATE OF DEATH

,

PLACE OF DEATH

Heinz Hopf

Heinz Hopf (November 19, 1894 – June 3, 1971) was a German mathematician born in Gräbschen, Germany (now Grabiszyn, part of Wrocław, Poland). He attended Dr. Karl Mittelhaus' higher boys' school from 1901 to 1904, and then entered the König-Wilhelm- Gymnasium in Breslau. He showed mathematical talent from an early age. In 1913 he entered the Silesian Friedrich Wilhelm University where he attended lectures by Ernst Steinitz, Kneser, Max Dehn, Erhard Schmidt, and Rudolf Sturm. When World War I broke out in 1914, Hopf eagerly enlisted. He was wounded twice and received the iron cross (first class) in 1918.

In 1920, Hopf moved to Berlin to continue his mathematical education. He studied under Ludwig Bieberbach, receiving his doctorate in 1925. In his dissertation, *Connections between topology and metric of manifolds* (German *Über Zusammenhänge zwischen Topologie und Metrik von Mannigfaltigkeiten*), he proved that any simply connected complete Riemannian 3-manifold of constant curvature is globally isometric to

Euclidean, spherical, or hyperbolic space. He also studied the indices of zeros of vector fields on hypersurfaces, and connected their sum to curvature. Some six months later he gave a new proof that the sum of the indices of the zeros of a vector field on a manifold is independent of the choice of vector field and equal to the Euler characteristic of the manifold. This theorem is now called the Poincaré-Hopf theorem.

Hopf spent the year after his doctorate at Göttingen, where David Hilbert, Richard Courant, Carl Runge, and Emmy Noether were working. While there he met Paul Alexandrov and began a lifelong friendship.

In 1926 Hopf moved back to Berlin, where he gave a course in combinatorial topology. He spent the academic year 1927/28 at Princeton University on a Rockefeller fellowship with Alexandrov. Solomon Lefschetz, Oswald Veblen and J.W. Alexander were all at Princeton at the time. At this time Hopf discovered the Hopf invariant of maps $S^3 \rightarrow S^2$. and proved that the Hopf fibration has invariant 1. In the summer of 1928 Hopf returned to Berlin and began working with Alexandrov, at the suggestion of Courant, on a book on topology. Three volumes were planned, but only one was finished. It was published in 1935.

In October 1928 Hopf married Anja von Mickwitz (1891 - 1967). The next year he declined a job offer from Princeton. In 1931 Hopf took Hermann Weyl's position at ETH, in Zürich.



Heinz Hopf (on the right) in Oberwolfach, together with Hellmuth Kneser

Hopf received another invitation to Princeton in 1940, but he declined it. Two years later, however, he was forced to file for Swiss citizenship after his property was confiscated by Nazi authorities.

In 1946/47 and 1955/56 Hopf visited the United States, staying at Princeton and giving lectures at New York University and Stanford University. He served as president of the International Mathematical Union from 1955 to 1958. He received honorary doctorates from Princeton, Freiburg i. Br., Manchester, Sorbonne at Paris, Brussels, and Lausanne.

See also

- Hopf algebra
- Hopf bifurcation (actually was done by Eberhard Hopf)
- Hopf bundle
- Hopf conjecture
- Hopf link
- H-space
- Hopf–Rinow theorem

External links

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- Heinz Hopf^[1] at the Mathematics Genealogy Project

Academic Genealogy	
Notable teachers	Notable students
Ludwig Bieberbach Erhard Schmidt	Beno Eckmann Hans Freudenthal Werner Gysin Friedrich Hirzebruch Heinz Huber Michel Kervaire Willi Rinow Hans Samelson Eduard Stiefel

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Maxim Kontsevich

Maxim Kontsevich	
	
Born	25 August 1964 Russia
Citizenship	France
Fields	Mathematics
Institutions	Institut des Hautes Études Scientifiques University of Miami
Alma mater	Moscow State University, mexmat
Doctoral advisor	Don Bernard Zagier
Notable awards	Fields Medal (1998) Crafoord Prize (2008)

Maxim Lvovich Kontsevich (Russian: Максим Львович Концевич) (born 25 August 1964) is a Russian mathematician. He received a Fields Medal in 1998, at the 23rd International Congress of Mathematicians in Berlin. He also received a Crafoord Prize in 2008.

Biography

Born into the family of Lev Rafailovich Kontsevich – Soviet orientalist and author of the Kontsevich system. After ranking second in the All-Union Mathematics Olympiads, he attended Moscow State University but left without a degree in 1985 to become a researcher at the Institute for Problems of Information Transmission in Moscow [1]. In 1992 he received his Ph.D. at the University of Bonn under Don Bernard Zagier. His thesis claims to prove a conjecture by Edward Witten that two quantum gravitational models are equivalent. Currently he is a Professor at the Institut des Hautes Études Scientifiques (IHÉS) in Bures-sur-Yvette, France and Distinguished Professor at University of Miami in Coral Gables, Florida, U.S..

His work concentrates on geometric aspects of mathematical physics, most notably on knot theory, quantization, and mirror symmetry. His most famous result is a formal deformation quantization that holds for any Poisson manifold. He also introduced knot invariants defined by complicated integrals analogous to Feynman integrals. In topological field theory, he introduced the moduli space of stable maps, which may be considered a mathematically rigorous formulation of the Feynman integral for topological string theory. These results are a part of his "contributions to four problems of geometry" for which he was awarded the Fields Medal in 1998.

See also

- Kontsevich integral
- Homological mirror symmetry
- Motivic integration

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 [3] <http://www.icm2002.org.cn/>
 [4] <http://genealogy.math.ndsu.nodak.edu/id.php?id=26861>
 [5] <http://www.ams.org/featurecolumn/archive/kontsevich.html>

Otto Hermann Künneth

Otto Hermann Lorenz Künneth (July 6, 1892 Neustadt an der Haardt – May 7, 1975 Erlangen) was a German mathematician and renowned algebraic topologist, best known for his contribution to what is now known as the Künneth theorem.

His 1922 doctoral thesis was titled "Über die Bettischen Zahlen einer Produktmannigfaltigkeit" (University of Erlangen, supervised by Heinrich Franz Friedrich Tietze).

Künneth participated in the First World War and was captured by British forces.

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-

Saunders Mac Lane

Saunders Mac Lane (4 August 1909, Taftville, Connecticut – 14 April 2005, San Francisco) was an American mathematician who cofounded category theory with Samuel Eilenberg.

Career

Mac Lane was christened "Leslie Saunders MacLane", but "Leslie" fell into disuse because his parents, Donald MacLane and Winifred Saunders, came to dislike it. He began inserting a space into his surname because his first wife found it difficult to type the name without a space.^[1]

Mac Lane earned a BA from Yale University in 1930, and an MA from the University of Chicago in 1931. During this period, he published his first scientific paper, in physics and co-authored with Irving Langmuir. He attended the University of Göttingen, 1931–1933, studying logic and mathematics under Paul Bernays, Emmy Noether and Hermann Weyl. Göttingen's Mathematisches Institut awarded him the Ph.D. in 1934.

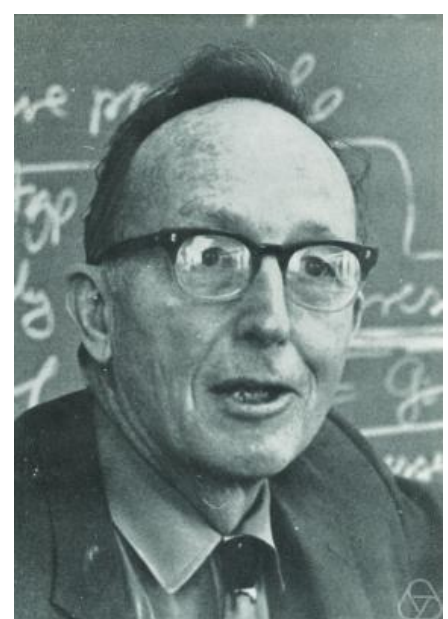
From 1934 through 1938, Mac Lane held short term appointments at Harvard University, Cornell University, and the University of Chicago. He then held a tenure track appointment at Harvard, 1938–1947, before spending the rest of his career at the University of Chicago. In 1944 and 1945, he also directed Columbia University's Applied Mathematics Group, which was involved in the war effort as a contractor for the Applied Mathematics Panel.

Mac Lane served as vice president of the National Academy of Sciences and the American Philosophical Society, and as president of the American Mathematical Society. While presiding over the Mathematical Association of America in the 1950s, he initiated its activities aimed at improving the teaching of modern mathematics. He was a member of the National Science Board, 1974–1980, advising the American government. In 1976, he led a delegation of mathematicians to China to study the conditions affecting mathematics there. Mac Lane was elected to the National Academy of Sciences in 1949, and received the National Medal of Science in 1989.

Contributions

After a thesis in mathematical logic, his early work was in field theory and valuation theory. He wrote on valuation rings and Witt vectors, and separability in infinite field extensions. He started writing on group extensions in 1942, and began his epochal collaboration with Samuel Eilenberg in 1943, resulting in what are now called Eilenberg–Mac Lane spaces $K(G, n)$, having a single non-trivial homotopy group G in dimension n . This work opened the way to group cohomology in general.

After introducing, via the Eilenberg–Steenrod axioms, the abstract approach to homology theory, he and Eilenberg originated category theory in 1945. He is especially known for his



Saunders Mac Lane

work on coherence theorems. A recurring feature of category theory, abstract algebra, and of some other mathematics as well, is the use of diagrams, consisting of arrows (morphisms) linking objects, such as products and coproducts. According to McLarty (2005), this diagrammatic approach to contemporary mathematics largely stems from Mac Lane (1948).

Mac Lane had an exemplary devotion to writing approachable texts, starting with his very influential *A Survey of Modern Algebra*, coauthored in 1941 with Garrett Birkhoff. From then on, it was possible to teach elementary modern algebra to undergraduates using an English text. His *Categories for the Working Mathematician* remains the definitive introduction to category theory.

Mac Lane supervised the Ph.Ds of, among many others, David Eisenbud, William Howard, Irving Kaplansky, Michael Morley, Anil Nerode, Robert Solovay, and John G. Thompson.

In addition to reviewing a fair bit of his mathematical output, the obituary articles McLarty (2005, 2007) clarify Mac Lane's contributions to the philosophy of mathematics. Mac Lane (1986) is an approachable introduction to his views on this subject.

Books by Mac Lane

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See also

- From Action to Mathematics per Mac Lane

Footnotes

[1] Mac Lane (2005), p. 6.

[2] <http://www.ams.org/bull/2000-37-01/S0273-0979-99-00847-2/S0273-0979-99-00847-2.pdf>

[3] <http://philmat.oxfordjournals.org/cgi/content/full/13/3/237>

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External links

- O'Connor, John J.; Robertson, Edmund F., <http://www-history.mcs.st-andrews.ac.uk/Biographies/MacLane.html>|"Saunders Mac Lane", *MacTutor History of Mathematics archive*.
- Obituary press release (<http://www-news.uchicago.edu/releases/05/050421.maclane.shtml>) from the University of Chicago.
- Photographs of Mac Lane, (<http://www.iti.cs.tu-bs.de/~koslowj/cgi-bin/p0454?MacLane-0/idx&256>)1984–99.
- Saunders Mac Lane (<http://genealogy.math.ndsu.nodak.edu/id.php?id=834>) at the Mathematics Genealogy Project

William Lawvere

Francis William Lawvere (b. February 9, 1937 in Muncie, Indiana) is a mathematician known for his work in category theory, topos theory and the philosophy of mathematics.

Biography

Born February 9, 1937 in Muncie, Indiana, Lawvere studied continuum mechanics as an undergraduate with Clifford Truesdell. While teaching a course on functional analysis for Truesdell he learned of category theory from the topology text of John L. Kelley. Lawvere found it a promising framework for simple rigorous axioms for the physical ideas of Truesdell and Walter Noll. Truesdell, who had an appointment in mathematics himself, supported Lawvere's application to study more pure mathematics with Samuel Eilenberg, a founder of category theory, at Columbia University in 1960.

Before completing the Ph.D. Lawvere spent a year in Berkeley as an informal student of model theory and set theory, following lectures by Alfred Tarski and Dana Scott. In his first teaching position at Reed College he was instructed to devise courses in calculus and abstract algebra from a foundational perspective. He tried to use the then current axiomatic set theory but found it unworkable for undergraduates, so he instead developed the first axioms for the more relevant composition of mappings of sets. He later streamlined those axioms into the Elementary Theory of the Category of Sets (1964) which became a key ingredient (the constant case) of elementary topos theory.

Work

Lawvere completed his Ph.D at Columbia in 1963 with Eilenberg. His dissertation introduced the Category of Categories in his thesis as a framework for the semantics of algebraic theories. During 1964-1967 at the Forschungsinstitut fuer Mathematik at the ETH in Zurich he worked on the Category of Categories and was especially influenced by Pierre Gabriel's seminars at Oberwolfach on Grothendieck's foundation of algebraic geometry. He then taught at the University of Chicago, working with Mac Lane, and at the City University of New York Graduate Center (CUNY), working with Alex Heller. His Chicago lectures on categorical dynamics were a further step toward topos theory and his CUNY lectures on hyperdoctrines advanced categorical logic especially using his 1963 discovery that existential and universal quantifiers can be characterized as special cases of adjoint functors.

Back in Zurich for 1968-69 he proposed elementary (first-order) axioms for toposes generalizing the concept of the Grothendieck topos (see background and genesis of topos theory and worked with the algebraic topologist Tierney to clarifying and applying this theory. Tierney discovered major simplifications in the description of Grothendieck "topologies". Kock later found further simplifications so that a topos can be described as a category with products and equalizers in which the notions of map space and subobject are representable. Lawvere had pointed out that a Grothendieck topology can be entirely described as an endomorphism of the subobject representor, and Tierney showed that the conditions it needs to satisfy are just idempotence and the preservation of finite intersections. These "topologies" are important in both algebraic geometry and model theory because they determine the subtoposes as sheaf-categories.

Dalhousie University in 1969 set up a group of 15 Killam-supported researchers with Lawvere at the head; but in 1971 it terminated the group. Lawvere was controversial for his political opinions, for example, his opposition to the 1970 use of the War Measures Act , and for teaching the history of mathematics without permission. But in 1995 Dalhousie hosted the celebration of 50 years of category theory with Lawvere and Saunders Mac Lane present.

Lawvere ran a seminar in Perugia, Italy (1972-1974) and especially worked on various kinds of enriched category. For example a metric space can be regarded as an enriched category. From 1974 until his retirement in 2000 he was professor of mathematics at University at Buffalo, often collaborating with Stephen Schanuel. In 1977 he was elected him to the Martin professorship in mathematics for 5 years, which made possible the meeting on "Categories in Continuum Physics" in 1982. Clifford Truesdell participated in that meeting, as did several other researchers in the rational foundations of continuum physics and in the synthetic differential geometry which had evolved from the spatial part of Lawvere's categorical dynamics program). Lawvere continues to work on his 50-year quest for a rigorous flexible base for physical ideas, free of unnecessary analytic complications. He is now professor emeritus of mathematics and adjunct professor emeritus of philosophy at Buffalo.

Selected books

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- 1997 *Conceptual Mathematics: A First Introduction to Categories* (with Stephen H. Schanuel). Cambridge Uni. Press. ISBN 0-521-47817-0
- 2003 (2002) *Sets for Mathematics* (with Robert Rosebrugh). Cambridge Uni. Press. ISBN 0-521-01060-8

External links

- [1] Includes reprints of seven of Lawvere's fundamental articles, among them his dissertation and his first full treatment of the category of sets. Those two had circulated only as mimeographs.
- Homepage.^[2] Includes bibliography and downloadable papers, Ph.D. thesis.
- William Lawvere^[3] at the Mathematics Genealogy Project
- Photograph^[4]
- John Baez's This Week's Finds in Mathematical Physics (Week 200)^[5]

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[2] <http://www.acsu.buffalo.edu/~wlawvere/>
[3] <http://genealogy.math.ndsu.nodak.edu/id.php?id=18947>
[4] http://andrej.com/mathematicians/L/Lawvere_William.html
[5] <http://math.ucr.edu/home/baez/week200.html>
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Egbert van Kampen

Egbert Rudolf van Kampen (May 28, 1908, Berchem, Belgium – February 11, 1942, Baltimore, Maryland) was a mathematician. He made important contributions to topology, especially to the study of fundamental groups.

Van Kampen received his Ph.D. degree from Leiden University in 1929. His dissertation, entitled *Die kombinatorische Topologie und die Dualitaetssaetze*, was written under the direction of Willem van der Woude.

In 1931 van Kampen left Europe and travelled to the United States to take up the position which he had been offered at Johns Hopkins University in Baltimore, Maryland. There he met Oscar Zariski who had taught at Johns Hopkins University as a Johnston Scholar from 1927 until 1929 when he had joined the Faculty. Zariski had been working on the fundamental group of the complement of an algebraic curve, and he had found generators and relations for the fundamental group but was unable to show that he had found sufficient relations to give a presentation for the group. Van Kampen solved the problem, showing that Zariski's relations were sufficient, and the result is now known as the Zariski-van Kampen theorem. This led van Kampen to formulate and prove what is nowadays known as the Seifert-van Kampen theorem.

External links

- Egbert van Kampen ^[1] at the Mathematics Genealogy Project
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Grigory Perelman

1. REDIRECTGrigori Perelman

Nicolae Popescu

Nicolae Popescu (born 22 September 1937, at Strehaia, Romania) is a Romanian mathematician and Emeritus Professor. Popescu was elected a Member of the Romanian Academy in 1992, and he is best known for his contributions to Algebra and the theory of abelian categories. Since 1964 he collaborated on the characterization of abelian categories with the well-known French mathematician Pierre Gabriel. His areas of expertise are: Category theory, abelian categories with Applications to Rings and Modules, Adjoint Functors ^[1] and Limits/Colimits ^[2], Theory of Rings, Fields and Polynomials, and Valuation Theory; he also has interests and published in the following areas: Algebraic Topology, Algebraic Geometry, Commutative Algebra, K-Theory, Class-Field theory, and Algebraic Function Theory. He published between 1962 and 2008 more than 102 papers in peer-reviewed, mathematics journals, several monographs on the theory of sheaves, and also six books on abelian category theory and abstract algebra. In a Grothendieck-like, energetic style, he initiated and provided scientific leadership to several seminars on category theory^[3], sheaves and abstract algebra which resulted in a continuous stream of high-quality mathematical publications in international, peer-reviewed mathematics journals by several members participating in his Seminar series. His book *Abelian Categories with Applications to Rings and Modules*^[4] continues to provide valuable information to mathematicians around the world. His recent contributions have also branched into valuation and number theory.

Biography

Popescu is married and has three children. He earned his M.S. degree in mathematics in 1964, and his Ph.D. degree in mathematics in 1967, both at the University of Bucharest. He was awarded a D. Phil. degree (Doctor Docent) in 1972 by the University of Bucharest.

Presently, he continues his mathematics studies at the Institute of Mathematics of the Romanian Academy in the Algebra research group ^[5] and also has international collaborations on three continents. One finds from conversations with Academician Popescu that he shares many moral, ethical and religious values with another famous mathematician French-German-Jewish Alexander Grothendieck who visited the School of Mathematics in Bucharest in 1968. Like Grothendieck he has a long-standing interest in category theory, number theory, practicing Yoga, and supporting promising young mathematicians in his fields of interest. He also supported the early developments of category theory applications in relational biology and mathematical biophysics/mathematical biology.

Academic positions

Popescu was appointed as a Lecturer at the University of Bucharest in 1968 where he taught graduate students until 1972. Since 1964, he also held a Research Professorship^[6] at the Institute of Mathematics of the Romanian Academy, which institute was ruthlessly eliminated by former dictator and president of S.R. Romania, Nicolae Ceausescu, in 1976 for reasons related to his daughter Zoe Cheausescu who was 'hired' by the Mathematics Institute in Bucharest two years before.

Notes

- [1] <http://planetmath.org/encyclopedia/AdjointFunctor.html>
- [2] <http://planetmath.org/encyclopedia/AdjointFunctor.html>
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- [4] Nicolae Popescu. Abelian Categories with Applications to Rings and Modules, Academic Press, L. M. S. Monograph No.3, London, 1973., ISBN 0125615507 ([http://www.lib.iastate.edu:81/ipac20/ipac.jsp?session=1WB345344H890.711450&profile=parks&source=~!horizon&view=items&uri=full=3100001~!47305~!0&ri=1&aspect=basic_search&menu=search&ipp=20&spp=20&staffonly=&term=Modules+\(Algebra\)&index=PSUBJ&uindex=&aspect=basic_search&menu=search&ri=1#focus](http://www.lib.iastate.edu:81/ipac20/ipac.jsp?session=1WB345344H890.711450&profile=parks&source=~!horizon&view=items&uri=full=3100001~!47305~!0&ri=1&aspect=basic_search&menu=search&ipp=20&spp=20&staffonly=&term=Modules+(Algebra)&index=PSUBJ&uindex=&aspect=basic_search&menu=search&ri=1#focus))
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92. Analytic Normal Basis Theorem (with V. Alexandru and A. Zaharescu) (to appear)

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1. Elements of the theory of sheaves (orig.title: Elemente de teoria fasciculelor I, *St. Cerc. Mat.* (1966) 267-296.
2. *Ibid.* Elemente de teoria fasciculelor II, *St. Cerc. Mat.* (1966) 407-456.
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External links

- Nicolae Popescu - Institute of Mathematics of the Romanian Academy (<http://www.imar.ro/~nipopesc/>)
- Nicolae Popescu --Biography (<http://planetphysics.org/?op=getobj&from=objects&id=810>)

Robert Rosen

See also arts and entertainment celebrity producer-writer-performer: Robert M. Rosen, Robert Ozn

Robert Rosen (27 June, 1934, - 28 December, 1998, Rochester, New York) was an American theoretical biologist and professor of Biophysics at Dalhousie University.

Biography

Robert Rosen was born on June 27, 1934 in Brownsville (a section of Brooklyn), in New York City. He studied biology, mathematics, physics, philosophy, and history-- especially the history of science-- and eventually became a student of physicist and theoretical biologist, Professor Nicholas Rashevsky at the University of Chicago. He received his PhD in Relational Biology from the University of Chicago in 1959 and remained there until 1964.^[1] In 1964 Rosen was offered a full professorship with tenure at the University of Buffalo, now known as the State University of New York (SUNY) at Buffalo, holding a joint appointment at the Center for Theoretical Biology. In 1970, he took a sabbatical and spent a year as a Visiting Fellow at Robert Hutchins' Center for the Study of Democratic Institutions, in Santa Barbara, California. It was a seminal year for him, leading to the conception and development of what he later called Anticipatory Systems Theory, a corollary of his larger theoretical work on relational complexity, in which it is embedded. In 1975, he left Buffalo and accepted a position at Dalhousie University, in Halifax, Nova Scotia, as a Killam Research Professor in the Department of Physiology & Biophysics, where he remained until he took early retirement in 1994.^[2]

He was president of the Society for General Systems Research, (now the ISSS), in 1980-81.



Robert Rosen

Work

Rosen's research was concerned with the most fundamental aspects of biology, specifically the question "What is life?" or "Why are living organisms alive?". Major themes in the work of Robert Rosen were:

- developing a specific definition of complexity that is based on relations and, by extension, principles of organization
- developing a rigorous theoretical foundation for living organisms as "anticipatory systems"

Rosen came to realize that the contemporary model of physics - which is still based on the Cartesian/Newtonian world of mechanisms - was inadequate to explain or describe the behavior of biological systems; that is, one could not properly answer the question "what is life?" from within a scientific foundation that is entirely reductionistic. Approaching organisms with reductionistic scientific methods and practices always sacrifices the whole in order to study the parts, but what Rosen discovered was that the whole could not be recaptured once the organization had been destroyed. His conclusion was that the very thing about living organisms biologists should be studying, the organization, was the first aspect of all biological systems to be thrown away in scientific analysis. This is a limitation of contemporary science when science regards the machine as a model for all systems in the universe. Rosen came to regard the machine metaphor as the single biggest impediment to scientific exploration of questions in biology and concluded that the paradigm needs to be expanded beyond purely reductionist capabilities. In order to do this properly, he said there must be a sound theoretical foundation underlying the expansion and that relational complexity provided such a foundation. So it was that, rather than biology being a mere subset of already-known physics, it turned out that biology had profound lessons for physics, and science in general.^[3]

Notion of the scientific model

The clarification of the notion of the scientific model: Rosen maintained that modeling is the essence of science and of thought. His book *Anticipatory Systems* describes, in detail, what he termed the modeling relation. He showed the deep differences between a true modeling relation and a simulation, which is not based on such a relation. In biology he is known by some for a class of relational models called "(M,R)-Systems" that he devised, which he said capture the minimal capabilities a material system would have to manifest to justify calling it a "alive". In this class of system, M stands for metabolism and R stands for Repair. Thus, his mode for determining life or defining life in any given system is a functional one, not a material one.

Relational biology

Rosen's work proposes a methodology he calls "relational analysis" which needs to be developed in addition to the current capability of reductionistic science. ("Relational" is a term he attributes to Nicholas Rashevsky.) Rosen's "relational biology" maintains that organisms, indeed all systems, have a distinct quality called "organization" not captured by the language of reductionism. It has to do with more than purely structural or material aspects. For example, organization includes all relations between material parts, relations between the effects of interactions of the material parts, and relations with time and environment, to name a few. Many people sum up this aspect of complex systems by saying

that "The whole is more than the sum of the parts". Relations between parts and between the effects of interactions must be considered as additional parts, in some sense. Organization, Rosen says, must be independent from the material particles which seemingly constitute a living system. As he put it: "The human body completely changes the matter it is made of roughly every 8 weeks, through metabolism and repair. Yet, you're still you--with all your memories, your personality... If science insists on chasing the particles, they will follow them right through an organism and miss the organism entirely," (as told to his daughter, Judith Rosen).

He goes very far in this direction claiming that when studying a complex system, we can "throw away the matter and study the organization" to learn essential things about an entire class of systems, in general. He supports this claim (actually it is a quote which he also attributes to Rashevsky) based on the fact that living organisms are a class of systems with an extremely wide range of material "ingredients", different structures, different habitats, different modes of living and reproducing, and yet we are somehow able to recognize them all as "living". In contrast, a study of the specific material details of any given organism, or even of a whole species, will only tell us about how that type of organism "does it". Such a study doesn't approach what is common to all living organisms, i.e.; life. Relational approaches in biology allow us to study organisms in ways that preserve the qualities we are trying to learn about.

Biochemistry and Genetics

Rosen also questioned many aspects of mainstream interpretations of biochemistry and genetics. He objects to the idea that functional aspects in biological systems can be investigated via a material focus. One example: Rosen disputes that the functional capability of a biologically active protein can be investigated purely using the genetically encoded sequence of amino acids. This is because, he said, a protein must undergo a process of "folding" to attain its characteristic three-dimensional shape before it can become functionally active in the system. Yet, only the amino acid sequence is genetically coded. The mechanisms by which proteins fold are not completely known. He concluded, based on examples such as this, that phenotype cannot always be directly attributed to genotype and that the chemically active aspect of a biologically active protein relies on more than the sequence of amino acids, from which it was constructed: There must be other factors at work.

Questions about Rosen's arguments were raised in a paper authored by Christopher Landauer and Kirstie L. Bellman which claims that some of the mathematical formulations used by Rosen are problematic. (Note, by Judith Rosen, who owns the copyrights to her father's books: Some of the confusion is due to known errata introduced into the book, "Life, Itself," by the publisher. For example, the diagram that refers to "(M,R)-Systems" has more than one error; errors which do not exist in Rosen's manuscript for the book. These errata were made known to Columbia University Press when the company switched from hardcover to paperback version of the book (in 2006) but the errors were not corrected and remain in the paperback version as well. The book "Anticipatory Systems; Philosophical, Mathematical, and Methodological Foundations" has the same diagram, correctly represented.)

See also

- Autopoiesis
- system theory
- philosophy of science

Publications

Rosen has written several books and articles. A selection: ^[4]

- 1970, *Dynamical Systems Theory in Biology* New York: Wiley Interscience.
- 1970, *Optimality Principles*, Rosen Enterprises
- 1978, *Fundamentals of Measurement and Representation of Natural Systems*, Elsevier Science Ltd,
- 1985, *Anticipatory Systems: Philosophical, Mathematical and Methodological Foundations*. Pergamon Press.
- 1991, *Life Itself: A Comprehensive Inquiry into the Nature, Origin, and Fabrication of Life*, Columbia University Press

Published posthumously:

- 2000, *Essays on Life Itself*, Columbia University Press.
- 2003, "Anticipatory Systems; Philosophical, Mathematical, and Methodical Foundations", Rosen Enterprises
- 2003, *Rosennean Complexity*, Rosen Enterprises.
- 2003, *The Limits of the Limits Of Science*, Rosen Enterprises

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- [3] Robert Rosen - Biology, Complexity and Physics (<http://www.panmere.com/rosen/rosensum.htm>)
- [4] A complete Bibliography (<http://users.viawest.net/~keirsey/rosenbiblio.html>) of Robert Rosens publications.

External links

- Rosen Enterprises (<http://www.rosen-enterprises.com>) Judith Rosen's website providing biographical information, discussion of, and reprints of the work of Robert Rosen.
- (<http://www.rosen-enterprises.com/RobertRosen/rrosenautobio.html>) Autobiographical Reminiscences of Robert Rosen; about his educational background, his philosophy of science, and his general point of view.
- Rosen: Complexity and Life (<http://www.panmere.com/>"Robert) A website exploring the work of Rosen.
- Robert Rosen: June 27, 1934 — December 30, 1998 (<http://www.people.vcu.edu/~mikuleck/Rosenreq.html>) by Aloisius Louie.
- *Robert Rosen: The well posed question and its answer: why are organisms different from machines?* (<http://www.people.vcu.edu/~mikuleck/PPRISS3.html>) An essay by Donald C. Mikulecky.
- Paper (http://content.aip.org/APCPCS/v627/i1/59_1.html) by Christopher Landauer and Kirstie L. Bellman criticising some of Rosen's mathematical formulations, followed by

attempts to improve the formulations.

Jean-Pierre Serre

Jean-Pierre Serre	
	
Born	15 September 1926 Bages, Pyrénées-Orientales, France
Residence	Paris, France
Nationality	France
Fields	Mathematics
Institutions	Centre National de la Recherche Scientifique Collège de France
Alma mater	École Normale Supérieure
Doctoral advisor	Henri Cartan
Doctoral students	Michel Broué John Labute
Notable awards	Abel Prize (2003) Fields Medal (1954) Wolf Prize in Mathematics (2000) Balzan Prize (1985)

Jean-Pierre Serre (born 15 September 1926) is a French mathematician in the fields of algebraic geometry, number theory and topology. He has received numerous awards and honors for his mathematical research and exposition, including the Fields Medal in 1954 and the Abel Prize in 2003.

Biography

Early years

Born in Bages, Pyrénées-Orientales, France, Serre was educated at the Lycée de Nîmes and then from 1945 to 1948 at the École Normale Supérieure in Paris. He was awarded his doctorate from the Sorbonne in 1951. From 1948 to 1954 he held positions at the Centre National de la Recherche Scientifique in Paris. In 1956 he was elected professor at the Collège de France, a position he held until his retirement in 1994.

Career

From a very young age he was an outstanding figure in the school of Henri Cartan, working on algebraic topology, several complex variables and then commutative algebra and algebraic geometry, in the context of sheaf theory and homological algebra techniques. Serre's thesis concerned the Leray-Serre spectral sequence associated to a fibration. Together with Cartan, Serre established the technique of using Eilenberg-MacLane spaces for computing homotopy groups of spheres, which at that time was considered as the major problem in topology.

In his speech at the Fields Medal award ceremony in 1954, Hermann Weyl praised Serre in seemingly extravagant terms, and also made the point that the award was for the first time awarded to an algebraist. Serre subsequently changed his research focus; he apparently thought that homotopy theory, where he had started, was already overly technical. However, Weyl's perception that the central place of classical analysis had been challenged by abstract algebra has subsequently been justified, as has his assessment of Serre's place in this change.

Algebraic geometry

In the 1950s and 1960s, a fruitful collaboration between Serre and the two-years-younger Alexander Grothendieck led to important foundational work, much of it motivated by the Weil conjectures. Two major foundational papers by Serre were *Faisceaux Algébriques Cohérents* (FAC), on coherent cohomology) and *Géométrie Algébrique et Géométrie Analytique* (GAGA).

Even at an early stage in his work Serre had perceived a need to construct more general and refined cohomology theories to tackle the Weil conjectures. The problem was that the cohomology of a coherent sheaf over a finite field couldn't capture as much topology as singular cohomology with integer coefficients. Amongst Serre's early candidate theories of 1954–55 was one based on Witt vector coefficients.

Around 1958 Serre suggested that isotrivial principal bundles on algebraic varieties — those that become trivial after pullback by a finite étale map — are important. This acted as one important source of inspiration for Grothendieck to develop étale topology and the corresponding theory of étale cohomology. These tools, developed in full by Grothendieck and collaborators in *Séminaire de géométrie algébrique* (SGA) 4 and SGA 5, provided the tools for the eventual proof of the Weil conjectures.

In later years Serre was sometimes a source of counterexamples to over-optimistic extrapolations. He also had a close working relationship with Pierre Deligne, who eventually finished the proof of the Weil conjectures.

Other work

From 1959 onward Serre's interests turned towards number theory, in particular class field theory and the theory of complex multiplication.

Amongst his most original contributions were: the concept of algebraic K-theory; the Galois representation theory for ℓ -adic cohomology and the conceptions that these representations were "large"; and the Serre conjecture on mod- p representations that made Fermat's last theorem a connected part of mainstream arithmetic geometry.

Honours and awards

Serre, at twenty-seven in 1954, is the youngest ever to be awarded the Fields Medal. In 1985, he went on to win the Balzan Prize, the Steele Prize in 1995, the Wolf Prize in Mathematics in 2000, and was the first recipient of the Abel Prize in 2003. Serre and John Thompson are the only laureates of all three of the Fields Medal, the Wolf Prize, and the Abel Prize.

See also

- Serre duality
- Serre's multiplicity conjectures
- Serre's property FA
- Serre spectral sequence
- Serre fibration
- Serre twist sheaf
- Thin set in the sense of Serre
- Quillen-Suslin theorem
- Nicolas Bourbaki

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- *Corps Locaux* (1962) as *Local Fields* (1980)
- *Cohomologie Galoisienne* (1964) Collège de France course 1962–3, as *Galois Cohomology* (1997)
- *Algèbre Locale, Multiplicités* (1965) Collège de France course 1957–8, as *Local Algebra* (2000)
- *Lie Algebras and Lie Groups* (1965) 1964 Harvard lectures
- *Algèbres de Lie Semi-simples Complexes* (1966) as *Complex Semisimple Lie Algebras* (1987)
- *Abelian ℓ -Adic Representations and Elliptic Curves* (1968)
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- *Cohomological Invariants in Galois Cohomology* (2003) with Skip Garibaldi and Alexander Merkurjev
- Serre, Jean-Pierre (2001), *Exposés de séminaires 1950–1999*, Société Mathématique de France, ISBN 2-85629-103-1
- *Grothendieck–Serre Correspondence* (2003) edited with Pierre Colmez

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- Jean-Pierre Serre ^[1] at the Mathematics Genealogy Project
- Jean-Pierre Serre ^[2] at the French Academy of Sciences, in French.
- Interview with Jean-Pierre Serre ^[3] in Notices of the American Mathematical Society.
- An Interview with Jean-Pierre Serre ^[4] by C.T. Chong and Y.K. Leong, National University of Singapore.
- How to write mathematics badly ^[5] a public lecture by Jean-Pierre Serre on writing mathematics.

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J.H.C.Whitehead

J. H. C. Whitehead	
 John Henry Constantine Whitehead	
Born	11 November 1904 Madras (Chennai), India
Died	8 May 1960 (aged 55) Princeton, New Jersey
Residence	 United Kingdom, U.S.
Nationality	 British
Fields	Mathematics
Institutions	Oxford University
Alma mater	Oxford University Princeton University
Doctoral advisor	Oswald Veblen
Doctoral students	Michael Barratt Ronald M. Brown Wilfred H. Cockroft Victor K. A. M. Gugenheim Graham Higman Peter Hilton Ioan James Brian Steer
Known for	CW complex Simple homotopy Whitehead group Whitehead manifold Whitehead product

John Henry Constantine Whitehead (11 November 1904–8 May 1960), known as Henry, was a British mathematician and was one of the founders of homotopy theory. He was born in Chennai (then known as Madras), in India, and died in Princeton, New Jersey, in 1960.

Life

J. H. C. (Henry) Whitehead was the son of the Right Rev. Henry Whitehead, Bishop of Madras and brother of A. N. Whitehead, and of Isobel Duncan, who had herself studied mathematics at Oxford. He was brought up in Oxford, went to Eton and read mathematics at Balliol College, Oxford. After a year working as a stockbroker, he started a Ph.D. in 1929 at Princeton University. His thesis, titled *The representation of projective spaces*, was

written under the direction of Oswald Veblen in 1930. While in Princeton, he also worked with Solomon Lefschetz.

He became a fellow of Balliol in 1933. In 1934 he married the concert pianist Barbara Smyth, great-great-granddaughter of Elizabeth Fry and a cousin of Peter Pears; they had two sons. During the Second World War he worked on operations research for submarine warfare. Later, he joined the codebreakers at Bletchley Park, and by 1945 was one of some fifteen mathematicians working in the "Newmanry", a section headed by Max Newman and responsible for breaking a German teleprinter cipher using machine methods.^[1] Those methods included the Colossus machines, early digital electronic computers.^[1]

From 1947 to 1960 he was the Waynflete Professor of Pure Mathematics at Magdalen College, Oxford.

He became president of the London Mathematical Society (LMS) in 1953, a post he held until 1955.^[2] The LMS established two prizes in memory of J. H. C. Whitehead. The first is the annually awarded, to multiple recipients, Whitehead Prize; the second a biennially awarded Senior Whitehead Prize.^[3]

In the late 1950s, Whitehead approached Robert Maxwell, then chairman of Pergamon Press, to start a new journal, *Topology*, but died before its first edition appeared in 1962.

Work

His definition of CW complexes gave a setting for homotopy theory that became standard. He introduced the idea of simple homotopy theory, which was later much developed in connection with algebraic K-theory. The Whitehead product is an operation in homotopy theory. The Whitehead problem on abelian groups was solved (as an independence proof) by Saharon Shelah. His involvement with topology and the Poincaré conjecture led to the creation of the Whitehead manifold. The definition of crossed modules is due to him. Whitehead also made important contributions in differential topology, particularly on triangulations and their associated smooth structures.

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- J. H. C. Whitehead, *On incidence matrices, nuclei and homotopy types*, *Ann. of Math.* (2) **42** (1941), 1197–1239.
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- J. H. C. Whitehead, *Combinatorial homotopy. II.*, *Bull. Amer. Math. Soc.* **55** (1949), 453–496
- J. H. C. Whitehead, *A certain exact sequence*, *Ann. of Math.* (2) **52** (1950), 51–110
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See also

- Simple homotopy
- Spanier-Whitehead duality
- Whitehead group
- Whitehead link
- Whitehead theorem
- Whitehead torsion

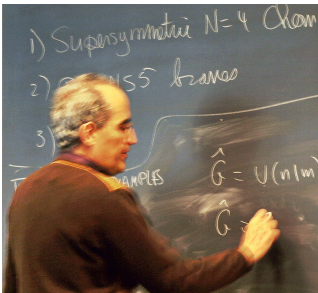
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Edward Witten

Edward Witten	
	
Born	August 26, 1951 Baltimore, Maryland
Nationality	American
Fields	Theoretical physicist
Institutions	Institute for Advanced Study
Alma mater	Brandeis University
Doctoral advisor	David Gross
Doctoral students	Cumrun Vafa, Eva Silverstein, Shamit Kachru
Known for	string theory, M-theory, quantum field theory
Notable awards	Fields Medal (1990) Crafoord Prize (2008)

Edward Witten (born August 26, 1951) is an American theoretical physicist and professor at the Institute for Advanced Study. He is one of the world's leading researchers in superstring theory. He has made extensive contributions to theoretical physics, and in 1990 he was awarded the Fields Medal for his influence on the development of mathematics. In 1995, he suggested the existence of M-theory at a conference at the University of Southern California, and used M-theory to explain a number of previously observed dualities, sparking a flurry of new research in string theory called the second superstring revolution.

Birth and education

Edward Witten was born in Baltimore, Maryland to a Jewish family, the son of Lorraine W. Witten and Louis Witten, a theoretical physicist specializing in gravitation and general relativity. He received his bachelor's degree in history (with a minor in linguistics) from Brandeis University. Witten planned to become a political journalist, and published articles in *The New Republic* and *The Nation*. He worked briefly for George McGovern's presidential campaign. Then, he attended the University of Wisconsin-Madison for one semester as an economics graduate student before dropping out. He then returned to academia, enrolling in applied mathematics at Princeton University before shifting departments and receiving a Ph.D. in physics in 1976 under David Gross, the 2004 Nobel laureate in Physics.

Academic career

After completing his PhD, he worked at Harvard University as a Junior Fellow and at Princeton as a professor. He was a Professor of Physics at Princeton University from 1980 to 1987. He also was briefly at Caltech for two years from 1999 to 2001. He is currently the Charles Simonyi Professor of Mathematical Physics at the Institute for Advanced Study in Princeton, New Jersey.

Witten has the highest h -index (110) of any living physicist.^{[1] [2]}

Research and achievements

Witten has made extensive contributions to theoretical physics, in work that has spawned a large number of highly mathematical results. He has been active primarily in quantum field theory and string theory, and in related areas of topology and geometry. His many contributions include a simplified proof of the positive energy theorem involving spinors in general relativity, his work relating supersymmetry and Morse theory, his introduction of topological quantum field theory and his related work on mirror symmetry and supersymmetric gauge theories, and his conjecture of the existence of M-theory.

Witten was awarded the Fields Medal^[3] by the International Mathematical Union in 1990, becoming the first physicist to win the prize. Sir Michael Atiyah said of Witten, "Although he is definitely a physicist, his command of mathematics is rivaled by few mathematicians... Time and again he has surprised the mathematical community by a brilliant application of physical insight leading to new and deep mathematical theorems... he has made a profound impact on contemporary mathematics. In his hands physics is once again providing a rich source of inspiration and insight in mathematics."^[4] One such example of his impact on pure mathematics is his framework for understanding the Jones polynomial using Chern-Simons theory. This had far reaching implications on low-dimensional topology and led to quantum invariants such as the Witten-Reshetikhin-Turaev invariants.

He is currently working on the possible relations between Gauge theories and Geometric Langlands program.

Personal life

He is married to Chiara Nappi, who is a professor of physics at Princeton University and they have two daughters Ilana and Daniela, and one son, Raphael (Rafi). His brother Matt Witten is a screenwriter and producer for several popular TV series including *L.A. Law* and *House*.

Awards and honors

Witten has been honored with numerous awards, including a MacArthur Grant (1982), a Fields Medal (1990), the National Medal of Science^[5] (2002), Pythagoras Award^[6] (Croton, 2005), and the Crafoord Prize (2008). Pope Benedict XVI also appointed Witten as a member of the Pontifical Academy of Sciences (2006). He also appeared in the list of *TIME* magazine's 100 most influential people of 2004. In 2000, he was awarded the Nemmers Prize in Mathematics.

See also

- Gromov-Witten invariant
- Seiberg-Witten gauge theory
- Seiberg-Witten invariant
- Vafa-Witten theorem
- Weinberg-Witten theorem
- Wess-Zumino-Witten model
- Witten index

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External links

- Witten's web page at the Institute (<http://www.sns.ias.edu/~witten/>)
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